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# FORBIDDEN SETS OF PLANAR RATIONAL SYSTEMS OF DIFFERENCE EQUATIONS WITH COMMON DENOMINATOR

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The forbidden sets of systems of first order rational difference equations in the plane in which the denominators are common for all the components of the system is studied. Such forbidden sets are composed of lines which, depending of some spectral properties of an associated matrix, can either be a finite number or lines or an infinity of lines converging to either an invariant line or to a finite number of lines itersecting in a fixed point or else it can be dense in a large subset of $\mathbb{R}^2$.

## 1. INTRODUCTION

Rational systems of difference equations have recently attracted a huge interest not only due to their apparent tractability in comparison with other non-linear equations but to the fact that they commonly appear in applications [**6, 9, 10, 11**]. In [**7**], the authors give some results on the behaviour of the solutions and provide some open questions for systems of equations defined by linear fractionals:

$$\left.\begin{aligned} x_{n+1} &= \frac{\alpha_1 x_n + \beta_1 y_n + \gamma_1}{A_1 x_n + B_1 y_n + C_1} \\ y_{n+1} &= \frac{\alpha_2 x_n + \beta_2 y_n + \gamma_2}{A_2 x_n + B_2 y_n + C_2} \end{aligned}\right\}, \quad n = 0, 1, \ldots$$

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where the parameters are considered non-negative. Such restriction on the coefficients ensures that the orbits starting with a positive initial condition are complete and, therefore, there always exists an open subset of $\mathbb{R}^2$ in which all the corresponding orbits are well defined. However, in many interesting cases coefficients may vary in sign or the initial conditions have to be taken negative. In those cases, the existence of full orbits is not obviously guaranteed and a so called *forbidden set* or *bad set* of points whose orbit is uncomplete appears. Obviously, the knowledge of such forbidden set is necessary to define the maximal open set of points in which the corresponding discrete dynamical system can be defined (if any). Nevertheless, the complete description of those forbidden sets in the general case is a complicated problem and, whenever it can be done, the techniques used for their determination should be adapted to the particularities of the equations [**13, 14**].

Linear fractionals in one variable give rise to the well known Riccati equations [**1, 12**], which arise from iteration of a Möbius transformation. Some higher order rational difference equations can be studied with the aid of such Riccati ones; for instance, the second order rational equation

$$x_{n+1} = \frac{x_{n-1}}{a + bx_{n-1}x_n}\,, \qquad n \geq 1$$

studied in [**5, 15**] can be transformed in the system of equations

$$\left.\begin{aligned} u_{n+1} &= \frac{u_n}{a + bu_n} \\ x_{n+1} &= \frac{u_{n+1}}{x_n} \end{aligned}\right\}$$

which is obviously composed of an autonomous Riccati equation and a non-autonomous one. Further examples of higher order systems of rational difference equations which are solvable and are, in certain way, related to the second order equation above or to the Riccati equation can be found in [**14, 16**] and references cited therein.

A higher dimensional generalization to Riccati equations may be done by considering systems of linear fractionals sharing denominator. Those special rational systems have been studied in [**4**] to obtain a large family of non-trivial examples of globally periodic dynamical systems. Certain biological applications of rational systems sharing denominator can be found in [**2**] where the authors study a particular case which yields a simple Leslie/Gower model for competition of biological species in a limited resource proving that competitive exclusion occurs and proposing stocking strategies to ensure the persistence of weaker species. The dynamics in the planar case has been partially studied in [**3**], while the work is focused in the long-time behaviour of complete solutions. The aim of this paper is to give a description of the forbidden set for all rational systems with common denominator in the plane.

A key fact in the study of this kind of rational systems is that they can somehow be expressed as a certain projection of a linear system and, as a consequence,

can be described in terms of certain square matrices. As it can be seen in the references cited above, the spectral properties of such matrices are substantial to describe the dynamics of the system. For example, fixed points of the system are related to certain eigenvalues of the associated matrix. However, usual matrix similarity does not, in general, preserve the dynamics of the system and, in particular, the corresponding forbidden sets can differ a lot between two of our rational systems with similar associated matrices. We will see however that affine transformations of the plane do preserve the dynamics and this will let us give a full description of the forbidden set in all the cases.

The paper is divided in three sections besides the introduction. The first section contains, mainly, the basic notions and the preliminary statement of the problem. As we see throughout the paper, in many cases the forbidden set accumulates on an invariant line and, thus, we begin Section 2 with a characterization of invariant lines for our rational systems. We also study in such section the reducibility to some simpler cases by means of affine changes of variables and we start the description of the forbidden sets in some cases where, under such a change of variables, the problem reduces to the characterization of the forbidden set of a scalar Riccati equation. Finally, in the last section we also use affine transformations to reduce the remaining cases to some canonical ones. It should be noticed that since the forbidden sets are composed of lines, affine transformations do not modify the general properties (as, for example, parallelism) of the set components.

The main results of the paper can be summarized, roughly speaking, as follows: In the cases in which the associated matrix has only real eigenvalues, the forbidden set is either a single line or an infinity of lines which converge pointwise to certain invariant line. When there exist complex eigenvalues and the forbidden set is composed of infinite lines, the lines can either converge to a single invariant line or converge to a finite number of lines or else are dense either in the whole plane or in the exterior of a parabola.

## 2. PRELIMINARIES

We consider planar systems of rational difference equations in which the denominators of both equations are equal. Thus, we deal with systems of the form

$$\left.\begin{aligned} x_{n+1} &= \frac{\alpha_1 x_n + \beta_1 y_n + \gamma_1}{\alpha x_n + \beta y_n + \gamma} \\ y_{n+1} &= \frac{\alpha_2 x_n + \beta_2 y_n + \gamma_2}{\alpha x_n + \beta y_n + \gamma} \end{aligned}\right\}, \quad n = 0, 1, \ldots \tag{1}$$

where the greek parameters represent fixed real numbers and $\alpha, \beta$ and $\gamma$ do not vanish simultaneously. Since the case in which $\alpha = \beta = 0$ corresponds to a linear system of equations, we always actually suppose that $(\alpha, \beta) \neq (0, 0)$.

