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ON A FIRST-ORDER SEMIPOSITONE BOUNDARY VALUE PROBLEM ON A TIME SCALE

Christopher S. Goodrich

We consider the existence of a positive solution to the first-order dynamic equation $y^{\Delta}(t)+p(t)y^{\sigma}(t) = \lambda f\left(t, y^{\sigma}(t)\right), t \in (a,b)_{\mathbb{T}}$, subject to the boundary condition $y(a) = y(b) + \int_{\tau_1}^{\tau_2} F(s, y(s))\ \Delta s$ for $\tau_1,\ \tau_2 \in [a,b]_{\mathbb{T}}$. In this setting, we allow f to take negative values for some (t, y). Our results generalize some recent results for this class of problems, and because we treat the problem on a general time scale $\mathbb{T}$ we provide new results for this problem in the case of differential, difference, and q-difference equations. We also provide some discussion of the applicability of our results.

1. INTRODUCTION

In this paper we consider the existence of at least one positive solution to the boundary value problem (BVP)

$$\begin{aligned} y^{\Delta}(t) + p(t)y^{\sigma}(t) &= \lambda f\left(t, y^{\sigma}(t)\right), \ t \in (a,b)_{\mathbb{T}} \\ y(a) &= y(b) + \int_{\tau_1}^{\tau_2} F(s, y(s))\ \Delta s, \end{aligned} \tag{1}$$

where $\mathbb{T}$ is a given time scale, $\lambda > 0$ is a parameter, the numbers τ_1 and τ_2 satisfy $\tau_1,\ \tau_2 \in [a,b]_{\mathbb{T}}$ with $\tau_1 < \tau_2$, and p and F are nonnegative functions on which we shall later place some additional hypotheses. We also allow the nonlinearity f to be negative for some values of t and y. Due to the presence of the integral

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in the boundary condition (BC) in (1) together with the fact that the function F appearing in the integrand may be nonlinear, problem (1.1) is, in fact, an example of a BVP with nonlinear, nonlocal BCs. Note that throughout this work we use the standard notational convention $E_{\mathbb{T}} := E \cap \mathbb{T}$ for some set E.

BVPs possessing either nonlinear and/or nonlocal BCs have seen a great deal of study recently, spanning such areas as first- and second-order problems, coupled systems of second-order problems, and higher-order problems – see, for example, recent works by Franco et al. [**10**], Goodrich [**11**–**17**], Graef et al. [**18**], Infante et al. [**22**–**25**], Kang et al. [**26**], Webb and Infante [**31**], Yang [**32, 33**], and the references therein. In addition, the study of BVPs with specifically integral boundary conditions has also seen much attention in recent years, and one may consult [**5, 19, 27**] and the references therein for some recent examples. It is also the case that the study of such BVPs on time scales has attracted a large research following over the past decade or so, and it would be impossible to mention all of the great many contributions to this emerging area. The papers [**2, 11, 28, 30**] and the references therein provide a broad introduction, nonetheless, to this line of research. In particular, the concept of analysis on a time scale was introduced by Hilger [**20**]. Part of the interest in this approach is due to its effectiveness in certain modeling situations as well as the fact that all manner of BVPs may be studied simultaneously, certain of the most important examples being on the time scales $\mathbb{R}$ (differential equations), $\mathbb{Z}$ (difference equations), and $q^{\mathbb{Z}}$ (q-difference equations). This approach then provides a unified treatment of several important types of equations rather than a disjointed, piecemeal approach.

In the specific case of problem (1), our results here generalize a class of results due to Anderson [**2**]. In particular, Anderson considered the multipoint problem

$$\begin{aligned} y^{\Delta}(t) + p(t)y^{\sigma}(t) &= \lambda f\left(t, y^{\sigma}(t)\right), \, t \in (a,b)_{\mathbb{T}} \\ y(a) &= y(b) + \sum_{i=2}^{n-1} \gamma_i y\left(t_i\right), \end{aligned} \tag{2}$$

where $\gamma_i \in [0, +\infty)$ for each i and $a < t_2 < \cdots < t_{n-1} < b \in \mathbb{T}^{\kappa}$. In this case, since f is allowed to take on negative values, Anderson considers the semipositone problem. The semipositone problem has been well studied on both the time scale $\mathbb{R}$ as well as more general time scales – see [**3, 4, 7, 8, 19, 21**] and the references therein. Furthermore, as Anderson remarks in [**2**, Remark 3.4], problem (2) can be generalized to

$$\begin{aligned} y^{\Delta}(t) + p(t)y^{\sigma}(t) &= \lambda f\left(t, y^{\sigma}(t)\right), \, t \in (a,b)_{\mathbb{T}} \\ y(a) &= y(b) + \int_{\tau_1}^{\tau_2} \gamma(t)y(t)\ \Delta t \end{aligned} \tag{3}$$

ith only trivial modifications to the proofs of the theorems given in [**2**]. The modification in (3) allows for a *linear* nonlocal integral boundary condition, and the linearity of the condition in (3) is precisely what makes the modification of An-

DERSON's results easy. If the boundary condition is permitted to be *nonlinear*, as we allow, then the analysis becomes more complicated.

With this background in mind our contribution in this work is to extend ANDERSON's analysis of problem (2) to the nonlinear boundary condition setting, as expressed in (1). Our result is obtained by assuming that F satisfies a sort-of asymptotic relatedness condition, which is further explained in Sections 2 and 3. Its use allows us to achieve our results whilst making few additional structural assumptions on the constituent functions. Indeed, in most all settings to our knowledge it is assumed that $F(t, y) = \alpha(t)\beta(y)$, for all admissible t and y, for some suitably restricted functions α and β, whereas here we make no such assumptions – cf., [**5, 9, 27, 29**], for example. Moreover, this strategy is applicable to other problems with the general type of boundary condition studied here, and so, we believe that the general techniques provided herein can be used to give generalizations of other results in the literature.

