Accepted article - prepared for print

Please cite this article as: Lihua Feng, Guihai Yu, Aleksandar Ilić, Dragan Stevanović: THE SIGNLESS LAPLACIAN SPECTRAL RADIUS OF BOUNDED DEGREE GRAPHS ON SURFACES, Appl. Anal. Discrete Math. Vol. 9 (2015) - to appear
DOI: 10.2298/AADM150722015Y

Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **9** (2015), xxx–xxx. doi:10.2298/AADM150722015Y

THE SIGNLESS LAPLACIAN SPECTRAL RADIUS OF BOUNDED DEGREE GRAPHS ON SURFACES

Guihai Yu, Lihua Feng, Aleksandar Ilić, Dragan Stevanović

Let G be an n-vertex ($n \geq 3$) simple graph embeddable on a surface of Euler genus γ (the number of crosscaps plus twice the number of handles). In this paper, we present upper bounds for the signless Laplacian spectral radius of planar graphs, outerplanar graphs and Halin graphs, respectively, in terms of order and maximum degree. We also demonstrate that our bounds are sometimes better than known ones. For outerplanar graphs without internal triangles, we determine the extremal graphs with the maximum and minimum signless Laplacian spectral radii.

1. INTRODUCTION

Spectral graph theory is a fast growing branch of algebraic graph theory. Within spectral graph theory, studying the properties of a graph using its signless Laplacian became very dynamic area of research in past decade. Recently, Cvetković and Simić defined in a series of three papers [**6, 7, 8**], entitled "Towards a spectral theory of graphs based on the signless Laplacian", the fundamentals of the spectral theory of graphs based on the signless Laplacian. The most studied problems within signless Laplacian theory are those of lower and upper bounding some particular eigenvalues such as the largest eigenvalues of the signless Laplacian, as well as the characterization of the extremal graphs achieving the bound. It is along these lines that our present work is done.

We follow the standard graph notations from [**1**]. Let G be a simple graph with the vertex set $V(G)$ and the edge set $E(G)$. Denote $n = |V(G)|$. The *adjacency*

2010 Mathematics Subject Classification. 05C50.
Keywords and Phrases. signless Laplacian matrix, spectral radius, Euler genus, outerplanar graph, Halin graph.

matrix of G is $A(G) = (a_{ij})$ where $a_{ij} = 1$ if two vertices i and j are adjacent in G and 0 otherwise. The largest eigenvalue of $A(G)$ is called the *spectral radius* of G and denoted by $\rho(G)$. The matrix $L(G) = D(G) - A(G)$ is called the *Laplacian matrix* of G, where $D(G) = \mathrm{diag}(d(u), u \in V)$ is the diagonal matrix of the vertex degrees of G. The largest eigenvalue of $L(G)$, denoted by $\lambda(G)$, is called the *Laplacian spectral radius* of G. For background on the Laplacian eigenvalues of a graph, the reader is referred to [**19**] and [**20**] and the references therein. We call the matrix $Q(G) = D(G) + A(G)$ the *signless Laplacian matrix* or the *Q-matrix* of G, and its largest eigenvalue is denoted by $\mu(G)$ or μ for simplicity. It is well known that $Q(G)$ is irreducible, entrywise nonnegative and positive definite, so from the Perron-Frobenius Theorem, if G is connected, there is a unique unit positive eigenvector corresponding to $\mu(G)$ whose entries sum to 1. We call this eigenvector the *Perron vector*. For more background on the signless Laplacian and spectral graph theory, see [**4, 5, 6, 7, 8**] and the references therein. We denote the maximum degree of a graph G by $\Delta = \max\{d(u), u \in V(G)\}$.

Let Σ be a compact surface and γ be the Euler genus (= the number of crosscaps plus twice the number of handles) of Σ. An embedding is k-representative if no noncontractible closed curve in the surface intersects the embedded graph at fewer than k points. An embedding is *cellular* if every face is homeomorphic to an open disk. As we usually consider a graph embedding in a surface with small genus, we always assume that γ is not too large.

In particular, if $\gamma = 0$, Σ is a plane. We call a graph G a *planar graph*, if G can be embedded in a plane such that no two edges intersect. An *outerplanar graph* is a planar graph that has a planar drawing with all vertices on the same face. From this definition, we know that a graph is outerplanar if it can be embedded in the plane such that all vertices lie on the outer boundary face. An edge of an outerplanar graph G is called a *chord*, if it joins two vertices of the outer face boundary of G but itself is not an edge of the outer boundary face. An outerplanar graph is *maximal outerplanar* if the graph obtained by adding an edge is not outerplanar. Namely, a maximal outerplanar graph on $n (n \geq 3)$ vertices has a plane representation as an n-gon triangulated by $n - 3$ chords. Thus, there are $2n - 3$ edges in a maximal outerplanar graph on n vertices, and there are at least two vertices of degree two.

Let T be a tree with $n \geq 4$ vertices and without vertices of degree 2. Suppose T is embedded in the plane with its end-vertices $v_1, v_2, \ldots, v_s$ in clockwise order, if we add new edges (v_i, v_{i+1}) (where $v_{s+1} = v_1$) in T, then T together with the cycle $(v_1, v_2, \ldots, v_s)$ forms a 3-connected planar graph G which is called *Halin graph*. Vertex v_i $(1 \leq i \leq s)$ is called an *outer vertex* and any other vertex is called *inner vertex*. Denote $IV(G) = \{v |\ v \text{ is an inner vertex of } V(G)\}$ and $t = |IV(G)|$.