As it was shown in the general higher dimensional case in [**4**], if one denotes by $q$ the mapping given by $q(a_1, a_2, a_3) = (a_1/a_3, a_2/a_3)$ for $(a_1, a_2, a_3) \in \mathbb{R}^3$ with $a_3 \neq 0$ and $\ell : \mathbb{R}^2 \to \mathbb{R}^3$ is given by $\ell(a_1, a_2) = (a_1, a_2, 1)^t$, where $M^t$ stands

for the transposed of a matrix $M$, then the system can be written in the form $(x_{n+1}, y_{n+1}) = q \circ A \circ \ell(x_n, y_n)$ for the square matrix

$$A = \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha & \beta & \gamma \end{pmatrix}. \tag{2}$$

When the matrix $A$ is singular and $(\alpha, \beta) \neq (0,0)$ it can be easily seen that there exists $C_1, C_2 \in \mathbb{R}$ such that, up to a permutation of the variables, $x_{n+1} = C_1 y_{n+1} + C_2$ holds for all $n \geq 1$. This means that our system can be reduced to a single Riccati equation in one of the variables and a direct substitution to obtain the other one. Therefore, in order to avoid this case, we consider that $A$ is invertible.

Since we frequently use the maps $q$ and $\ell$, we recall the following from [**4**]:

**Lemma 1.** *Let $q$ and $\ell$ be defined as above. One has:*

(a) $q\big(\ell(a_1, a_2)\big) = (a_1, a_2)$ *for all* $(a_1, a_2) \in \mathbb{R}^2$.

(b) $\ell\big(q(a_1, a_2, a_3)\big) = (1/a_3)(a_1, a_2, a_3)$ *for all* $(a_1, a_2, a_3) \in \mathbb{R}^3$ *with* $a_3 \neq 0$.

(c) *If* $a_3 \neq 0 \neq b_3$, *then* $q(a_1, a_2, a_3) = q(b_1, b_2, b_3)$ *is equivalent to* $(a_1, a_2, a_3) = \lambda(b_1, b_2, b_3)$ *for some* $\lambda \neq 0$.

(d) *If $A$ is an $3 \times 3$ matrix, then* $q(A\ell(q(v))) = q(Av)$ *for all* $v \in \mathbb{R}^3$ *such that* $q(v)$ *and* $q(Av)$ *exist.*

The description of our rational system by means of the matrix $A$ let us completely determine its solutions in terms of the powers of $A$. Actually, the explicit solution to the system with initial condition $(x_0, y_0)$ is given by

$$(x_{n+1}, y_{n+1}) = q \circ A^n (x_0, y_0, 1)^t. \tag{3}$$

Therefore, our system can be completely solved and the solution starting at $(x_0, y_0)$ is just the projection by $q$ of the solution of the linear system $v_{n+1} = Av_n$ with initial condition $v_0 = (x_0, y_0, 1)^t$ whenever such projection exists. If this is not the case, then the orbit starting at $(x_0, y_0)$ is not complete and we say that $(x_0, y_0)$ lies in the forbidden set. Explicitly, we have

**Definition 2.** The *forbidden set* of the system (1) is given by the union of lines:

$$\mathcal{F} = \bigcup_{n \geq 1} \left\{ \, (x, y) \in \mathbb{R}^2 \, : \, (0,0,1)A^n(x, y, 1)^t = 0 \, \right\}, \tag{4}$$

We call the first of those lines *principal forbidden line*; namely,

$$\mathcal{PF} = \left\{ \, (x, y) \in \mathbb{R}^2 \, : \, (0,0,1)A(x, y, 1)^t = 0 \, \right\} = \left\{ \, (x, y) \in \mathbb{R}^2 \, : \, \alpha x + \beta y + \gamma = 0 \, \right\}$$

.

Remark 3. In [**4**] we studied global periodicity of $n$-dimensional rational systems of equations sharing denominator. They can be described similarly by a $n \times n$ matrix $A$ and one sees that global periodicity of the system is equivalent to $A^k = \lambda I$ for some $k \in \mathbb{N}$. This implies that the forbidden set of such systems is obviously composed of a finite number of hyperplanes. Note that, in particular, when the map is globally 2-periodic then the forbidden set reduces to the first forbidden hyperplane.

## 3. INVARIANT LINES, AFFINE TRANSFORMATIONS AND THE CASES REDUCIBLE TO SCALAR EQUATIONS

From now on, we denote by $\mathrm{pr} : \mathbb{R}^3 \to \mathbb{R}$ the projection on the third component $\mathrm{pr}(x, y, z) = z$ and by $U_0$ the subspace of vectors in $\mathbb{R}^3$ with vanishing third component $U_0 = \mathrm{pr}^{-1}(\{0\})$. It is obvious that $U_0$ is an $A$-invariant subspace if and only if $(\alpha, \beta) = (0, 0)$. Since this case was excluded because it corresponds to a linear system of equations, we can always assume that $U_0$ is not stable by $A$.

Remark 4. It has been proved in [**4**] that a point $(a, b) \in \mathbb{R}^2$ is an equilibrium of the rational system if and only if $(a, b, 1)$ is an eigenvector of the associated matrix $A$ for a non null eigenvalue. This obviously suggests a relation for a regular matrix $A$ between certain $A$-invariant subspaces of $\mathbb{R}^3$ and geometrical objects which are invariant under the rational system. In fact, when we consider $F = q \circ A \circ \ell$ then every $F$-invariant line in $\mathbb{R}^2$ is the projection by $q$ of an $A$-invariant 2-dimensional subspace of $\mathbb{R}^3$, as it is shown in the following result. Note that since a line in $\mathbb{R}^2$ may intersect the principal forbidden line, we will consider that it is $F$-invariant if it is not the principal forbidden line and the image of its points which do not lie on $\mathcal{P}F$ remains on the line.

