

ON THE GRACEFULNESS OF THE GRAPH $P_{2m,2n}$

Zhihe Liang, Huijuan Zuo

Let $P_{a,b}$ denotes a graph obtained by identifying the end vertices of b internally disjoint paths each of length a . Kathiresan conjectured that graph $P_{a,b}$ is graceful except when a is odd and $b \equiv 2 \pmod{4}$. In this paper we show that the graph $P_{a,b}$ is graceful when both a and b are even.

1. INTRODUCTION

The following notations are used frequently. The symbol Z_n denotes the residue ring of integers modulo n . Let Z be the set of all integers. Then the symbol $[a, b]$ is defined by $\{x \mid x \in Z, a \leq x \leq b\}$, $[a, b]_k$ is defined by $\{x \mid x \in Z, a \leq x \leq b, x \equiv a \pmod{k}\}$, and the symbol $\lfloor x \rfloor$ ($\lceil x \rceil$) denotes the greatest (least) integer y such that $y \leq x$ ($y \geq x$). When S is a set, let $f(S)$ denotes the set $\{f(x) \mid x \in S\}$ and $S + a = \{s + a \mid s \in S\}$.

Graphs labeling, where the vertices are assigned values subject to certain conditions, have often been motivated by practical problems. Labeled graphs serves as useful mathematical models for a broad range of applications such as coding theory, including the design of good radar type codes, synch-set codes, missile guidance codes and convolution codes with optimal autocorrelation properties. They facilitate the optimal nonstandard encoding of integers. A systematic presentation of diverse applications of graph labelings is presented in [1]. A function f is called a graceful labeling of a (p, q) -graph $G=(V, E)$ if f is an injection from the vertices of G to the set $[0, q]$, such that the induced mapping $f^*(uv)=|f(u) - f(v)|$ is a bijection from $E(G)$ onto $[1, q]$. ROSA [5] introduced this concept in 1967 and also defined an α -labeling of a graph G as a graceful labeling f of G such that for each edge uv of G either $f(u) \leq c < f(v)$ or $f(v) \leq c < f(u)$ for some integer c , called characteristic of f .

Let $P_{a,b}$ denotes a graph obtained by identifying the end vertices of b internally disjoint paths each of length a . KATHIRESAN [4] conjectured that graph $P_{a,b}$

2000 Mathematics Subject Classification. 05C78

Keywords and Phrases. Graph $P_{a,b}$, graph labelings, graceful graph, matrix

is graceful except when a is odd and $b \equiv 2 \pmod{4}$. He showed that the conjecture is true for the case a even and b odd. SEKER [7] has shown that $P_{a,b}$ is graceful when $a \neq 4r + 1$, $r > 1$; $b=4m$, $m > r$. YANG et al. [6, 8] showed that $P_{2r+1,2m-1}$ is graceful for any positive integer r and any positive integer m , and $P_{2r,2m}$ is graceful for any positive integer m and $r=1, 2, 3, 4, 5, 6, 7$, and 9. DAI et al. [2] showed that $P_{3,4m}$ is graceful for any positive integer m . GALLIAN [3] surveyed the results on graceful labeling of graphs. In this paper, we investigate the gracefulness of graph $P_{a,b}$ when both a and b are even.

2. MAIN RESULTS

For a $(k+1)$ -dimensional vector $A=(a_1, a_2, \dots, a_{k+1})$, the k -dimensional vector $DA=(|a_1 - a_2|, |a_2 - a_3|, \dots, |a_{k-1} - a_k|, |a_k - a_{k+1}|)$ is called a difference vector of A . A vector (or matrix) is called an EPD vector (or matrix) if its elements are pairwise distinct. The symbol $\langle A \rangle$ denotes the set of all elements of vector (or matrix) A . We obtain the following two lemmas.

Lemma 2.1. *Let $A=(a_1, a_2, \dots, a_{k+1})$, where*

$$a_i = \begin{cases} s + (i - 1)/2, & i \in [1, k + 1]_2 \\ t - (i - 2)/2, & i \in [2, k + 1]_2. \end{cases}$$

Then $\langle DA \rangle = [t - s - k + 1, t - s]$ when $t - s \geq k$ and $\langle DA \rangle = [s - t, s - t + k - 1]$ when $s > t$.