Schwenk and Wilson [**23**] initiated the study of the eigenvalues of planar graphs. After that, much attention is paid to this question. Cao and Vince [**3**], independently Boots and Royle [**2**] by using computer studies, conjectured that the planar graph of a given order with largest spectral radius is $K_2 \bigvee P_{n-2}$, the join of K_2 and P_{n-2}. It is noted in [**2**] that the conjecture is not true for $n = 7$ and 8, but suggested it was true for all $n \geq 9$. Up to now, this conjecture is still open. Cvetković [**9**] conjectured that $K_1 \bigvee P_{n-1}$ is the unique graph that has the maximal spectral radius among all outerplanar graphs. As far as we know, this conjecture is also unsolved. Hong [**14, 15**] obtained some significant results for planar graphs. In those papers, Hong also gave the bounds on the spectral radius of graphs on an arbitrary surface. Shu and Hong [**24**] studied the spectral radius of outerplanar graphs and Halin graphs. In [**22**], the spectral radius of a special class of outerplanar graphs is considered. Ellingham and Zha [**11**] presented several new upper bounds on the spectral radius of graphs embeddable on a given compact surface. Dvořák and Mohar [**10**], using graph decompositions, derived several upper bounds for the spectral radius of planar graphs in terms of maximum degree.

The following results were obtained recently on the signless Laplacian spectral radius of graphs embedded on surfaces.

Theorem 1.1. *Let G be a connected graph of order $n \geq 3$, with maximum degree Δ. Suppose G can be embedded on a surface of Euler genus γ. Then*
(i) (*see* [**17**])

$$\mu(G) \leq \frac{1}{2}\big(\Delta + 4 + \sqrt{(\Delta+4)^2 + 8(2n + 8\gamma - 10)}\,\big). \tag{1}$$

(ii) (*see* [**12**])

$$\mu(G) \leq \frac{1}{2}\big(n + 6 + \sqrt{n^2 + 4n - 20 + 64\gamma}\,\big). \tag{2}$$

Theorem 1.2. *Let G be a connected outerplanar graph of order $n \geq 2$, with maximum degree Δ. Then*
(i) (*see* [**17**])

$$\mu(G) \leq \frac{1}{2}\big(\Delta + 3 + \sqrt{(\Delta+3)^2 + 8(n-4)}\,\big). \tag{3}$$

(ii) (*see* [**12**])

$$\mu(G) \leq n + 2. \tag{4}$$

Theorem 1.3. *Let G be a Halin graph of order $n \geq 7$, with maximum degree Δ. If G has t (≥ 1) inner vertices, then*
(i) (*see* [**17**])

$$\mu(G) \leq \frac{1}{2}\big(\Delta + 2 + \sqrt{(\Delta+2)^2 + 8(n - 2t + 1)}\,\big). \tag{5}$$

(ii) (*see* [**12**])

$$\mu(G) \leq \frac{1}{2}\big(n - 2t + 6 + \sqrt{(n - 2t + 2)^2 + 24}\,\big). \tag{6}$$

In this paper, we present several new upper bounds for the signless Laplacian spectral radius of planar graphs, outerplanar graphs and Halin graphs, in terms of order and maximum degree. We also demonstrate that our bounds are sometimes better when compared with known ones. At last, for a special class of outerplanar graphs, we determine the extremal graphs with the maximal and minimal signless Laplacian spectral radii.

2. PRELIMINARIES

We present here a few results that will be needed in later proofs.

Lemma 2.4 ([**13**])**.** *Let $M = (m_{ij})$ be an $n \times n$ irreducible nonnegative matrix with spectral radius $\rho(M)$, and let $s_i(M)$ be the ith row sum of M, i.e., $s_i(M) = \sum_{j=1}^{n} m_{ij}$. Then*

$$\min\{s_i(M) : 1 \leq i \leq n\} \leq \rho(M) \leq \max\{s_i(M) : 1 \leq i \leq n\}.$$

Moreover, the equalities hold if and only if the row sums of M are all equal.

Lemma 2.5 ([**11**])**.** *Let G be a connected graph of order n, $Q = Q(G)$ be the signless Laplacian matrix and P be any polynomial. Then*

$$\min_{v \in V(G)} s_v(P(Q)) \leq \mu(P(Q)) \leq \max_{v \in V(G)} s_v(P(Q)).$$

Let G be a connected graph and $v \in V(G)$. Denote by $N_i(v, G)$ the set of vertices at distance i from v and let $n_i(v, G) = |N_i(v, G)|$.

Lemma 2.6 ([**11**])**.** *Let G be a connected graph on at least two vertices, with adjacency matrix A and with a cellular embedding D in a surface of Euler genus γ. Let $v \in V(G)$. If $n_1(v, G) \geq 3$, then we have*

$$s_v(A^2) \leq 6n_1(v, G) + 2n_2(v, G) + 8\gamma - 8.$$

The following relation between the spectral radii of the signless Laplacian and the adjacency matrices will be useful in Section 3.