**Proposition 5.** *Let $A$ be an invertible real matrix of order 3 such that $U_0$ is not $A$-invariant and put $F = q \circ A \circ \ell$.*

(1) *There is a one-to-one correspondence between $F$-invariant lines in $\mathbb{R}^2$ and $A$-invariant 2-dimensional subspaces of $\mathbb{R}^3$.*

(2) *There exists at least an $F$-invariant line.*

(3) *In particular, if $F$ admits two different equilibria, then the line passing through them is $F$-invariant.*

**Proof.** Suppose first that $U$ is a 2-dimensional subspace in $\mathbb{R}^3$ which remains stable by $A$. Since $U$ and $U_0$ are 2-dimensional and unequal, they have a one-dimensional intersection and, hence, we may find a basis $\{u_0, u_1\}$ of $U$ such that $u_0 = (a_0, b_0, 0)^t$ and $u_1 = (a_1, b_1, 1)^t$. It is straightforward to see that the line $L = \{\ (a_1, b_1) + t(a_0, b_0)\ :\ t \in \mathbb{R}\ \}$ is independent of the choice of $a_0, b_0, a_1, b_1$ and

$$F(a_1 + ta_0, b_1 + tb_0) = q\big(A(u_1) + tA(u_0)\big).$$

Since $U$ is stable by $A$, there exist $\xi_0, \xi_1 \in \mathbb{R}$ (depending on $t$) such that $A(u_1) + tA(u_0) = \xi_0 u_0 + \xi_1 u_1$. If $\xi_1 = 0$, then $(a_1 + ta_0, b_1 + tb_0)$ is in the prinicipal forbidden

line whereas, if $\xi_1 \neq 0$, then

$$F(a_1 + ta_0, b_1 + tb_0) = q\big(A(u_1) + tA(u_0)\big) = u_1 + \frac{\xi_0}{\xi_1} u_0 \in L.$$

Conversely, if a line $L$ is $F$-invariant, then we may choose a couple of distinct points $(x_1, y_1), (x_2, y_2)$ within the line which do not belong to $\mathcal{P}F$. Consider $U$ the linear span of $\{(x_1, y_1, 1)^t, (x_2, y_2, 1)^t\}$. Since $L$ is $F$-invariant, for $i \in \{1, 2\}$, one has that there exist $\xi \in \mathbb{R}$ such that

$$q\big(A(x_i, y_i, 1)^t\big) = (x_1, y_1) + \xi(x_2 - x_1, y_2 - y_1) = q\big((1-\xi)(x_1, y_1, 1)^t + \xi(x_2, y_2, 1)^t\big),$$

which is only possible if

$$A(x_i, y_i, 1)^t = \lambda\big((1-\xi)(x_1, y_1, 1)^t + \xi(x_2, y_2, 1)^t\big)$$

for some $\lambda \in \mathbb{R}$. This proves that $U$ is stable by $A$.

Recall that every matrix of order three admits, at least, one 2-dimensional invariant subspace. This fact is clear when the Jordan has (at least) two Jordan blocks but also in the case of a unique real eigenvalue $\lambda$ of index 3, where the subspace $\ker(A - \lambda I)^2$ is obviously $A$-invariant and 2-dimensional. The second assertion is, hence, straightforward.

Finally, if $F$ admits two different fixed points $(a_1, b_1)$ and $(a_2, b_2)$ then the vectors $(a_1, b_1, 1), (a_2, b_2, 1)$ are eigenvalues of $A$. Therefore, the subspace of $\mathbb{R}^3$ spanned by them is $A$-invariant and it is straightforward to see that the corresponding $F$-invariant line passes through the fixed points. □

Obviously, there is an evident strong relation between the spectral properties of the matrix $A$ and the dynamics of the corresponding system of difference equations. However, several facts should be noticed. Firstly, that proportional non-null matrices generate the same system of equations and, secondly, that matrices with identical Jordan form may lead to different dynamics. This is obvious if one considers, for instance, the three similar matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 3 \end{pmatrix}.$$

The three of them have the same Jordan form but there are substantial differences among the corresponding dynamical systems. The first matrix gives rise to a linear system of difference equations with a unique fixed point and, obviously, no forbidden set. The systems obtained for the other two matrices do have a forbidden set but they have a different number of equilibria: one has two and the other one three. This occurs because the second matrix has an eigenvector whose third coordinate is null.

This suggests that, although the Jordan form of the matrix $A$ will be of great importance, one also has to take into account if it is possible or not to find

eigenvectors (or, more precisely, Jordan basis) of $A$ outside the subspace $U_0$. Thus, in order to reduce our problem to the study of some canonical matrices, we cannot use the Jordan matrix itself but a canonical form obtained by a stronger similarity. The key to do this is the use of affine transformations.

**Definition 6.** An *affine transformation* on $\mathbb{R}^2$ is the composition of a linear transformation and a translation on $\mathbb{R}^2$. Explicitly, an affine transformation $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $\phi(X) = MX + X_0$ for all $X \in \mathbb{R}^2$, where $M$ is a $2\times 2$ matrix an $X_0 \in \mathbb{R}^2$. If one considers the $3 \times 3$ matrix

$$M_\phi = \left(\begin{array}{c|c} M & X_0 \\ \hline 0 & 1 \end{array}\right),$$

then the affine transformation may be expressed as:

$$\phi(X) = q \circ M_\phi \circ \ell(X).$$

Obviously, an affine transformation is invertible if and only if $M_\phi$ is so and, in this case,

$$M_{\phi^{-1}} = M_\phi^{-1} = \left(\begin{array}{c|c} M^{-1} & -M^{-1}X_0 \\ \hline 0 & 1 \end{array}\right).$$