2. PRELIMINARIES

In this section we collect some preliminary lemmas that we shall use in Section 3 to deduce the existence result that we present. In addition, we collect some basic results from the theory of the calculus on time scales. For the most part we assume a general familiarity with time scales, and we invite the reader to consult the excellent textbook by BOHNER and PETERSON [**6**] for additional information on the theory and application of time scales. Nonetheless, we do state a few basic results here since their use is rather frequent in the sequel. We begin with some properties of the time scales exponential function. For further discussion regarding the time scales exponential function, please consult [**6**, Chapter 2].

Lemma 2.1. *Suppose that $p : \mathbb{T} \to \mathbb{R}$ is regressive and rd-continuous. Then each of the following holds for all t, s, r, a, $b \in \mathbb{T}$:*

1. $e_0(t, s) \equiv 1$ *and* $e_p(t, t) \equiv 1$;
2. $e(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
3. $\dfrac{1}{e_p(t, s)} = e_{\ominus p}(t, s)$;
4. $e_p(t, s)e_p(s, r) = e_p(t, r)$;
5. *if p is nonnegative and $a \le t \le b$, then $e_p(a, t) \le e_p(b, t)$; and*
6. *if p is nonnegative and $a \le t \le b$, then $e_p(a, t) \le e_p(a, a) = 1$.*

Let us next state the cone in which we search for positive solutions of problem (1). In particular, let $\mathcal{B}$ be the Banach space $\mathcal{C}_{\mathrm{rd}}([a, b]_{\mathbb{T}})$ when equipped with the usual supremum norm $\|\cdot\|$. We then define the cone $\mathcal{K} \subseteq \mathcal{B}$ by

$$\mathcal{K} := \{y \in \mathcal{B} \ : \ y(t) \ge e_p(a, b)\|y\| \text{ for each } t \in [a, b]_{\mathbb{T}}\}.$$

We next collect the hypotheses that we impose on the various functions in problem (1). We will discuss later the use of these conditions in some remarks.

H1: The function $F : [a,b]_{\mathbb{T}} \times [0,+\infty) \to [0,+\infty)$ is continuous, and there exists a continuous function $H : [0,+\infty) \to [0,+\infty)$ such that for each $\varepsilon > 0$ given, there exists a number $M_\varepsilon \geq 0$ such that

$$|F(t,y) - H(y)| < \varepsilon H(y),$$

for each $t \in [a,b]_{\mathbb{T}}$, whenever $y \geq M_\varepsilon$. In addition, the function H satisfies the growth condition $H(y) \leq C_1 y$, for some constant $C_1 \geq 0$ and all $y \geq 0$.

H2: The function $p : [a,b]_{\mathbb{T}} \to (0,+\infty)$ satisfies $p \in \mathcal{C}_{\mathrm{rd}}([a,b]_{\mathbb{T}})$.

H3: Assume that

$$e_p(b,a) - 1 > 0.$$

H4: Assume that the nonlinearity $f : (a,b)_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is continuous such that it is not identically zero on any subinterval of $(a,b)_{\mathbb{T}}$.

H5: Assume that there is $[\alpha_1,\alpha_2]_{\mathbb{T}} \subset (a,b)_{\mathbb{T}}$ such that $\lim_{y\to+\infty} f(t,y) = +\infty$ uniformly for $t \in [\alpha_1,\alpha_2]_{\mathbb{T}}$, with $\alpha_1,\ \alpha_2 \in \mathbb{T}$ satisfying $\alpha_1 < \alpha_2$.

H6: Assume that $\lim_{y\to+\infty} \dfrac{f(t,y)}{y} = 0$ uniformly for $t \in [a,b]_{\mathbb{T}}$.

H7: Assume that there is a function $u : (a,b)_{\mathbb{T}} \to (0,+\infty)$ with $u \in \mathcal{C}_{\mathrm{rd}}((a,b)_{\mathbb{T}})$ such that

$$-u(t) \leq f(t,y),$$

where we assume that

$$\int_a^b u(t)\ \Delta t < +\infty.$$

Before proceeding let us make some remarks regarding certain of these conditions.

Remark 2.2. Observe that condition (H1) permits a broad variety of functions F. For example, each of the following pairs of functions F, H are asymptotically related in the sense of condition (H1):

$$\begin{array}{lll} F(t,y) := 3y + 2t^2 + 5t\sqrt{y} & \text{and} & H(y) := 3y \\ F(t,y) := y + 2te^t\sqrt[3]{y} & \text{and} & H(y) := y \\ F(t,y) := \ln(y+1) + y & \text{and} & H(y) := y. \end{array}$$

Now, as is the typical strategy for a problem such as (1), we will need to study the auxiliary problem

$$\begin{aligned} w^\Delta + p(t)w^\sigma &= \lambda u(t) \\ w(a) &= w(b). \end{aligned} \tag{4}$$

Regarding this auxiliary problem, we state the following lemma, which essentially is the content of [**2**, Lemma 2.1].

Lemma 2.3. *Assume that each of conditions* (H2)–(H3) *and* (H7) *holds. Then the unique solution of problem* (4) *is given by*

$$w(t) := \lambda \int_a^b G(t,s)u(s)\ \Delta s, \tag{5}$$

where $G\ :\ [a,b]_{\mathbb{T}} \times [a,b]_{\mathbb{T}} \to \mathbb{R}$ *is the Green's function defined by*

$$G(t,s) := \begin{cases} \dfrac{1}{e_p(b,a)-1} e_p(s,t), & a \le t \le s \le b \\ \Big(\dfrac{1}{e_p(b,a)-1} + 1\Big) e_p(s,t), & a \le s < t \le b. \end{cases} \tag{6}$$

Furthermore, it holds that $G(t,s) > 0$ *for each admissible pair* (t,s). *Finally, the function* w *as defined by* (5) *is a continuous function of* t *for each* $t \in [a,b]_{\mathbb{T}}$.