Lemma 2.7. *Let G be a connected graph with maximum degree Δ. Then*

$$\mu(G) \leq \Delta + \rho(G).$$

The equality holds if and only if G is a regular graph.

Proof. Let X be the Perron vector corresponding to $\mu(G)$. Then we have

$$\mu(G) = X^t D(G)X + X^t A(G)X \leq \rho(D(G)) + \rho(A(G)) = \Delta + \rho(G).$$

If the equality holds, then $X^t D(G)X = \Delta$ and $X^t A(G)X = \rho(G)$. Note that $X^t D(G)X = \Delta$ implies that G is regular in light of Lemma 2.4, and thus X is also the Perron vector of $A(G)$. The converse is easy to verify since $\mu(G) = 2\Delta$, $\rho(G) = \Delta$ for a regular graph G.

3. UPPER BOUNDS ON THE SIGNLESS LAPLACIAN SPECTRAL RADIUS

We first present a new bound involving the maximum vertex degree for the signless Laplacian spectral radius of a graph embeddable on a surface of given genus.

Theorem 3.8. *Let G be a connected graph of order $n \geq 5$ with maximum degree Δ. Suppose G can be embedded in a surface of Euler genus γ. If $\Delta \geq 2+\sqrt{2n+8\gamma-6}$, then*

$$\mu(G) \leq \frac{1}{2}\big(\Delta+5+\sqrt{(\Delta+1)^2+8(2n+8\gamma-7)}\,\big). \tag{7}$$

Proof. Note that $\Delta \geq 2+\sqrt{2n+8\gamma-6} \geq 4$ for $n \geq 5$. For a vertex $v \in V(G)$, we have $s_v(Q)=2d(v)$, $s_v(D^2)=s_v(DA)=d(v)^2$ and $s_v(AD)=s_v(A^2)=d(v)m(v)$. Further, we abbreviate $n_i(v,G)$ to n_i (note that $n_1=d(v)$).

We first consider a more general case. For a suitable real number c,

$$\begin{aligned}
&s_v(Q^2)-(n_1+c)s_v(Q)\\
&=s_v(D^2+DA+AD+A^2)-(n_1+c)s_v(Q)\\
&=2d(v)^2+2d(v)m(v)-2(n_1+c)d(v)\\
&\leq 2n_1^2+2(6n_1+2n_2+8\gamma-8)-2(n_1+c)n_1 \quad \text{(from Lemma 2.6)}\\
&\leq 2n_1^2+2\big(6n_1+2(n-1-n_1)+8\gamma-8\big)-2(n_1+c)n_1 \quad (\text{as } n \geq 1+n_1+n_2)\\
&=2\big(n_1^2+(4-n_1-c)n_1+2n-10+8\gamma\big)=2\big((4-c)n_1+2n+8\gamma-10\big).
\end{aligned}$$

Hence, we have

$$s_v(Q^2)-(n_1+c)s_v(Q)-2\big((4-c)n_1+2n+8\gamma-10\big) \leq 0.$$

Note that Lemma 2.5 cannot be applied directly to the above inequality, as n_1 is not a constant term, but depends on v. However, $n_1 \leq \Delta$ and $s_v(Q)=2d(v) \geq 2$ for each v, as the graph is connected. Therefore, if $c \leq 5$ then $(\Delta-n_1)[s_v(Q)+2(4-c)] \geq 0$ and one immediately gets

$$\begin{aligned}
&s_v(Q^2)-(\Delta+c)s_v(Q)-2\big((4-c)\Delta+2n+8\gamma-10\big)\\
&\leq s_v(Q^2)-(n_1+c)s_v(Q)-2\big((4-c)n_1+2n+8\gamma-10\big) \leq 0.
\end{aligned}$$

Now, Lemma 2.5 implies that

$$\mu^2-(\Delta+c)\mu-2\big((4-c)\Delta+2n+8\gamma-10\big) \leq 0,$$

wherefrom

$$\mu(G) \leq \frac{1}{2}\big((\Delta+c)+\sqrt{(\Delta+c)^2+8((4-c)\Delta+2n+8\gamma-10)}\,\big) =: f(c).$$

A direct computation shows that

$$f'(c) = \frac{1}{2}\left(1 + \frac{2c - 6\Delta}{2\sqrt{(\Delta+c)^2 + 8((4-c)\Delta + 2n + 8\gamma - 10)}}\right).$$

Since $\Delta \geq 2 + \sqrt{2n + 8\gamma - 6}$, we have $f'(c) \leq 0$ for all $c \leq 5$, hence

$$f(c) \geq f(5) = \frac{1}{2}\big(\Delta + 5 + \sqrt{(\Delta+1)^2 + 8(2n + 8\gamma - 7)}\big),$$

and Eq. (7) follows.

Remark. In case that $\Delta \leq 2 + \sqrt{2n + 8\gamma - 6}$, we have that $f'(c) \geq 0$ for all $c \leq 5$, so that the best bound is obtained when $c \to -\infty$. Since

$$\lim_{c \to -\infty} f(c) = 2\Delta,$$

we conclude that

$$\mu(G) \leq 2\Delta$$

in this case, which is a trivial and well known bound (a direct consequence of Lemma 2.4).