**Proposition 7.** *Let $A$ be a real matrix of order 3, $F = q \circ A \circ \ell$ and let $\phi$ be an invertible affine transformation on $\mathbb{R}^2$. The change of variables $X_n = \phi(\widetilde{X}_n)$, $n \geq 0$, transforms the system $X_{n+1} = F(X_n)$ into $\widetilde{X}_{n+1} = \widetilde{F}(\widetilde{X}_n)$ with $\widetilde{F} = q \circ (M_\phi^{-1} A M_\phi) \circ \ell$.*

**Proof.** If $X_n = \phi(\widetilde{X}_n)$, for all $n \geq 0$, then using assertion (d) of Lemma 1 we have

$$\phi(\widetilde{X}_{n+1}) = F\big(\phi(\widetilde{X}_n)\big) = q\big(A\ell(qM_\phi\ell)(\widetilde{X}_n)\big) = q\big(AM_\phi\ell(\widetilde{X}_n)\big)$$

and, therefore,

$$\widetilde{X}_{n+1} = \phi^{-1}\Big(q\big(AM_\phi\ell(\widetilde{X}_n)\big)\Big) = q\Big(M_\phi^{-1}\ell q\big(AM_\phi\ell(\widetilde{X}_n)\big)\Big) = q\big(M_\phi^{-1}AM_\phi\ell(\widetilde{X}_n)\big),$$

where Lemma 1 was also used in the last equality.

Remark 8. According to the proposition above, we can transform one of our rational systems in a simpler one by performing a conjugation of the associated matrix by the matrix of an affine transformation. Recall that affine transformations preserve parallelism and, therefore, every conclusion on parallel lines for one of the systems is clearly applicable to the transformed one.

A conjugation of a matrix $A$ by a matrix $M_\phi$ is obviously equivalent to a change of coordinates in the linear map defined by $A$ to a new basis $\{v_1, v_2, v_3\}$ with $\mathrm{pr}(v_3) = 1$ and $\mathrm{pr}(v_1) = \mathrm{pr}(v_2) = 0$.

Next result describes the case in which there exist eigenvectors of the associated matrix with null third component. We say that a first order Riccati equation is *proper* if it is neither linear nor constant.

**Proposition 9.** *Let $A$ be an invertible matrix given by* (2) *with* $(\alpha, \beta) \neq (0,0)$ *and* $F = q \circ A \circ \ell$. *If $A$ admits a real eigenvector $v_1$ such that* $\mathrm{pr}(v_1) = 0$ *then the system* $X_{n+1} = F(X_n)$ *can be reduced by an affine transformation to a first order proper Riccati equation in one variable and a first order linear equation in the other one.*

**Proof.** Let us consider that $A$ admits an eigenvector $v_1 \in \mathbb{R}^3$ such that $\mathrm{pr}(v_1) = 0$ and suppose that $\lambda_1$ is the corresponding eigenvalue. Choose $v_2, v_3 \in \mathbb{R}^3$ with $\mathrm{pr}(v_2) = 0$, $\mathrm{pr}(v_3) = 1$ and such that $\{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$. The associated matrix to the linear map defined by $A$ in this new basis has the form

$$\begin{pmatrix} \lambda_1 & a_1 & b_1 \\ 0 & a_2 & b_2 \\ 0 & a & b \end{pmatrix}$$

This means that, under the change of variables defined by the affine transformation $\phi$ such that $M_\phi = (v_1|v_2|v_3)$ one obtains the new system

$$\begin{aligned} \tilde{y}_{n+1} &= \frac{a_2 \tilde{y}_n + b_2}{a\tilde{y}_n + b} \\ \tilde{x}_{n+1} &= \frac{\lambda_1}{a y_n + b}\, \tilde{x}_n + \frac{a_1 \tilde{y}_n + b_1}{a\tilde{y}_n + b}, \end{aligned}$$

as claimed. Note that $a$ cannot be null and, therefore, the Riccati equation is proper.

Remark 10. Notice that in the case of the proposition above the description of the forbidden set reduces by the corresponding affine transformation to the one of the forbidden set for the Riccati equation. Forbidden sets for Riccati equations are well-known (see [**8**] and references therein) and, therefore, the forbidden lines for our planar systems can be easily deduced. For the sake of completeness, we give the explicit description in the following result.

**Corollary 11.** *Let $A$ be an invertible matrix given by* (2) *with* $(\alpha, \beta) \neq (0,0)$ *and suppose that it admits an eigenvector $v_1$ associated with a real eigenvalue $\lambda_1$ such that* $\mathrm{pr}(v_1) = 0$.

*The forbidden set of the dynamical system defined by $F = q \circ C \circ \ell$ is composed of lines which are parallel to $\mathcal{PF}$ and, further*:

(a) *If the eigenvalues of $A$ are real and* $\mathrm{trace}(A) = \lambda_1$, *then $\mathcal{F} = \mathcal{PF}$*

(b) *If the eigenvalues of $A$ are real and* $\mathrm{trace}(A) \neq \lambda_1$, *then $\mathcal{F}$ has an infinity of lines converging to an invariant line.*

(c) *If $A$ has complex eigenvalues whose argument is a rational multiple of $\pi$ then $\mathcal{F}$ consists of a finite number of lines.*

(d) *If $A$ has complex eigenvalues whose argument is not a rational multiple of $\pi$ then $\mathcal{F}$ is a dense set in $\mathbb{R}^2$.*

**Proof.** As we have shown in the proof of the proposition above, by an affine change of variables we can consider that

$$A = \begin{pmatrix} \lambda_1 & a_1 & b_1 \\ 0 & a_2 & b_2 \\ 0 & a & b \end{pmatrix}.$$