Proof. When $t - s \geq k$, we have $A_1 = \{|a_{2i} - a_{2i-1}| \mid i \in [1, \lfloor (k+1)/2 \rfloor]\} = \{t - s - 2i + 2 \mid i \in [1, \lfloor (k+1)/2 \rfloor]\} = [t - s - 2\lfloor (k+1)/2 \rfloor + 2, t - s]_2$,

$A_2 = \{|a_{2i+1} - a_{2i}| \mid i \in [1, \lfloor k/2 \rfloor]\} = \{t - s - 2i + 1 \mid i \in [1, \lfloor k/2 \rfloor]\} = [t - s - 2\lfloor k/2 \rfloor + 1, t - s - 1]_2$. Therefore, $\langle DA \rangle = A_1 \cup A_2 = [t - s - k + 1, t - s]$.

When $s > t$, we have $A_1 = \{|a_{2i} - a_{2i-1}| \mid i \in [1, \lfloor (k+1)/2 \rfloor]\} = \{s - t + 2i - 2 \mid i \in [1, \lfloor (k+1)/2 \rfloor]\} = [s - t, s - t + 2\lfloor (k+1)/2 \rfloor - 2]_2$,

$A_2 = \{|a_{2i+1} - a_{2i}| \mid i \in [1, \lfloor k/2 \rfloor]\} = \{s - t + 2i - 1 \mid i \in [1, \lfloor k/2 \rfloor]\} = [s - t + 1, s - t + 2\lfloor k/2 \rfloor - 1]_2$. Therefore, $\langle DA \rangle = A_1 \cup A_2 = [s - t, s - t + k - 1]$. \square

Lemma 2.2. *Let $A=(a_1, a_2, \dots, a_{k+1})$, where*

$$a_i = \begin{cases} s - (i - 1)/2, & i \in [1, k + 1]_2 \\ t + (i - 2)/2, & i \in [2, k + 1]_2. \end{cases}$$

Then $\langle DA \rangle = [s - t - k + 1, s - t]$ when $s - t \geq k$ and $\langle DA \rangle = [t - s, t - s + k - 1]$ when $s < t$.

Proof. When $s - t \geq k$, we have $A_1 = \{|a_{2i} - a_{2i-1}| \mid i \in [1, \lfloor (k+1)/2 \rfloor]\} = \{s - t - 2i + 2 \mid i \in [1, \lfloor (k+1)/2 \rfloor]\} = [s - t - 2\lfloor (k+1)/2 \rfloor + 2, s - t]_2$,

$A_2 = \{|a_{2i+1} - a_{2i}| \mid i \in [1, \lfloor k/2 \rfloor]\} = \{s - t - 2i + 1 \mid i \in [1, \lfloor k/2 \rfloor]\} = [s - t - 2\lfloor k/2 \rfloor + 1, s - t - 1]_2$. Therefore, $\langle DA \rangle = A_1 \cup A_2 = [s - t - k + 1, s - t]$.

When $s < t$, we have $A_1 = \{|a_{2i} - a_{2i-1}| \mid i \in [1, \lfloor (k+1)/2 \rfloor]\} = \{t - s + 2i - 2 \mid i \in [1, \lfloor (k+1)/2 \rfloor]\} = [t - s, t - s + 2\lfloor (k+1)/2 \rfloor - 2]_2$,

$A_2 = \{|a_{2i+1} - a_{2i}| \mid i \in [1, \lfloor k/2 \rfloor]\} = \{t - s + 2i - 1 \mid i \in [1, \lfloor k/2 \rfloor]\} = [t - s + 1, t - s + 2\lfloor k/2 \rfloor - 1]_2$. Therefore, $\langle DA \rangle = A_1 \cup A_2 = [t - s, t - s + k - 1]$. \square

Let $A_i = (a_{i,1}, a_{i,2}, \dots, a_{i,2m-1})$, $i \in [1, 2n]$ and

$$(1) \quad A = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_{2n} \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,2m-1} \\ a_{2,1} & a_{2,2} & \dots & a_{2,2m-1} \\ & & \dots & \\ a_{2n,1} & a_{2n,2} & \dots & a_{2n,2m-1} \end{pmatrix}.$$

The difference matrix of the matrix A , denoted by DA , is a matrix whose row vectors are $DA_1, DA_2, \dots, DA_{2n}$. That is

$$(2) \quad DA = \begin{pmatrix} DA_1 \\ DA_2 \\ \vdots \\ DA_{2n} \end{pmatrix}.$$

Lemma 2.3. *In the matrix (1), let*

$$\begin{aligned} a_{2i-1,2j-1} &= 4m(n-i) + 4m - j + 1 \text{ if } i \in [1, n] \text{ and } j \in [1, m], \\ a_{2i,2j-1} &= 4m(n-i) + 2m + j \text{ if } i \in [1, n] \text{ and } j \in [1, m], \\ a_{2i-1,2j} &= 4m(i-1) + j \text{ if } i \in [1, n] \text{ and } j \in [1, m-1], \\ a_{2i,2j} &= 4m(i-1) + 2m - j + 1 \text{ if } i \in [1, n] \text{ and } j \in [1, m-1]. \end{aligned}$$