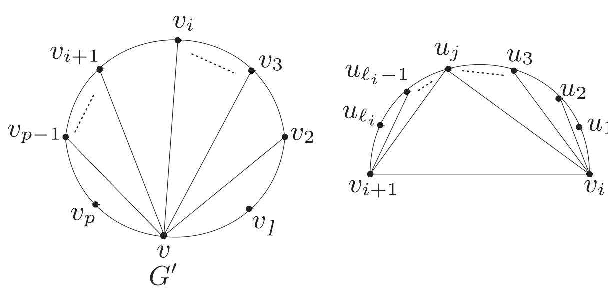

Figure 1. Graphs used in Theorem 3.9.

The next result is for the signless Laplacian spectral radius of an outerplanar graph using maximum degree and order.

Theorem 3.9. *Let G be a connected outerplanar graph of order $n \geq 4$ with maximum degree $\Delta \geq 3$. Then*

$$\mu(G) \leq \frac{1}{2}\big((\Delta+5) + \sqrt{(\Delta+5)^2 + 8(n - 2\Delta - 4)}\big). \tag{8}$$

Proof. By adding edges to G, one can get a maximal outerplanar graph G', for which $\mu(G) < \mu(G')$. As G' contains a Hamiltonian cycle, consider a planar embedding of G' such that the boundary face is the Hamiltonian cycle. Note that any edge of G' is either on the boundary face or a chord, and no two chords intersect.

Let $v \in V(G')$. Suppose $N_{G'}(v) = \{v_1, v_2, \ldots, v_p\}$ (in a clockwise order, as depicted in Figure 1), so that $d(v) = p \geq 2$. Note that $v_1, v_2, \ldots, v_p$ are in the boundary face, and there are no vertices between v and v_1 or between v_1 and v_p.

Suppose there are ℓ_i $(1 \leq i \leq p-1)$ vertices between v_i and v_{i+1} such that they are not adjacent to v. Note that

$$p + 1 + \sum_{i=1}^{p-1} \ell_i = n.$$

Suppose on the boundary face, the set of the ℓ_i vertices between v_i and v_{i+1} are $U = \{u_1, u_2, \ldots, u_{\ell_i}\}$, then the vertices in U are either adjacent to v_i, v_{i+1}, or adjacent to each other, and any two chords cannot intersect inside G'. Therefore there is at most one vertex in U, say u_j, that may be adjacent to both v_i and v_{i+1} (see Fig. 1). Hence, the number of edges between the vertices of U and $\{v_i, v_{i+1}\}$ is at most $\ell_i + 1$. It follows that

$$\begin{aligned} d_v m_v = \sum_{i=1}^{p} d(v_i) \leq\ & p && \text{(the edges between } v \text{ and } v_i) \\ & + 2(p-1) && \text{(the edges between } v_i \text{ and } v_{i+1}) \\ & + \sum_{i=1}^{p-1} (\ell_i + 1) \\ & = n + 3p - 4. \end{aligned}$$

As in the proof of Theorem 3.8, we first consider a more general case. For a real number c,

$$\begin{aligned} s_v(Q^2) - (\Delta + c)s_v(Q) &= s_v(D^2 + DA + AD + A^2) - (\Delta + c)s_v(Q) \\ &= 2d(v)^2 + 2d(v)m(v) - 2(\Delta + c)d(v) \\ &\leq 2p^2 + 2(n + 3p - 4) - 2(\Delta + c)p \\ &= 2p^2 - 2(\Delta + c - 3)p + 2(n - 4) =: f(p). \end{aligned}$$

Since $p \in \{2, \ldots, \Delta\}$, we have that $f(p) \leq f(2) = 2(n - 2\Delta - 2c + 6)$ provided that $c \geq 5$ and

$$s_v(Q^2) - (\Delta + c)s_v(Q) - 2(n - 2\Delta - 2c + 6) \leq 0.$$

From Lemma 2.5, it follows that

$$\mu^2 - (\Delta + c)\mu - 2(n - 2\Delta - 2c + 6) \leq 0,$$

which implies

$$\mu(G) \leq \frac{1}{2}\Big((\Delta + c) + \sqrt{(\Delta + c)^2 + 8(n - 2\Delta - 2c + 6)}\Big) =: g(c).$$

It is immediate to see that $g'(c) > 0$ for $c \geq 5$ and, therefore, $g(c) \geq g(5)$. Hence for $c = 5$, we get

$$\mu(G) \leq \frac{1}{2}\Big((\Delta + 5) + \sqrt{(\Delta + 5)^2 + 8(n - 2\Delta - 4)}\Big).$$

The proof is completed

Remark. On the other hand, if $c \leq 5$, then $f(p) \leq f(\Delta) = 2(n-4-(c-3)\Delta)$. As above, we similarly get

$$\mu(G) \leq \frac{1}{2}\Big((\Delta+c) + \sqrt{(\Delta+c)^2 + 8(n-4-(c-3)\Delta)}\,\Big) =: h(c).$$

If $\Delta \leq \frac{3}{2} + \sqrt{n - \frac{7}{4}}$, then $h'(c) \geq 0$ and the best upper bound is obtained when $c \to -\infty$. Since

$$\lim_{c \to -\infty} h(c) = 2\Delta$$

we conclude that

$$\mu(G) \leq 2\Delta$$

in this case, which is a trivial and well known bound (a direct consequence of Lemma 2.4). If $\Delta \geq \frac{3}{2} + \sqrt{n - \frac{7}{4}}$, then $h'(c) \leq 0$ and $h(c) \geq h(5)$ and

$$\mu(G) \leq \frac{1}{2}\Big((\Delta+5) + \sqrt{(\Delta+5)^2 + 8(n-4-2\Delta)}\,\Big),$$

but since $h(5) = g(5)$, we cannot obtain a better bound in the case $c \leq 5$.