It is obvious that if $\{\mu_n\}_{n\geq 1}$ denote the forbidden points for the proper Riccati equation

$$\tilde{y}_{n+1} = \frac{a_2\tilde{y}_n + b_2}{a\tilde{y}_n + b},$$

then the forbidden set of $F$ is given by $\mathcal{F} = \bigcup_{n\geq 1}\{\ (x,y) \in \mathbb{R}^2\ :\ y = \mu_n\ \}$. The assymptotic behaviour of the set $\{\mu_n\}$ in terms of the eigenvalues of the matrix

$$B = \begin{pmatrix} a_2 & b_2 \\ a & b \end{pmatrix}$$

are well-known and can also be easily deduced from the description of the forbidden set given, for instance, in [**8**]. Explicitly, one has that when the eigenvalues of $B$ are real and distinct in modulus, $\{\mu_n\}$ converges to the fixed point corresponding to the eigenvalue of minimal modulus; when they are real and equal, the forbidden set accumulates in the unique fixed point; when they are real with the same modulus and distinct sign, the equation is globally 2-periodic and there is only one forbidden point; if the eigenvalues are complex numbers whose argument is a rational multiple of $\pi$, the equation is globally periodic and there are, at most, $m-1$ forbidden points where $m$ denotes the minimal period; and, finally, if they are complex and their argument is not a rational multiple of $\pi$, then $\{\mu_n\}$ is dense in the real line. Therefore, assertions (a),(c) and (d) are obvious and, in order to prove (b) it suffices to recall that if $\mu$ is a fixed point of $\tilde{y}_{n+1} = \dfrac{a_2\tilde{y}_n + b_2}{a\tilde{y}_n + b}$, then $y = \mu$ is clearly an invariant line for

$$F(x,y) = \left(\frac{\lambda_1}{ay_n + b}\,\tilde{x}_n + \frac{a_1\tilde{y}_n + b_1}{a\tilde{y}_n + b}, \frac{a_2\tilde{y}_n + b_2}{a\tilde{y}_n + b}\right).$$

Actually, it is the invariant line obtained by the projection of the $A$-invariant subspace spanned by $\{(1,0,0),(0,\mu,1)\}$.

## 4. FORBIDDEN SETS IN THE CASES WHICH ARE NOT REDUCIBLE TO A SCALAR EQUATION

In this section we always consider that the $3\times 3$ matrix $A$ given by (2) is non singular and its eigenvectors lie outside the subspace $U_0$. In such case, the geometric multiplicity of each eigenvalue must be 1, otherwise the corresponding eigenspace would intersect the 2-dimensional subspace $U_0$. Besides, since $A$ has odd order, it admits at least a real eigenvalue $\lambda$. We begin with the following result which reduces our problem to a certain "canonical" matrix:

**Proposition 12.** *Let $A$ be non singular and suppose that its eigenvectors lie outside the subspace $U_0$. There is an invertible affine transformation $\phi$ on $\mathbb{R}^2$ such that*

$$M_\phi^{-1} A M_\phi = \left(\begin{array}{c|c} J & 0 \\ \hline h & \lambda \end{array}\right)$$

*where $J$ is a $2 \times 2$ real Jordan matrix and $h$ is non-null.*

*Moreover, when $A$ has a unique eigenvalue of multiplicity 3, $M_\phi^{-1} A M_\phi$ can be taken to be the lower triangular Jordan form; in the remaining cases, if $J$ has two real eigenvalues one can choose $h = (1,1)(J - \lambda I)$ whereas if it has complex eigenvalues or a unique real eigenvalue and $J$ is lower triangular, one can choose $h = (0,1)(J - \lambda I)$.*

**Proof.** In case that there is a unique eigenvalue of multiplicity 3 (which means a unique Jordan block), one chooses a basis $\{w_1, w_2, v\}$ for which the Jordan form is lower triangular and $\mathrm{pr}(v) = 1$ (dividing the three vectors by the third component of the eigenvector if necessary). Now, let us consider

$$v_1 = w_1 - \mathrm{pr}(w_2) w_2 + \big(\mathrm{pr}(w_2)^2 - \mathrm{pr}(w_1)\big) v\,, \quad v_2 = (A - \lambda I) v_1.$$

A direct calculation proves that $\{v_1, v_2, v\}$ is again a Jordan basis for a lower triangular Jordan form and that $\mathrm{pr}(v_1) = \mathrm{pr}(v_2) = 0$. As a consequence, $M_\phi = (v_1|v_2|v)$ is the matrix associated with an affine transformation, and one obviously has

$$M_\phi^{-1} A M_\phi = \left(\begin{array}{cc|c} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ \hline 0 & 1 & \lambda \end{array}\right).$$

When the Jordan form of $A$ has, at least, two Jordan blocks, we can find a real Jordan basis $\{w_1, w_2, v\}$ such that $v$ is an eigenvector associated with $\lambda$ and the real Jordan form is

$$P^{-1} A P = \left(\begin{array}{c|c} J & 0 \\ \hline 0 & \lambda \end{array}\right),$$

where $P = (w_1|w_2|v)$. Since $v$ is an eigenvector, we have $\mathrm{pr}(v) \neq 0$ and may suppose that $\mathrm{pr}(v) = 1$ (otherwise, it suffices to divide $v$ by its third component). Hence, if we consider the new basis $\{v_1, v_2, v\}$ defined by $v_i = w_i - \mathrm{pr}(w_i) v$ for $i = 1, 2$, one immediately sees that $\mathrm{pr}(v_i) = 0$, which means that $M_\phi = (v_1|v_2|v)$ is the matrix associated with an affine transformation, and

$$M_\phi = P \left(\begin{array}{c|c} I & 0 \\ \hline -p & 1 \end{array}\right),$$

where $p = \big(\mathrm{pr}(w_1), \mathrm{pr}(w_2)\big)$ and $I$ denotes the $2 \times 2$ identity matrix. This implies that

$$M_\phi^{-1} A M_\phi = \left(\begin{array}{c|c} I & 0 \\ \hline p & 1 \end{array}\right) \left(\begin{array}{c|c} J & 0 \\ \hline 0 & \lambda \end{array}\right) \left(\begin{array}{c|c} I & 0 \\ \hline -p & 1 \end{array}\right) = \left(\begin{array}{c|c} J & 0 \\ \hline p(J - \lambda I) & \lambda \end{array}\right).$$

Notice that $p \neq (0,0)$ because $U_0$ is not $A$-invariant and since the geometric multiplicity of $\lambda$ is 1, it is obvious that $J - \lambda I$ is invertible; hence, $h \neq (0,0)$.