Proof. The derivation of (5) is as in [**2**, Lemma 2.1]. Therefore, we do not restate the proof here. Moreover, the strict positivity of G is obvious. It remains to prove that w is a continuous function of t. We give a brief argument of the continuity of $t \mapsto \lambda \int_a^b G(t,s)u(s)\ \Delta s$. To this end, fix $t_0 \in [a,b]_{\mathbb{T}}$. We assume, for simplicity, that $t_0 \in (a,b)_{\mathbb{T}}$ and that t_0 is not an isolated point of $\mathbb{T}$. (If these facts are not true, then the following argument is very easily appropriately modified.) Now, recall that u is L^1 in the sense that

$$M_0 := \int_a^b u(s)\ \Delta s < +\infty$$

holds. Let $\varepsilon > 0$ be given. Consider the open set $(t_0-\delta, t_0+\delta)_{\mathbb{T}}$ for $\delta > 0$ sufficiently small and to be selected later. Let $t \in (t_0-\delta, t_0+\delta)_{\mathbb{T}}$ and, for notational convenience in the sequel, define the functions G_1 and G_2 by $G_1(t,s) := \dfrac{1}{e_p(b,a)-1} e_p(s,t)$ and $G_2(t,s) := \Big(\dfrac{1}{e_p(b,a)-1} + 1\Big) e_p(s,t)$, where G_1 is defined when the pair (t,s) satisfies $a \le t \le s \le b$ and G_2 is defined when $a \le s \le t \le b$. Notice that for $\delta < \delta_1$, say,

$$\begin{aligned} \int_a^{\rho(t_0-\delta)} \left|G(t,s) - G\left(t_0,s\right)\right| u(s)\ \Delta s &= \int_a^{\rho(t_0-\delta)} \left|G_2(t,s) - G_2\left(t_0,s\right)\right| u(s)\ \Delta s \\ &\le \frac{\varepsilon}{3M_0+1} \int_a^{\rho(t_0-\delta)} u(s)\ \Delta s < \frac{\varepsilon}{3}, \end{aligned}$$

where we have used the continuity of $t \mapsto G_2(t,\cdot)$. In a similar way, we deduce that for $\delta < \delta_2$, say,

$$\int_{\sigma(t_0+\delta)}^b \left|G(t,s) - G\left(t_0,s\right)\right| u(s)\ \Delta s = \int_{\sigma(t_0+\delta)}^b \left|G_1(t,s) - G_1\left(t_0,s\right)\right| u(s)\ \Delta s < \frac{\varepsilon}{3}.$$

Finally, we see that for $\delta < \delta_3$, say,

$$\int_{\rho(t_0-\delta)}^{\sigma(t_0+\delta)} \left|G(t,s) - G\left(t_0,s\right)\right| u(s)\ \Delta s \le M_1 \int_{\rho(t_0-\delta)}^{\sigma(t_0+\delta)} u(s)\ \Delta s < M_1 \frac{\varepsilon}{3M_1+1} < \frac{\varepsilon}{3},$$

where we put

$$M_1 := \max_{(t,s)\in(t_0-\delta,t_0+\delta)_{\mathbb{T}}\times[\rho(t_0-\delta),\sigma(t_0+\delta)]_{\mathbb{T}}} |G(t,s) - G(t_0,s)| < +\infty,$$

and use the fact that since u is L^1 on $[a,b]_{\mathbb{T}}$, it holds that $\int_{\rho(t_0-\delta)}^{\sigma(t_0+\delta)} u(s)\ \Delta s$ can be made arbitrarily small by making δ sufficiently small; recall here that t_0 is either left- and/or right-dense, by assumption. Therefore, putting all of the preceding estimates together, we deduce that for $0<\delta<\min\{\delta_1,\delta_2,\delta_3\}$, it holds, for $0 \leq |t-t_0| < \delta$, that

$$\begin{aligned}\left|\int_a^b [G(t,s)-G(t_0,s)]\,u(s)\ \Delta s\right| &\leq \int_a^b |G(t,s)-G(t_0,s)|\,u(s)\ \Delta s \\ &< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon.\end{aligned} \tag{7}$$

Consequently, it follows from (7) that the map

$$t \mapsto \lambda \int_a^b G(t,s)u(s)\ \Delta s$$

is continuous, and this completes the proof. □

We next record a result regarding bounds on the Green's function appearing in (6) above. This result is similar to a result stated in Anderson – see [**2**, Lemma 2.2]. Unfortunately, the proof and result as stated in [**2**] contains a slight error. Therefore, due to this, we give a proof in full of the following lemma, which corrects the minor error in the proof of [**2**, Lemma 2.2].

Lemma 2.4. *Assume that each of conditions* (H2) *and* (H3) *holds. Then the Green's function in* (6) *satisfies*

$$0 < G(s,s) \leq G(t,s) \leq e_p(b,a)G(s,s), \tag{8}$$

for each $s,\ t \in [a,b]_{\mathbb{T}}$.

Proof. First of all, note that by (6) it follows that

$$G(s,s) = \frac{1}{e_p(b,a)-1},$$

which is well defined since condition (H4) is assumed. Fix $s \in [a,b]_{\mathbb{T}}$. If $t \leq s$, then (8) is obvious, for we calculate, just as in the proof of [**2**, Lemma 2.2],

$$\frac{1}{e_p(b,a)-1} = \frac{1}{e_p(b,a)-1}e_p(s,s) \leq \frac{1}{e_p(b,a)-1}e_p(s,t) \leq \frac{1}{e_p(b,a)-1}e_p(s,a). \tag{9}$$

Rewriting (9) by means of (6), we estimate

$$0 < G(s,s) \leq G(t,s) \leq e_p(s,a)G(s,s) \leq e_p(b,a)G(s,s).$$

So, (8) holds in case $t \le s$.