4. COROLLARIES AND COMPARISONS

We deduce here several corollaries of earlier results by Lemma 2.7, and then present comparisons between earlier results and Theorems 3.8 and 3.9.

Corollary 4.10. *Let G be a connected graph of order $n \geq 3$, with maximum degree Δ. Suppose G can be embedded on a surface of Euler genus γ. Then*

$$(9)\quad \mu(G) \leq \begin{cases} \Delta + 2 + \sqrt{2n + 8\gamma - 6}, & \textit{if } n \geq 7 + 2\gamma + 4\sqrt{1+3\gamma} \textit{ or } n = 3 \textit{ and } \gamma = 0; \\ \Delta + 1 + \sqrt{3n + 6\gamma - 8}, & \textit{if } 3 \leq n \leq 7 + 2\gamma + 4\sqrt{1+3\gamma}. \end{cases}$$

Proof. In [**11**], it is obtained that for a connected graph G of order $n \geq 3$ which can be embedded on a surface of Euler genus γ, it holds

$$\rho(G) \leq \begin{cases} 2 + \sqrt{2n + 8\gamma - 6}, & \text{if } n \geq 7 + 2\gamma + 4\sqrt{1+3\gamma} \text{ or } n = 3 \text{ and } \gamma = 0; \\ 1 + \sqrt{3n + 6\gamma - 8}, & \text{if } 3 \leq n \leq 7 + 2\gamma + 4\sqrt{1+3\gamma}. \end{cases}$$

The result follows from Lemma 2.7. □

n	Δ	γ	conclusion
100	5	1	$(1) = 24.90 > (9) = 21.21$
100	50	1	$(9) = 66.21 > (1) = 60.54$
100	5	1	$(2) = 104.10 > (9) = 21.21$
100	97	1	$(9) = 113.21 > (2) = 104.10$

Table 1. Comparison between bounds (1), (2) and (9).

n	Δ	γ	conclusion
1000	998	1	(7) = 1004.99 > (2) = 1004.01
100	49	1	(2) = 104.10 > (7) = 59.05

Table 2. Comparison between bounds (2) and (7) for $\Delta \geq 2 + \sqrt{2n + 8\gamma - 6}$.

Table 1 reveals that the bound (9) is incomparable with bounds (1) and (2). Here, (1) > (9) means the value given by the bound (1) is larger than the value given by the bound (9). Table 2 further reveals that the bounds (2) and (7) are incomparable. If $\Delta \geq 2 + \sqrt{2n + 8\gamma - 6}$, direct computation shows that (7) is less than or equal to (1), and if further $n \geq 7 + 2\gamma + 4\sqrt{1 + 3\gamma}$, then (7) is less than or equal to (9).

We further have the following corollaries for planar graphs.

Corollary 4.11. *Let G be a connected planar graph of order $n \geq 3$, with maximum degree Δ. Then*

$$\mu(G) \leq \begin{cases} \Delta + 2 + \sqrt{2n - 6}, & \text{if } n \geq 9; \\ \Delta + 1 + \sqrt{3n - 8}, & \text{if } 3 \leq n \leq 9. \end{cases} \tag{10}$$

Proof. Set $\gamma = 0$ in Corollary 4.10.

Corollary 4.12. *Let G be a connected planar graph with maximum degree Δ. Then*

$$\mu(G) \leq \begin{cases} 2\Delta, & \text{if } \Delta \leq 5; \\ \Delta + \sqrt{12\Delta - 36}, & \text{if } 6 \leq \Delta \leq 36; \\ \Delta + \sqrt{8\Delta - 16} + 2\sqrt{3}, & \text{if } \Delta \geq 37. \end{cases} \tag{11}$$

Proof. In [**10**], it is obtained that for a connected planar graph G, the following best known bounds hold

$$\rho(G) \leq \begin{cases} \Delta, & \text{if } \Delta \leq 5; \\ \sqrt{12\Delta - 36}, & \text{if } 6 \leq \Delta \leq 36; \\ \sqrt{8\Delta - 16} + 2\sqrt{3}, & \text{if } \Delta \geq 37. \end{cases}$$

The result now follows from Lemma 2.7. □

Table 3 shows that the bounds (10) and (11) are incomparable for large Δ.

n	Δ	conclusion
100	40	(11) = 60.90 > (10) = 55.93
1000	40	(10) = 86.65 > (11) = 60.90

Table 3. Comparison between bounds (10) and (11).