The final assertion concerning $h$ can be obtained by an adequate election of the Jordan basis. If $J$ has two real eigenvalues one has to choose the corresponding eigenvectors $w_1, w_2$ such that $\mathrm{pr}(w_i) = 1$, which is always possible. If $J$ has a unique eigenvalue $\lambda_1$, one chooses $w_1 \in \mathrm{Ker}(A - \lambda_1 I)^2 \cap U_0$ and $w_2 = (A - \lambda_1 I)v_1$ and divide both of them by $\mathrm{pr}(w_2)$ if necessary. Finally, in the case that $J$ has complex eigenvalues, it suffices to take a basis of the 2-dimensional invariant subspace such that $w_1 \in U_0$ and $\mathrm{pr}(w_2) = 1$, which is always possible. □

Next we give the explicit description of the forbidden sets for the canonical systems. We shall study the case with only real eigenvalues and the case admitting complex eigenvalues separately.

**Theorem 13.** *Let $\lambda \in \mathbb{R}$, $\lambda \neq 0$ and consider that the matrix*

$$C = \left(\begin{array}{c|c} J & 0 \\ \hline h & \lambda \end{array}\right)$$

*where $J$ is an invertible lower triangular Jordan form with real eigenvalues and $h$ is given as in the proposition above.*

*The forbidden set of the dynamical system defined by $F = q \circ C \circ \ell$ is given as follows:*

(a) *If $C$ has the unique eigenvalue $\lambda$ with multiplicity 3, then*

$$\mathcal{F} = \bigcup_{n \geq 1} \left\{ (x,y) \ : \ \frac{n^2 - n}{2\lambda^2}\, x + \frac{n}{\lambda}\, y + 1 = 0 \right\},$$

*which consist of an infinite number of non-parallel lines converging to the unique $F$-invariant line.*

(b) *If $C$ has two distinct eigenvalues $\lambda_1, \lambda$ and the multiplicity of $\lambda_1$ is 2, then*

$$\mathcal{F} = \bigcup_{n \geq 1} \left\{ (x,y) \ : \ \frac{n\lambda_1^{n-1}}{\lambda^n}\, x + \frac{\lambda_1^n - \lambda^n}{\lambda^n}\, y + 1 = 0 \right\}.$$

*It consists of an infinite number of lines converging to one of the $F$-invariant lines. More precisely,*

(b.1) *If $|\lambda| > |\lambda_1|$, then the forbidden lines are not parallel and converge to the $F$-invariant line defined by the $A$-invariant subspace* $\mathrm{Ker}(A - \lambda_1 I)^2$.

(b.2) *If $|\lambda| < |\lambda_1|$, then the forbidden lines are not parallel and converge to the $F$-invariant line passing through the fixed points.*

(b.3) *If $\lambda = -\lambda_1$, then the forbidden lines corresponding to an odd $n$ are non-parallel and the ones corresponding to an even $n$ are parallel. Both subsets of lines converge to the $F$-invariant line connecting the fixed points.*

(c) *If $C$ admits three distinct eigenvalues $\lambda_1, \lambda_2, \lambda$, then*

$$\mathcal{F} = \bigcup_{n\geq 1} \left\{ \ (x,y) \ : \ \big((\lambda_1/\lambda)^n - 1\big)x + \big((\lambda_2/\lambda)^n - 1\big)y + 1 = 0 \ \right\}.$$

(c.1) *If there is a unique eigenvalue of maximal modulus, $\mathcal{F}$ consists of an infinite number of non-parallel lines converging to the $F$-invariant line passing through the fixed points associated with the other two eigenvalues.*

(c.2) *If two eigenvalues have maximal modulus, the lines corresponding to an odd $n$ are non-parallel and converge to the line passing through the fixed point associated with the eigenvalue of minimal modulus and the midpoint of the other two equilibria, whereas the ones corresponding to an even $n$ are parallel to the $F$-invariant line connecting the fixed points related to eigenvalues of maximal modulus and tend to the one passing through the other fixed point.*

**Proof.** When $C$ has a unique eigenvalue of multiplicity 3 then, according to the proposition above, $C$ is exactly its lower triangular form and, therefore,

$$C^n = \begin{pmatrix} \lambda^n & 0 & 0 \\ n\lambda^{n-1} & \lambda^n & 0 \\ \dfrac{n^2-n}{2}\lambda^{n-2} & n\lambda^{n-1} & \lambda^n \end{pmatrix}.$$

It is then obvious that

$$\mathcal{F} = \bigcup_{n\geq 1} \left\{ \ (x,y) \ : \ \frac{n^2-n}{2\lambda^2}\, x + \frac{n}{\lambda}\, y + 1 = 0 \ \right\}.$$

As $n$ tends to $\infty$, the lines in the forbidden set approach the line $x = 0$, which is the $F$-invariant line given by the unique 2-dimensional $C$-invariant subspace: $\mathrm{Ker}(C - \lambda I)^2 = \{(x,y,z) \ : \ x = 0\}$.

In the remaining cases, one easily sees that

$$C^n = \left(\begin{array}{c|c} J^n & 0 \\ \hline hS_n & \lambda^n \end{array}\right)$$

where $S_n = \displaystyle\sum_{k=0}^{n-1} \lambda^k J^{n-1-k}$. Since $\lambda$ cannot be an eigenvalue of $J$, $(J - \lambda I)$ is invertible and we actually have $S_n = (J-\lambda I)^{-1}(J^n - \lambda^n I)$. Recall that $h$ was given by $h = (1,1)(J - \lambda I)$ if $J$ had two different real eigenvalues and $h = (0,1)(J - \lambda I)$ otherwise. Thus, one immediately gets

$$\mathcal{F} = \bigcup_{n\geq 1} \left\{ \ (x,y) \ : \ (1,1)(J^n - \lambda^n I)\lambda^{-n}(x,y)^t + 1 = 0 \ \right\}$$

in the case when $J$ admits two different real eigenvalues, and

$$\mathcal{F} = \bigcup_{n\geq 1} \left\{ \ (x,y) \ : \ (0,1)(J^n - \lambda^n I)\lambda^{-n}(x,y)^t + 1 = 0 \ \right\}$$

in the cases in which $J$ has a unique real Jordan block. The explicit form of $\mathcal{F}$ in each case is now directly deduced by the explicit calculation of $J^n$. Thus it only remains to study the asymptotic behavior of the forbidden lines.