On the other hand, in case $t > s$ we reason as follows. First of all, we note that we cannot merely repeat the proof given in [**2**, Lemma 2.2], for that argument possesses a slight error. (In particular, the value of $G(s,s)$ is recorded and used incorrectly in the second part of the proof there.) Rather, we first note that

$$G(t,s) = \left(1 + \frac{1}{e_p(b,a)-1}\right)e_p(s,t) = \frac{e_p(b,a)e_p(s,t)}{e_p(b,a)-1} \ge \frac{1}{e_p(b,a)-1} = G(s,s), \tag{10}$$

where we use the fact that

$$e_p(b,a)e_p(s,t) \ge e_p(b,a)e_p(a,b) = 1$$

so that

$$e_p(b,a)e_p(s,t) \ge 1.$$

Hence, (10) implies that

$$G(s,s) \le G(t,s) \quad \text{for} \quad t > s. \tag{11}$$

Next, put

$$d := e_p(b,a) - 1 > 0.$$

We wish to prove that the inequality

$$G(t,s) = \left(\frac{1}{d} + 1\right) e_p(s,t) = \left(\frac{d+1}{d}\right) e_p(s,t) \le e_p(b,a)G(s,s) = (d+1)G(s,s) \tag{12}$$

holds. But observe that (12) is true if and only if

$$\left(\frac{d+1}{d}\right) e_p(s,t) \le (d+1)G(s,s) \tag{13}$$

if and only if

$$\frac{1}{d}e_p(s,t) \le G(s,s). \tag{14}$$

Since

$$e_p(s,t) \le dG(s,s) = 1, \tag{15}$$

we see that (12) holds if and only if $e_p(s,t) \le 1$. But this latter inequality is clearly true, for $s < t$. Therefore, because the steps in (12)–(15) are reversible, it follows that (12) holds. That is to say,

$$G(t,s) \le (d+1)G(s,s) = e_p(b,a)G(s,s). \tag{16}$$

Putting (11) and (16) together, we conclude that

$$0 < G(s,s) \le G(t,s) \le e_p(b,a)G(s,s)$$

holds in case $t > s$, which completes the proof. □

We next require some *a priori* bounds on the solution of problem (4). Such bounds will become important in Section 3.

Lemma 2.5. *Suppose that w is the unique solution of problem* (4). *Then w satisfies*

$$e_p(a,b)\|w\| \leq w(t) \leq \lambda\xi, \tag{17}$$

for each $t \in [a,b]_{\mathbb{T}}$, where ξ is defined here and in the sequel by

$$\xi := \int_a^b e_p(b,a)G(s,s)u(s)\ \Delta s.$$

Proof. On the one hand we see that

$$w(t) = \lambda \int_a^b G(t,s)u(s)\ \Delta s \leq \lambda \int_a^b e_p(b,a)G(s,s)u(s)\ \Delta s = \lambda\xi. \tag{18}$$

On the other hand,

$$\begin{aligned} w(t) &\geq \lambda \int_a^b G(s,s)u(s)\ \Delta s = e_p(a,b)\left[\lambda \int_a^b e_p(b,a)G(s,s)u(s)\ \Delta s\right] \\ &\geq e_p(a,b)\|w\|. \end{aligned} \tag{19}$$

Since each of (18) and (19) holds for each $t \in [a,b]_{\mathbb{T}}$, the conclusion of the lemma follows. □

We provide next a simple lemma that nonetheless will be important in the existence proofs.

Lemma 2.6. *Suppose that $y \in \mathcal{K}$ is given. Then it holds that*

$$\left(1 - \frac{\lambda\xi}{e_p(a,b)\|y\|}\right) y(t) \geq \left(1 - \frac{\lambda\xi}{\lambda(\xi+1)}\right) y(t), \tag{20}$$

for each $t \in [a,b]_{\mathbb{T}}$, whenever $\|y\| \geq e_{\ominus p}(a,b)\lambda(\xi+1)$.

Proof. Suppose that $\|y\| \geq e_{\ominus p}(a,b)\lambda(\xi+1)$. It then follows that

$$\lambda\xi + \lambda \leq e_p(a,b)\|y\|. \tag{21}$$

From (21) we estimate

$$\lambda^2\xi^2 + \lambda^2\xi \leq \lambda\xi e_p(a,b)\|y\|. \tag{22}$$

Finally, (22) implies that

$$1 - \frac{\lambda\xi}{e_p(a,b)\|y\|} \geq 1 - \frac{\lambda\xi}{\lambda(\xi+1)},$$

which, since $y(t) \geq 0$, implies that

$$\left(1 - \frac{\lambda\xi}{e_p(a,b)\|y\|}\right) y(t) \geq \left(1 - \frac{\lambda\xi}{\lambda(\xi+1)}\right) y(t),$$

for each $t \in [a,b]_{\mathbb{T}}$, which is the desired inequality. □

A simple consequence of the preceding lemma is the following.

Lemma 2.7. *Let w be the solution of the auxiliary problem* (4). *Moreover, let $y \in \mathcal{K}$ be a given function satisfying $\|y\| \geq e_{\ominus p}(a,b)\lambda(\xi+1)$. Then it holds that*

$$\min_{t\in[a,b]_{\mathbb{T}}} (y(t)-w(t)) \geq \frac{e_p(a,b)}{\xi+1}\|y\|.$$

Proof. Let w and y be as in the statement of the lemma. Note that since $y \in \mathcal{K}$ it holds that

$$y(t) \geq e_p(a,b)\|y\|,$$

for each $t \in [a,b]_{\mathbb{T}}$, so that

$$-1 \leq -\frac{e_p(a,b)\|y\|}{y(t)},$$

for each $t \in [a,b]_{\mathbb{T}}$. In particular, it holds that

$$-\frac{y(t)}{e_p(a,b)\|y\|} \leq -1, \tag{23}$$

for each $t \in [a,b]_{\mathbb{T}}$. Thus, using (17), (20), and (23) we estimate

$$\begin{aligned} y(t)-w(t) &\geq y(t)-\lambda\xi \geq y(t)-\frac{\lambda\xi y(t)}{e_p(a,b)\|y\|} = \left[1-\frac{\lambda\xi}{e_p(a,b)\|y\|}\right]y(t) \\ &\geq \left[1-\frac{\lambda\xi}{\lambda(\xi+1)}\right]y(t) \geq \left[1-\frac{\lambda\xi}{\lambda(\xi+1)}\right]\min_{t\in[a,b]_{\mathbb{T}}} y(t) \\ &= \frac{1}{\xi+1}\min_{t\in[a,b]_{\mathbb{T}}} y(t) \geq \frac{e_p(a,b)}{\xi+1}\|y\|, \end{aligned} \tag{24}$$

which, since (24) holds for each $t \in [a,b]_{\mathbb{T}}$, completes the proof. □