A cycle C in a connected graph G is separating if $G - V(C)$ is disconnected. Clearly, a 5-connected graph contains no separating cycle of length at most four.

Corollary 4.13. *Let G be a connected planar graph with maximum degree Δ, without separating 4-cycle. Then*

$$\mu(G) \leq \Delta + 2\sqrt{\Delta - 9} + 2\sqrt{19} + 2\sqrt{21}.$$

Proof. In [**10**], it is obtained that for a connected planar graph G without separating 4-cycles,

$$\rho(G) \leq 2\sqrt{\Delta - 9} + 2\sqrt{19} + 2\sqrt{21}.$$

The result now follows from Lemma 2.7.

Corollary 4.14. *Let G be a connected outerplanar graph of order n with maximum degree Δ. Then*

$$\mu(G) \leq \Delta + \frac{3}{2} + \sqrt{n - \frac{7}{4}}. \tag{12}$$

Proof. In [**24**], it is obtained that for a connected outerplanar graph G, it holds

$$\rho(G) \leq \frac{3}{2} + \sqrt{n - \frac{7}{4}}.$$

The result now follows from Lemma 2.7. □

n	Δ	conclusion
100	10	(3) = 21.81 > (12) = 21.41
100	50	(12) = 61.41 > (3) = 56.40
100	3	(4) = 102 > (12) = 14.41
100	95	(12) = 106.41 > (4) = 102
32	6	(8) = 13.39 > (12) = 13
32	10	(12) = 17 > (8) = 16
15	10	(3) = 14.52 > (8) = 13.68
100	10	(8) = 21.93 > (3) = 21.81

Table 4. Comparison between bounds (3), (4), (8) and (12).

Table 4 reveals that the bound (12) is incomparable with the bounds (3), (4) and (8), and that the bounds (3) and (8) are also incomparable. Further, direct computation shows that (8) $\leq$ (4) with equality if $\Delta = n - 1$.

Corollary 4.15. *Let G be a Halin graph of order $n \geq 7$ with maximum degree Δ. If G has t (≥ 1) inner vertices, then*

$$\mu(G) \leq \Delta + 1 + \sqrt{n + 2 - 2t}. \tag{13}$$

Proof. In [**24**], it is obtained that for a Halin graph of order $n \geq 7$, if G has t (≥ 1) inner vertices, then

$$\rho(G) \leq 1 + \sqrt{n + 2 - 2t}.$$

The result now follows from Lemma 2.7. □

n	t	Δ	conclusion
100	4	20	$(13) = 30.69 > (5) = 28.5$
100	4	4	$(5) = 17 > (13) = 14.69$
100	4	95	$(13) = 105.70 > (6) = 96.06$
100	4	5	$(6) = 96.06 > (13) = 15.70$

Table 5. Comparison between bounds (5), (6) and (13).

Table 5 reveals that the bound (13) is incomparable with the bounds (5) and (6).

5. A SPECIAL CLASS OF OUTERPLANAR GRAPHS

In [**12**], it is conjectured that $K_1 \bigvee P_{n-1}$ is the unique graph that has the maximal signless Laplacian spectral radius among all outerplanar graphs. Recently, this conjecture is proved to be true by Yu et al. [**25**]. In this section, we focus our attention on a special class of outerplanar graphs containing no internal triangles, which are of independent interest, as stated at the end of this section, they are just the so called "2-paths".

For $n \geq 4$, let $\mathcal{H}_n$ be the set of all maximal outerplanar graphs on n vertices. Thus if $G \in \mathcal{H}_n$, then G has a planar represeentation as an n-gon triangulated by $n-3$ chords, and the boundary of this graph is the unique Hamiltonian cycle Z of G. Let G^* be the graph obtained from G by deleting all edges of Z. Then G^* has a unique nontrivial component G^{**} and at least two isolated vertices. A 3-cycle in G^{**} is called an internal triangle of G, in other words, an internal triangle is a triangle with no edge on the outer face. Note that the following statements are equivalent:

(a) G has no internal triangle;
(b) G^* has exactly two isolated vertices;
(c) G^{**} is a tree.

We are interested in the set of maximal outerplanar graphs having no internal triangles.

Lemma 5.16 ([**16**]). *Let u, v be two vertices of the connected graph G, suppose $v_1, v_2, \ldots, v_s \in N(v) \setminus N(u)$ for $1 \leq s \leq d(v)$, where $v_1, v_2, \ldots, v_s$ are different from u. Let $X = (x_1, x_2, \ldots, x_n)$ be the Perron vector of $Q(G)$, where x_i corresponds to v_i $(1 \leq i \leq n)$. Let H be the graph obtained from G by deleting the edges vv_i and adding the edges uv_i $(1 \leq i \leq s)$. If $x_u \geq x_v$, then $\mu(G) < \mu(H)$.*

Let P_n^2 denote the graph obtained from the path P_n by adding new edges joining all pairs of vertices at distance 2 in P_n. The graph P_9^2 is illustrated in Figure 3.