Let us first study the case of three distinct eigenvalues. In such case we have

$$C = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ \lambda_1 - \lambda & \lambda_2 - \lambda & \lambda \end{pmatrix}.$$

The fixed points are $(1,0),(0,1),(0,0)$, which are respectively associated with the eigenvalues $\lambda_1, \lambda_2, \lambda$ and the $F$-invariant lines are $x+y=1$, $y=0$ and $x=0$. If we suppose that $|\lambda| > |\lambda_i|$ for $i=1,2$ then the lines of the forbidden set clearly tend to the line $x+y=1$. But when we suppose $|\lambda| > \lambda_1$ and $\lambda_2 = -\lambda$ we have that the forbidden lines are $\big((\lambda_1/\lambda)^n - 1\big)x + 1 = 0$ for an even $n$ and $\big((\lambda_1/\lambda)^n - 1\big)x - 2y + 1 = 0$ for an odd $n$. They clearly converge, respectively, to $x=1$ and $x = 1-2y$, as claimed in (c).

In the case of two distinct eigenvalues $\lambda_1, \lambda$ where the multiplicity of $\lambda_1$ is 2, the matrix $C$ has the form

$$C = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 1 & \lambda_1 & 0 \\ 1 & \lambda_1 - \lambda & \lambda \end{pmatrix}.$$

The $F$-invariant line passing through the fixed points $(0,1),(0,0)$ is clearly $x=0$ and the $F$-invariant line obtained by projection of $\mathrm{Ker}(C - \lambda_1 I)$ is $y=1$. When $|\lambda| > |\lambda_1|$ the forbidden lines obviously tend to $-y+1=0$ and if $|\lambda| < |\lambda_1|$ then they tend to $x=0$. Finally, for $\lambda = -\lambda_1$, the forbidden lines are given by

$$(-1)^{n-1}\frac{n}{\lambda}\, x + \big((-1)^n - 1\big)y + 1 = 0,$$

which converge to $x=0$ and are parallel to such line whenever $n$ is even. □

Next theorem describes the forbidden set when complex eigenvalues arise. Since $q \circ C \circ \ell = q \circ (-C) \circ \ell$, we can suppose that the real eigenvalue es actually positive.

**Theorem 14.** *Let $\lambda, r \in (0,\infty)$ and $\theta \in (0,\pi) \cup (\pi, 2\pi)$ and consider the matrix*

$$C = \left(\begin{array}{c|c} J & 0 \\ \hline h & \lambda \end{array}\right)$$

*where $h = (0,1)(J - \lambda I)$ and*

$$J = \begin{pmatrix} r\cos(\theta) & r\sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{pmatrix}.$$

*The forbidden set of the dynamical system defined by $F = q \circ C \circ \ell$ is*

$$\mathcal{F} = \bigcup_{n\geq 1} \left\{ \ (x,y) \ : \ (r/\lambda)^n \sin(n\theta)x - \big((r/\lambda)^n \cos(n\theta) - 1\big)y - 1 = 0 \ \right\}.$$

(a) *If $r < \lambda$, then the forbidden lines converge to the unique $F$-invariant line.*

(b) *If $r > \lambda$ and $\theta$ is a rational multiple of $\pi$, then $\mathcal{F}$ is the union of a finite number of sheaves of lines. Each of these sheaves is composed of an infinity of non-parallel lines converging to a line passing through the fixed point.*

(c) *If $r > \lambda$ and $\theta$ is not a rational multiple of $\pi$, then $\mathcal{F}$ is a dense subset of $\mathbb{R}^2$.*

(d) *If $r = \lambda$ and $\theta$ is a rational multiple of $\pi$, then $\mathcal{F}$ is globally periodic and $\mathcal{F}$ is a finite number of non-parallel lines.*

(e) *If $r = \lambda$ but $\theta$ is not a rational multiple of $\pi$, denote by $\mathcal{P}$ the parabola with focus at the fixed point and whose directrix is the $F$-invariant line and by $\mathcal{P}_{ext}$ the connected component of $\mathbb{R}^2 \setminus \mathcal{P}$ not containing the equilibrium. Then $\mathcal{F}$ is a dense subset in $\Omega = \mathcal{P} \cup \mathcal{P}_{ext}$.*

**Proof.** As in the proof of the theorem above,

$$C^n = \left( \begin{array}{c|c} J^n & 0 \\ \hline hS_n & \lambda^n \end{array} \right)$$

where $S_n = (J - \lambda I)^{-1}(J^n - \lambda^n I)$. Hence the forbidden lines are given by

$$(0,1)(J^n - \lambda^n I)(x,y)^t + \lambda^n = 0\,, \quad n \geq 1$$

and, thus, the explicit expression of $\mathcal{F}$ is directly deduced from

$$J^n = \begin{pmatrix} r^n \cos(n\theta) & r^n \sin(n\theta) \\ -r^n \sin(n\theta) & r^n \cos(n\theta) \end{pmatrix}.$$

It should be noticed that the unique $F$-invariant line is the line $y = 1$ and the fixed point is $(0,0)$. Clearly, if $r < |\lambda|$, then the forbidden lines tend to $y = 1$.