Finally, we construct the operator $T : \mathcal{B} \to \mathcal{B}$ that we shall use to find positive solutions of problem (1). First of all, as is a standard approach when seeking positive solutions of semipositone BVPs, we consider a modified BVP, namely the problem

$$\begin{aligned} y^{\Delta}+p(t)y^{\sigma} &= \lambda\left[f\left(t,\max\left\{y^{\sigma}(t)-w^{\sigma}(t),0\right\}\right)+u(t)\right] \\ y(a) &= y(b)+\int_{\tau_1}^{\tau_2} F\left(s,\max\{y(s)-w(s),0\}\right)\,\Delta s. \end{aligned} \tag{25}$$

Note that in both (25) and the sequel the function w is the unique solution to the auxiliary problem (4).

We next give the following lemma. Note that in the statement of this lemma the quantity $1-e_p(a,b) \neq 0$ since $e_p(b,a) > 1$ by condition (H3). Since the proof of this lemma is obvious from [**2**, Lemma 2.1] and the fact that the function $Ae_{\ominus p}(t,a)$ is a general solution of $y^{\Delta}+p(t)y^{\sigma}=0$ for $t \in (a,b)_{\mathbb{T}}$, we omit the proof.

Lemma 2.8. *Suppose that each of* (H2)–(H3) *and* (H7) *holds. Let* $T : \mathcal{B} \to \mathcal{B}$ *be the operator defined by*

$$
\begin{aligned}
(Ty)(t) &:= \lambda \int_a^b G(t,s)\left[f\left(s, \max\left\{y^\sigma(s) - w^\sigma(s), 0\right\}\right) + u(s)\right]\ \Delta s \\
&\quad + \frac{e_p(a,t)}{1 - e_p(a,b)} \int_{\tau_1}^{\tau_2} F(s, \max\{y(s) - w(s), 0\})\ \Delta s,
\end{aligned} \tag{26}
$$

where $G : [a,b]_{\mathbb{T}} \times [a,b]_{\mathbb{T}} \to \mathbb{R}$ *is the Green's function defined in* (6) *above. If* y_0 *is a fixed point of* T*, then* y_0 *is a solution of the modified problem* (25)*.*

In the existence proof in Section 3, the strategy will be to show that T has a fixed point $y \in \mathcal{K}$ and then to show that if we put $x(t) := y(t) - w(t)$ and if $\|y\|$ is bounded below by a sufficiently large positive number, then $x(t)$ is a positive solution of the original problem (1). To facilitate this, we state and prove two final lemmas.

Lemma 2.9. *Assume that each of* (H2)–(H3) *and* (H7) *holds, and let* T *be the operator defined in* (26)*. Then* $T(\mathcal{K}) \subseteq \mathcal{K}$*.*

Proof. We argue very much as in [**2**, Lemma 3.1]. First note that

$$
e_p(a,b) \le e_p(a,t) \le 1,
$$

for each $t \in [a,b]_{\mathbb{T}}$. So, it follows that

$$
\begin{aligned}
(Ty)(t) &= \lambda \int_a^b G(t,s)\left[f\left(s, \max\left\{y^\sigma(s) - w^\sigma(s), 0\right\}\right) + u(s)\right]\ \Delta s \\
&\quad + \frac{e_p(a,t)}{1 - e_p(a,b)} \int_{\tau_1}^{\tau_2} F(s, \max\{y(s) - w(s), 0\})\ \Delta s \\
&\ge e_p(a,b)\Bigg(\lambda \int_a^b e_p(b,a) G(s,s)\left[f\left(s, \max\left\{y^\sigma(s) - w^\sigma(s), 0\right\}\right) + u(s)\right] \Delta s \\
&\quad + \frac{1}{1 - e_p(a,b)} \int_{\tau_1}^{\tau_2} F(s, \max\{y(s) - w(s), 0\})\ \Delta s\Bigg),
\end{aligned} \tag{27}
$$

whilst

$$
\begin{aligned}
(Ty)(t) &= \lambda \int_a^b G(t,s)\left[f\left(s, \max\left\{y^\sigma(s) - w^\sigma(s), 0\right\}\right) + u(s)\right]\ \Delta s \\
&\quad + \frac{e_p(a,t)}{1 - e_p(a,b)} \int_{\tau_1}^{\tau_2} F(s, \max\{y(s) - w(s), 0\})\ \Delta s \\
&\le \lambda \int_a^b e_p(b,a) G(s,s)\left[f\left(s, \max\left\{y^\sigma(s) - w^\sigma(s), 0\right\}\right) + u(s)\right]\ \Delta s \\
&\quad + \frac{1}{1 - e_p(a,b)} \int_{\tau_1}^{\tau_2} F(s, \max\{y(s) - w(s), 0\})\ \Delta s.
\end{aligned} \tag{28}
$$

But combining (27) and (28) and using the fact that the right-hand side of (28) is independent of t we deduce that

$$(Ty)(t) \geq e_p(a,b)\|Ty\|,$$

for each $t \in [a,b]_{\mathbb{T}}$, whence $T(\mathcal{K}) \subseteq \mathcal{K}$, as desired. $\square$

Lemma 2.10. *Suppose that y is a fixed point of the operator T defined in* (26)*. Define the function $x\ :\ [a,b]_{\mathbb{T}} \to \mathbb{R}$ by*

$$x(t) := y(t) - w(t),$$

where w is the solution of the auxiliary problem (4)*. If $x(t) \geq 0$ on its domain, then x is a positive solution of problem* (1)*.*