Theorem 5.17. *Let $\mathcal{G}_n$ be the set of maximal outerplanar graphs having n vertices and no internal triangles. The graph $K_1 \bigvee P_{n-1}$ is the unique graph in $\mathcal{G}_n$ with maximum signless Laplacian spectral radius and the graph P_n^2 is the unique graph in $\mathcal{G}_n$ with minimum signless Laplacian spectral radius.*

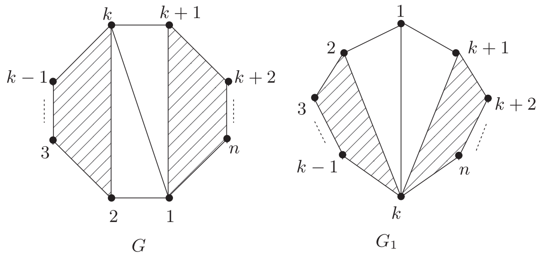

Figure 2. The graphs G and G_1.

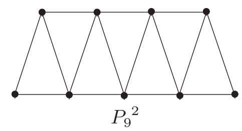

Figure 3. The graph P_9^2.

Proof. If $n = 4, 5$, it is easy to find that $K_1 \bigvee P_{n-1} \cong P_n^2$ is the unique graph in $\mathcal{G}_n$. Hence we may suppose $n \geq 6$ in the sequel. Let $G \in \mathcal{G}_n$ and suppose that $G \neq K_1 \bigvee P_{n-1}$, we will show that there exists a graph $G_1 \in \mathcal{G}_n$ such that $\mu(G) < \mu(G_1)$.

Let the vertices of the Hamiltonian cycle in G be labeled by $1, 2, \ldots, n$ in the cyclic order. Since G^{**} is not a star, the labeling may be chosen so that some vertex k is adjacent to vertices 1 and 2, while vertex $k+1$ is adjacent to 1 $(3 < k < n-1)$. A plane representation of G is illustrated in Figure 2, where only the subgraph induced by the vertices 1,2, k, $k+1$ is shown in full. Let e be the edge $(1, k)$. Then the graph G_1 is obtained by twisting G along the edge e. Formally, let H be the subgraph induced by vertices $1, 2, , \ldots, k$ and K be the subgraph induced by vertices $1, k, k+1, \ldots, n$. We may regard G as a triangulated polygon constructed from H and K by coalescence of the edge e. The graph G_1 is constructed from H and K by coalescence of the edge e after first reorienting one of H or K.

Let $X = (x_1, x_2, \cdots, x_n)$ be the Perron vector of $Q(G)$, where x_i corresponds to the vertex i. Without loss of generality, we may suppose that $x_k \geq x_1$. In G, let J be the set of neighbors of vertex 1 between $k+2$ and n inclusive. Then G_1 may be obtained from G by replacing the edge $(1, j)$ with (k, j) for each $j \in J$. A plane representation of G_1 is illustrated in Figure 2. From Lemma 5.16, we conclude that $\mu(G) < \mu(G_1)$.

To prove the second assertion of the theorem, let G_2 be a graph in $\mathcal{G}_n$ other than P_n^2 with the smallest signless Laplacian spectral radius. Then G_2 has a vertex v of degree at least 5. As above, the vertices of the Hamiltonian cycle in G_2 are labeled in cyclic order, so we take $v = 1$ without loss of generality. Then there exists $h \in \{4, 5, \ldots, n-2\}$ such that vertex 1 is adjacent to $h, h-1, h+1$, due to the fact that G_2 is a maximal outerplanar graph. Let e be the edge $(1, h)$ and let G be the graph obtained from G_2 by twisting along e, so that G has the form shown in Figure 2. Note that G_2 is obtained from G by twisting along e. The previous argument now shows that $\mu(G) < \mu(G_2)$. It follows that P_n^2 is the unique graph with the smallest signless Laplacian spectral radius. $\square$

Remark. A vertex $v \in G$ is said to be *simplicial* if $N(v)$ induces a clique in G. A *k-path graph* can be inductively defined as follows [**21**]:

- Every complete graph with $k+1$ vertices is a k-path graph.

- If $G = (V, E)$ is a k-path graph, $Q \subseteq V$ is a k-clique of G containing at least one simplicial vertex and $v \notin V$, then the graph $G' = (V \cup \{v\}, E \cup \{\{v, w\} | w \in Q\})$ is also a k-path graph.
- Nothing else is a k-path graph.

As pointed out by one of the referees, the set of 2-path graphs and the set of maximal outerplanar graphs without internal triangles are identical. Thus the results in this section also can be restated as the maximum and minimum spectral radius of 2-path graphs. For more detailed discussion on k-path graphs, one may refer to [**18, 21**] and their references. Their eigenvalue problems would be also an interesting topic, see, for example, the recent related work on k-trees in [**26**].

Acknowledgements. This work was supported by NSFC (11101245, 11271208, 61202362, 11301302), NSF of Shandong (No. BS2013SF009), Mathematics and Interdisciplinary Sciences Project of Central South University, Research Grant ON174033 of the Ministry of Education and Science of Serbia and Research Programme P1-0285 and Research Project J1-4021 of Slovenian Agency for Research. The authors thank the referees for bringing the reference [**21**] to our attention, and for their valuable comments, corrections and suggestions which led to an improvement of this paper.