Let us suppose now that $r > |\lambda|$. If $\theta$ is a rational multiple of $\pi$, let $s$ be the smallest positive integer such that $s\theta = 2m\pi$ for some $m \in \mathbb{N}$, then

$$\mathcal{F} = \bigcup_{k=1}^{s} \mathcal{F}_k$$

$$\mathcal{F}_k = \bigcup_{j\geq 0} \left\{ \ (x,y) \ : \ (r/\lambda)^{k+jms} \sin(k\theta)x - \big((r/\lambda)^{k+jms} \cos(k\theta) - 1\big)y - 1 = 0 \ \right\}$$

It is straightforward to see that the lines of each $\mathcal{F}_k$ approach the line $\sin(k\theta)x = \cos(k\theta)y$, which obviously contains the fixed point $(0,0)$. It $\theta$ is not a rational multiple of $\pi$ then the set $\big\{\big(\cos(n\theta), \sin(n\theta)\big) \ : \ n \in \mathbb{N}, n \geq 1\big\}$ is dense in the unit

circle. Let us consider a point $(x_0, y_0) \in \mathbb{R}^2$, with $y_0 \neq 0$ and take an increasing sequence $\{n_k\}_{k\geq 1} \subset \mathbb{N}$ such that

$$\lim_{k\to\infty} \big(\cos(n_k\theta), \sin(n_k\theta)\big) = \frac{1}{||(x_0, y_0)||}(x_0, y_0).$$

The corresponding forbidden line for $n_k$ is

$$\sin(n_k\theta)x - \cos(n_k\theta)y - (\lambda/r)^n y - (\lambda/r)^n = 0.$$

Hence, if we take

$$x_k = \frac{\cos(n_k\theta)y_0 + (\lambda/r)^{n_k} y_0 + (\lambda/r)^{n_k}}{\sin(n_k\theta)},$$

then $(x_k, y_0) \in \mathcal{F}$ and

$$\lim_{k\to\infty} x_k = \lim_{k\to\infty} \frac{\cos(n_k\theta)y_0}{\sin(n_k\theta)} = x_0.$$

This proves that $(x_0, y_0)$ is adherent to $\mathcal{F}$. Therefore, the closure of $\mathcal{F}$ contains the closure of the set $\{(x_0, y_0) \ : \ y_0 \neq 0\}$ which is, evidently, the whole $\mathbb{R}^2$.

Finally, let us consider that $r = \lambda$. It is clear that if $\theta$ is a rational multiple of $\pi$, then $F$ is globally periodic and the forbidden set is

$$\mathcal{F} = \bigcup_{k=1}^{s} \Big\{ \ (x, y) \ : \ \sin(k\theta)x - \big(\cos(k\theta) - 1\big)y - 1 = 0 \ \Big\}$$

where $s$ is the minimum period. When $\theta$ is not so, we obviously have $\sin(n\theta) \neq 0$ for all $n \geq 1$ and the forbidden lines are:

$$x = \frac{(\cos(n\theta) - 1)y + 1}{\sin(n\theta)}, \quad n \geq 1.$$

Hence, if $(x, y) \in \mathcal{F}$, then we have

$$\begin{aligned} x^2 + 2y - 1 &= \frac{\big(\cos(n\theta) - 1\big)^2 y^2 + 2y\big(\cos(n\theta) - 1 + \sin^2(n\theta)\big) + 1 - \sin^2(n\theta)}{\sin^2(n\theta)} \\ &= \frac{\big(\cos(n\theta) - 1\big)^2 y^2 - 2y\big(\cos^2(n\theta) - \cos(n\theta)\big) + \cos^2(n\theta)}{\sin^2(n\theta)} \\ &= \frac{\big((\cos(n\theta) - 1)y - \cos(n\theta)\big)^2}{\sin^2(n\theta)} \geq 0. \end{aligned}$$

This means that $\mathcal{F}$ is contained in the subset $\Omega = \{(x, y) \ : \ x^2 + 2y - 1 \geq 0\}$, which is the set of points $\mathcal{P} \cup \mathcal{P}_{ext}$. To see that $\mathcal{F}$ is actually dense in $\Omega$, let us consider $(x_0, y_0)$ such that $x_0^2 + 2y_0 - 1 > 0$ and choose

$$a = \frac{y_0^2 - y_0 + |x_0|\sqrt{x_0^2 + 2y_0 - 1}}{x_0^2 + y_0^2}.$$

One easily sees that $a$ is one of the solutions to the equation

$$(1-a^2)x_0^2 = \big((a-1)y_0+1\big)^2.$$

Therefore, one has that $|a| \leq 1$ and, further, one can use both equations to see that $|a| \neq 1$. This implies that the following identity holds for one of the two possible signs:

$$x_0 = \frac{(a-1)y_0+1}{\pm\sqrt{1-a^2}}.$$

Again, the density of $\big\{\big(\cos(n\theta), \sin(n\theta)\big) \ : \ n \in \mathbb{N}, n \geq 1\big\}$ in the unit circle shows that we can find an increasing sequence $\{n_k\}_{k\geq 1} \subset \mathbb{N}$ such that

$$\lim_{k\to\infty} \big(\cos(n_k\theta), \sin(n_k\theta)\big) = \big(a, \pm\sqrt{1-a^2}\,\big),$$

where the sign is the same as in the expression of $x_0$ above. Thus, if we take

$$x_k = \frac{\big(\cos(n_k\theta)-1\big)y_0+1}{\sin(n_k\theta)},$$

then $\{(x_k, y_0)\}_{k\geq 1}$ is a sequence of points in $\mathcal{F}$ which clearly converge to $(x_0, y_0)$. This shows that the adherence of $\mathcal{F}$ contains $\mathcal{P}_{ext}$ and, thus, also contains its closure $\Omega$.

Remark 15. An affine transformation transforms a parabola in another parabola but, in general, it does not preserve the directrix and the focus. Thus, when one transforms a general system with a matrix $A$ in a new one with a matrix $C$ as the one given in the theorem above, the assertion (e) in the theorem is still valid except in which respects to the focus and directrix of $\mathcal{P}$.

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