Proof. Suppose as in the statement of the lemma that $x(t) \geq 0$, for each $t \in [a,b]_{\mathbb{T}}$. Obviously, x is a nonnegative function. Furthermore, it holds that

$$\begin{aligned} x^{\Delta} + p(t)x^{\sigma} &= y^{\Delta} - w^{\Delta} + p(t)y^{\sigma} - p(t)w^{\sigma} \\ &= \lambda f\left(t, y^{\sigma}(t) - w^{\sigma}(t)\right) + \lambda u(t) - w^{\Delta} - p(t)w^{\sigma} \\ &= \lambda f\left(t, x^{\sigma}(t)\right) + \lambda u(t) - \lambda u(t) = \lambda f\left(t, x^{\sigma}(t)\right). \end{aligned}$$

Finally, we compute

$$\begin{aligned} x(a) - x(b) &= y(a) - y(b) - w(a) + w(b) \\ &= \int_{\tau_1}^{\tau_2} F(s, y(s) - w(s))\ \Delta s - 0 = \int_{\tau_1}^{\tau_2} F(s, x(s))\ \Delta s. \end{aligned}$$

Thus, x is a positive solution of problem (1), as desired. $\square$

We conclude this section with the statement of Krasnosel'skiĭ's fixed point theorem – see [**1**]. We shall use this result to prove the existence theorem of Section 3.

Lemma 2.11. *Let $\mathcal{B}$ be a Banach space and let $\mathcal{K} \subseteq \mathcal{B}$ be a cone. Assume that Ω_1 and Ω_2 are bounded, open sets contained in $\mathcal{B}$ such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subseteq \Omega_2$. Assume, further, that $T\ :\ \mathcal{K} \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right) \to \mathcal{K}$ is a completely continuous operator. If either*

1. $\|Ty\| \leq \|y\|$ *for* $y \in \mathcal{K} \cap \partial\Omega_1$ *and* $\|Ty\| \geq \|y\|$ *for* $y \in \mathcal{K} \cap \partial\Omega_2$*; or*

2. $\|Ty\| \geq \|y\|$ *for* $y \in \mathcal{K} \cap \partial\Omega_1$ *and* $\|Ty\| \leq \|y\|$ *for* $y \in \mathcal{K} \cap \partial\Omega_2$*;*

then T has at least one fixed point in $\mathcal{K} \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right)$.

3. MAIN RESULT AND CONCLUDING REMARKS

We now present our existence result for problem (1). Following the statement and proof of the result, we conclude with some final remarks regarding its application. Note that throughout we use the following notation.

Notation 3.12. Given $R > 0$, we denote by $\Omega_R \subseteq \mathcal{B}$ the open set

$$\Omega_R := \{y \in \mathcal{B} : \|y\| < R\}.$$

Theorem 3.13. *Assume that each of conditions* (H1)–(H7) *holds. Furthermore, assume that*

$$\frac{C_1}{1 - e_p(a,b)} \int_{\tau_1}^{\tau_2} 1\ \Delta s < 1. \tag{29}$$

Then there exist $\lambda_2 > \lambda_1 > 0$ such that for each $\lambda \in [\lambda_1, \lambda_2]$ problem (1) *has at least one positive solution.*

Proof. First of all, $\mathcal{K}$ is invariant under T due to Lemma 2.9. In addition, in light of Lemma 2.3 and the continuity of F, it is standard to show that T is completely continuous, and so, we omit the details. Therefore, it remains to show that T is alternatively a cone expansion and compression on appropriate sets.

To this end, since $\dfrac{(\xi+1)e_{\ominus p}(a,b)}{\int_{\alpha_1}^{\alpha_2} G(s,s)\ \Delta s} < +\infty$, select $K_0 \in (0, +\infty)$ such that $\dfrac{(\xi+1)e_{\ominus p}(a,b)}{\int_{\alpha_1}^{\alpha_2} G(s,s)K_0\ \Delta s} < 1$ holds. Furthermore, notice that by condition (H5), it follows that there exists a number $r_1 > 0$ sufficiently large such that

$$f(t,y) \geq K_0 \tag{30}$$

uniformly for $t \in [\alpha_1, \alpha_2]_{\mathbb{T}}$, whenever $y \in [r_1, +\infty)$. In addition, with K_0 chosen as above, select $\lambda > 0$ such that

$$\lambda \in \left[\frac{(\xi+1)r_1}{e_p(a,b)\int_{\alpha_1}^{\alpha_2} G(s,s)K_0\ \Delta s}, r_1\right]. \tag{31}$$

Let us interrupt to note that the set indicated in (31) is nonempty. In particular, this follows from the observation that since

$$\frac{(\xi+1)e_{\ominus p}(a,b)}{\int_{\alpha_1}^{\alpha_2} G(s,s)K_0\ \Delta s} < 1,$$

by hypothesis, it follows at once that

$$\frac{(\xi+1)r_1}{e_p(a,b)\int_{\alpha_1}^{\alpha_2} G(s,s)K_0\ \Delta s} < r_1.$$

Thus

$$\left[\frac{(\xi+1)r_1}{e_p(a,b)\int_{\alpha_1}^{\alpha_2} G(s,s)K_0\ \Delta s}, r_1\right] \neq \emptyset.$$