REFERENCES

1. J. A. Bondy, U. S. R. Murty: *Graph Theory with Applications.* New York, Macmilan Press, 1976.

2. B. N. Boots, G. F. Royle: *A conjecture on the maximum value of the principal eigenvalue of a planar graph.* Geograph. Anal., **23** (1991), 276–282.

3. D. Cao, A. Vince: *The spectral radius of a planar graph.* Linear Algebra Appl., **187** (1993), 251–257.

4. D. Cvetković, P. Rowlinson, S. K. Simić: *Signless Laplacians of finite graphs.* Linear Algebra Appl., **423** (2007), 155–171.

5. D. Cvetković, M. Doob, H. Sachs: *Spectra of Graphs.* Academic Press, New York, 1980.

6. D. Cvetković, S. K. Simić: *Towards a spectral theory of graphs based on signless Laplacian* I. Publ. Inst. Math. (Beograd) (N.S.), **85** (99) (2009), 19–33.

7. D. Cvetković, S. K. Simić: *Towards a spectral theory of graphs based on signless Laplacian* II. Linear Algebra Appl., **432** (2010), 2257–2272.

8. D. Cvetković, S. K. Simić: *Towards a spectral theory of graphs based on signless Laplacian* III. Appl. Anal. Discrete Math., **4** (2010), 156–166.

9. D. Cvetković, P. Rowlinson: *The largest eigenvalue of a graph: A survey.* Linear Multilinear Algebra, **28** (1990), 3–33.

10. Z. Dvorak, B. Mohar: *Spectral radius of finite and infinite planar graphs and of graphs of bounded genus.* J. Combin. Theory Ser. B, **100** (2010), 729–739.

11. M. N. Ellingham, X. Y. Zha: *The spectral radius of graphs on surfaces.* J. Combin. Theory Ser. B, **78** (2000), 45–56.

12. L. Feng, G. Yu, A. Ilić, D. Stevanović: *The signless Laplacian spectral radius of graphs on surfaces.* Linear Multilinear Algebra, **61** (2013), 573–581.

13. R.A. Horn, C. R. Johnson: *Matrix Analysis.* Cambridge University Press, New York, 1985.

14. Y. Hong: *On the spectral radius and the genus of graphs.* J. Combin. Theory Ser. B, **65** (1995), 262–268.

15. Y. Hong: *Sharp upper bounds of the spectral radius of graphs in terms of genus.* J. Combin. Theory Ser. B, **74** (1998), 153–159.

16. Y. Hong, X.-D. Zhang: *Sharp upper and lower bounds for largest eigenvalue of the Laplacian matrices of trees.* Discrete Math., **296** (2005), 187–197.

17. L. Lin: *The Laplacian spectral radius of graphs on surfaces.* Linear Algebra Appl., **428** (2008), 973–977.

18. L. Markenzon, C. Justel, N. Paciornik: *Subclasses of k-trees: characterization and recognition.* Discrete Appl. Math., **154** (2006), 818–825.

19. R. Merris: *Laplacian matrices of graphs: a survey.* Linear Algebra Appl., **197-198** (1994), 143–176.

20. B. Mohar: *The Laplacian spectrum of graphs. Graph theory, Combinartorics and applications*, Vol 2, Ed. Y. Alavi, et al, Wiley, 1991, pp.871–898.

21. P. Pereira, A. Garcia, L. Markenzon: *Generating and counting unlabeled k-path graphs.* Discrete Applied Math., **164** (2014), 297–303.

22. P. Rowlinson: *On the index of certain outerplanar graphs.* Ars Comb., **29C** (1990), 221–225.

23. A. J. Schwenk, R. J. Wilson: *On the eigenvalues of a graph,* in "Selected Topics in Graph Theory" (L. W. Beineke and R. J. Wilson, Eds.), Chap. 11, pp. 307–336, Academic Press, London, 1978.

24. J.L. Shu, Y. Hong: *Upper bounds of the spectral radius for outerplanar graphs and Halin graphs.* Chin. Ann. Math. Ser. A, **21** (2001), 677–682.

25. G. Yu, S. Guo, Y. Wu: *Maxima of the Q-index for outer-planar graphs.* Linear and Multilinear Algebra, **63** (2015), 1837–1848.

26. M.J. Zhang, S.C. Li: *On the signless Laplacian spectra of k-trees*, Linear Algebra Appl., **467** (2015), 136–148.

(Received December 7, 2014)
(Revised June 23, 2015)

School of Mathematics,
Shandong Institute of Business and Technology,
191 Binhaizhong Road,
Yantai, Shandong, 264005
China

and

School of Mathematics and Statistics,
Central South University,
New Campus, Changsha, Hunan, 410083
China
E-mail: yuguihai@126.com

School of Mathematics and Statistics,
Central South University,
New Campus, Changsha, Hunan, 410083
China
E-mail: fenglh@163.com

Mathematical Institute,
Serbian Academy of Science and Arts,
Knez Mihajlova 36, 11000, Belgrade
Serbia
E-mail: aleksandari@gmail.com

Mathematical Institute,
Serbian Academy of Science and Arts,
Knez Mihajlova 36, 11000, Belgrade
Serbia

and

University of Primorska,
Institute Andrej Marušič,
Muzejski trg 2, 6000 Koper
Slovenia
E-mail: dragance106@yahoo.com