Now let

$$y \in \mathcal{K} \cap \partial\Omega_{\frac{\xi+1}{e_p(a,b)} r_1}$$

be arbitrary but fixed. Then since

$$\lambda \leq r_1 = \frac{r_1(\xi+1)}{e_p(a,b)(\xi+1)e_{\ominus p}(a,b)} = \frac{\|y\|}{(\xi+1)e_{\ominus p}(a,b)},$$

it follows that

$$\|y\| \geq \lambda(\xi+1)e_{\ominus p}(a,b). \tag{32}$$

In addition, since

$$\lambda \geq \frac{(\xi+1)r_1}{e_p(a,b)\int_{\alpha_1}^{\alpha_2} G(s,s)K_0\ \Delta s},$$

it likewise follows for such y that

$$\lambda \geq \frac{\|y\|}{\int_{\alpha_1}^{\alpha_2} G(s,s)K_0\ \Delta s}, \tag{33}$$

whence

$$\lambda \int_{\alpha_1}^{\alpha_2} G(s,s)K_0\ \Delta s \geq \|y\|.$$

Notice, moreover, that due to (32) and Lemma 2.7, it holds that

$$y^\sigma(t) - w^\sigma(t) \geq 0, \tag{34}$$

for each $t \in [a, \rho(b)]_{\mathbb{T}}$, whence

$$\max\left\{y^\sigma(t) - w^\sigma(t), 0\right\} = \left(y^\sigma - w^\sigma\right)(t),$$

for each $t \in [a, \rho(b)]_{\mathbb{T}}$. In addition, Lemma 2.7 implies that

$$\min_{t\in[a,\rho(b)]_{\mathbb{T}}} \left(y^\sigma(t) - w^\sigma(t)\right) \geq \frac{e_p(a,b)}{\xi+1}\|y\|. \tag{35}$$

Consequently, it follows from both (30) and (35) that if

$$\|y\| = \frac{\xi+1}{e_p(a,b)} r_1,$$

then

$$f\left(t, y^\sigma(t) - w^\sigma(t)\right) \geq K_0,$$

for each $t \in [a, \rho(b)]_{\mathbb{T}}$ and thus for each $t \in [\alpha_1, \alpha_2]_{\mathbb{T}}$. Putting all of this together and using the fact that $F(x, y) \geq 0$ on its domain, the fact that $\frac{e_p(a,t)}{1-e_p(a,b)} \geq 0$, and the fact that $u(t) \geq 0$ on its domain, for $y \in \mathcal{K} \cap \partial\Omega_{\frac{\xi+1}{e_p(a,b)} r_1}$ we estimate

$$\begin{aligned}(Ty)(t) &= \lambda \int_a^b G(t,s)\left[f\left(s, \max\left\{y^\sigma(s) - w^\sigma(s), 0\right\}\right) + u(s)\right] \Delta s \\ &\quad + \frac{e_p(a,t)}{1-e_p(a,b)} \int_{\tau_1}^{\tau_2} F(s, \max\{y(s) - w(s), 0\}) \Delta s \\ &\geq \lambda \int_{\alpha_1}^{\alpha_2} G(t,s)\left[f\left(s, \max\left\{y^\sigma(s) - w^\sigma(s), 0\right\}\right) + u(s)\right] \Delta s \\ &\geq \lambda \int_{\alpha_1}^{\alpha_2} G(s,s) K_0 \Delta s \geq \|y\|,\end{aligned}$$

for each $t \in [a, b]_{\mathbb{T}}$, where to obtain the final inequality we have used estimate (33). Hence, we conclude that $\|Ty\| \geq \|y\|$.

Conversely, let λ as selected in (31) be henceforth fixed. Moreover, we assume throughout this part of the proof that

$$\|y\| \geq \frac{\xi+1}{e_p(a,b)} r_1$$

so that each of (34) and (35) holds. By hypothesis we have that

$$\frac{C_1}{1-e_p(a,b)} \int_{\tau_1}^{\tau_2} 1 \Delta s < 1.$$

Therefore, select $\varepsilon_1 > 0$ sufficiently small such that

$$\left[\frac{C_1}{1-e_p(a,b)} \int_{\tau_1}^{\tau_2} 1 \Delta s\right](1+\varepsilon_1) < 1.$$

In particular, for this choice of ε_1, it holds by condition (H2) that there exists $M_{\varepsilon_1} > 0$ such that for each $s \in [a, b]_{\mathbb{T}}$

$$\begin{aligned}|F(s, y(s) - w(s)) - H(y(s) - w(s))| &< \varepsilon_1 H(y(s) - w(s)) \\ &\leq \varepsilon_1 C_1 [y(s) - w(s)] \leq \varepsilon_1 C_1 \|y\|\end{aligned} \tag{36}$$

whenever

$$y(s) - w(s) \geq M_{\varepsilon_1}.$$

But recalling estimate (35), it follows that (36) holds provided that

$$\|y\| \geq \frac{\xi+1}{e_p(a,b)} M_{\varepsilon_1}.$$

Define θ_0 by

$$\theta_0 := \left[\frac{C_1}{1-e_p(a,b)} \int_{\tau_1}^{\tau_2} 1 \Delta s\right](1+\varepsilon_1),$$

and select $\varepsilon_2 > 0$ such that

$$\theta_0 + \varepsilon_2 < 1.$$

By then selecting $\eta_1 > 0$ such that

$$\left[\lambda \int_a^b e_p(b,a)G(s,s)\ \Delta s\right]\eta_1 < \varepsilon_2,$$

it follows from condition (H6) that we may find $r_2 > 0$ sufficiently large such that

$$f\left(t, \left(y^\sigma - w^\sigma\right)(t)\right) \le \eta_1 \left(y^\sigma - w^\sigma\right)(t) \le \eta_1 y^\sigma(t) \le \eta_1 \|y\| \tag{37}$$

whenever $\left(y^\sigma - w^\sigma\right)(t) > r_2$, for each $t \in [a, \rho(b)]_{\mathbb{T}}$. But, once again, in light of estimate (35), estimate (37) is seen to hold provided that

$$\|y\| \ge \frac{\xi+1}{e_p(a,b)} r_2.$$

Finally, select $\varepsilon_3 > 0$ sufficiently small such that

$$\theta_0 + \varepsilon_2 + \varepsilon_3 < 1.$$

Thus, if we also require that

$$\|y\| > \frac{1}{\varepsilon_3}\lambda \int_a^b e_p(b,a)G(s,s)u(s)\ \Delta s,$$

then

$$\lambda \int_a^b e_p(b,a)G(s,s)u(s)\ \Delta s < \varepsilon_3\|y\|.$$

Now define the number r_2 by

$$r_2 := \max\left\{\frac{2(\xi+1)}{e_p(a,b)} r_1, \frac{\xi+1}{e_p(a,b)} M_{\varepsilon_1}, \frac{\xi+1}{e_p(a,b)} r_2, \frac{1}{\varepsilon_3}\lambda \int_a^b e_p(b,a)G(s,s)u(s)\ \Delta s\right\}$$

and let

$$y \in \mathcal{K} \cap \partial\Omega_{r_2}.$$

Then putting all of the preceding estimates together, for each $t \in [a,b]_{\mathbb{T}}$ we estimate

$$\begin{aligned}
(Ty)(t) &\le \lambda \int_a^b e_p(b,a)G(s,s)\left[f\left(s, y^\sigma(s) - w^\sigma(s)\right) + u(s)\right]\ \Delta s \\
&\quad + \frac{e_p(a,t)}{1-e_p(a,b)} \int_{\tau_1}^{\tau_2} |F(s,y(s)-w(s)) - H(y(s)-w(s))| + |H(y(s)-w(s))|\ \Delta s \\
&\le \eta_1 \left[\lambda \int_a^b e_p(b,a)G(s,s)\ \Delta s\right] \|y\| \\
&\quad + (1+\varepsilon_1)C_1 \max_{t\in[a,b]_{\mathbb{T}}} \left[\frac{e_p(a,t)}{1-e_p(a,b)} \int_{\tau_1}^{\tau_2} 1\ \Delta s\right] \|y\| + \lambda \int_a^b e_p(b,a)G(s,s)u(s)\ \Delta s \\
&\le \eta_1 \left[\lambda \int_a^b e_p(b,a)G(s,s)\ \Delta s\right] \|y\| \\
&\quad + (1+\varepsilon_1)\left[\frac{C_1}{1-e_p(a,b)} \int_{\tau_1}^{\tau_2} 1\ \Delta s\right] \|y\| + \lambda \int_a^b e_p(b,a)G(s,s)u(s)\ \Delta s \\
&\le (\theta_0 + \varepsilon_2 + \varepsilon_3)\|y\| < \|y\|,
\end{aligned}$$

whence $\|Ty\| \le \|y\|$, for each $y \in \mathcal{K} \cap \partial\Omega_{r_2}$.

Consequently, by invoking Lemma 2.11 we find that there exists

$$y_0 \in \mathcal{K} \cap \left(\overline{\Omega}_{r_2} \setminus \Omega_{\frac{\xi+1}{e_p(a,b)} r_1}\right)$$

such that $Ty_0 = y_0$, with y_0 a positive solution of the modified problem (25). Now define $x : [a,b]_{\mathbb{T}} \to \mathbb{R}$ by $x(t) := y_0(t) - w(t)$. Since

$$\|y_0\| \ge \frac{\xi+1}{e_p(a,b)} r_1 \ge \lambda(\xi+1)e_{\ominus p}(a,b),$$

it follows that $(y_0 - w)(t) \ge 0$, for each $t \in [a,b]_{\mathbb{T}}$. Thus, invoking Lemma 2.10, we conclude that the function x is a positive solution of the original problem (1). And this completes the proof. $\square$

Remark 3.14. Let $\mathbb{T} = \mathbb{R}$, $C_1 = 3$, $\tau_1 = \dfrac{2}{5}$, $\tau_2 = \dfrac{1}{2}$, $a = 0$, $b = 1$, and $p(t) \equiv 1$. In this case, problem (1) becomes

$$\begin{aligned}
y'(t) + y(t) = \lambda f\left(t, y(t)\right),\ t \in (0,1) \\
y(0) = y(1) + \int_{2/5}^{1/2} F(s, y(s))\ \mathrm{d}s,
\end{aligned}$$

for some suitably chosen functions f and F. Note that condition (29) in Theorem 3.13 is $\dfrac{3}{1-e^{-1}} \int_{2/5}^{1/2} 1\ \mathrm{d}s = \dfrac{3}{10\left(1-e^{-1}\right)} \approx 0.475 < 1$. So, this condition is satisfied in this case.

Remark 3.15. Let $\mathbb{T}=\mathbb{Z}$, $C_1=\dfrac{1}{5}$, $\tau_1=2$, $\tau_2=5$, $a=0$, $b=10$, and $p(t)\equiv 1$. In this case, problem (1) becomes

$$\begin{aligned}&\Delta y(t)+y(t+1)=\lambda f\left(t,y(t+1)\right),\ t\in(0,10)_{\mathbb{Z}}\\ &y(0)=y(10)+\int_2^5 F(s,y(s))\ \Delta s=y(10)+\sum_{s=2}^{4}F(s,y(s))\end{aligned}\tag{38}$$

for some suitably chosen functions f and F; note that in (38) we use the fact that on the time scale $\mathbb{Z}$ it holds that

$$\int_a^b f(s)\ \Delta s=\sum_{s=a}^{b-1}f(s).$$

Recall that $e_p\left(t,t_0\right)=(1+p(t))^{t-t_0}$ on this time scale – see [**6**]. Consequently, condition (29) in Theorem 3.13 is $\dfrac{1/5}{1-2^{-10}}\displaystyle\sum_{s=2}^{4}1=\dfrac{3}{5\left(1-2^{-10}\right)}<1$, and so, this condition is satisfied in this case, too.

Remark 3.16. We note that it is possible to provide an existence result very similar to that presented in [**2**, Theorem 3.3] – in particular, a result in which we obtain a set $(0,\lambda_0)$ such that for each $\lambda\in(0,\lambda_0)$ problem (1) has at least one positive solution. This can be accomplished both by replacing condition (H1) with the condition that $\displaystyle\lim_{y\to 0^+}\frac{F(t,y)}{y}=0$, uniformly for $t\in[a,b]_{\mathbb{T}}$ and by replacing conditions (H5)–(H6) with the condition that $\displaystyle\lim_{y\to+\infty}\frac{f(t,y)}{y}=+\infty$, uniformly on some set $[\alpha_1,\alpha_2]_{\mathbb{T}}$. Since the proof of this result is rather similar to that of [**2**, Theorem 3.3], we have elected to omit it here.

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