Then the matrix A is an EPD matrix and $\{\langle DA \rangle, B\}$ is a partition of set $[1, 4mn]$, where $B = [2m+1, 4mn-2m+1]_{4m} \cup [4m, 4mn]_{4m} \cup [2m, 4mn-2m]_{4m} \cup [2m-1, 4mn-2m-1]_{4m}$.

Proof. It is easy to see that the matrix A and the matrix DA have $2n(2m-1)$ and $4n(m-1)$ elements, respectively. We have

$A_1 = \{a_{2i-1,2j-1}, a_{2i,2j-1} \mid i \in [1, n], j \in [1, m]\} = \bigcup_{i=1}^n [4m(n-i) + 2m + 1, 4m(n-i) + 4m]$, and

$A_2 = \{a_{2i-1,2j}, a_{2i,2j} \mid i \in [1, n], j \in [1, m-1]\} = \bigcup_{i=1}^n 4m(i-1) + ([1, 2m] \setminus \{m, m+1\})$. Therefore, $\langle A \rangle = [1, 4mn] \setminus (\bigcup_{i=1}^n \{4m(i-1) + m, 4m(i-1) + m + 1\})$, and there are $2n(2m-1)$ elements in $\langle A \rangle$. This implies the matrix A is an EPD matrix.

Let $s = 4mn - 4mi + 2m + 1$, $t = 4mi - 2m$, $k = 2m - 2$. When $1 \leq i \leq \lfloor (n+1)/2 \rfloor$, there is $s > t$. By Lemma 2.1, we obtain $\langle DA_{2i} \rangle = [s - t, s - t + k - 1] = [4mn - 8mi + 4m + 1, 4mn - 8mi + 6m - 2]$.

When $\lfloor (n+1)/2 \rfloor < i \leq n$, there is $s < t$. By Lemma 2.1, we obtain $\langle DA_{2i} \rangle = [t - s - k + 1, t - s] = [8mi - 4mn - 6m + 2, 8mi - 4mn - 4m - 1]$.

Let $s=4mn-4mi+4m$, $t=4mi-4m+1$, $k=2m-2$. When $1 \leq i < \lceil (n+2)/2 \rceil$, there is $s > t$. By Lemma 2.2, we obtain

$$\langle DA_{2i-1} \rangle = [s-t-k+1, s-t] = [4mn-8mi+6m+2, 4mn-8mi+8m-1].$$

When $\lceil (n+2)/2 \rceil \leq i \leq n$, there is $s < t$. By Lemma 2.2, we obtain

$$\langle DA_{2i-1} \rangle = [t-s, t-s+k-1] = [8mi-4mn-8m+1, 8mi-4mn-6m-2].$$

When n is odd, we have

$$\begin{aligned} \cup_{i=1}^{2n} \langle DA_i \rangle &= (\cup_{i=1}^{\lceil (n+1)/2 \rceil} [4mn-8mi+4m+1, 4mn-8mi+6m-2]) \\ &\cup (\cup_{i=\lceil (n+3)/2 \rceil}^n [8mi-4mn-6m+2, 8mi-4mn-4m-1]) \\ &\cup (\cup_{i=1}^{\lceil (n+1)/2 \rceil} [4mn-8mi+6m+2, 4mn-8mi+8m-1]) \\ &\cup (\cup_{i=\lceil (n+3)/2 \rceil}^n [8mi-4mn-8m+1, 8mi-4mn-6m-2]) \\ &= (\cup_{i=0}^{\lceil (n-1)/2 \rceil} (8mi + [1, 2m-2])) \cup (\cup_{i=0}^{\lceil (n-3)/2 \rceil} (8mi + [6m+2, 8m-1])) \\ &\cup (\cup_{i=0}^{\lceil (n-1)/2 \rceil} (8mi + [2m+2, 4m-1])) \cup (\cup_{i=0}^{\lceil (n-3)/2 \rceil} (8mi + [4m+1, 6m-2])) \\ &= [1, 4mn] \setminus B. \end{aligned}$$

When n is even, we have

$$\begin{aligned} \cup_{i=1}^{2n} \langle DA_i \rangle &= (\cup_{i=1}^{n/2} [4mn-8mi+4m+1, 4mn-8mi+6m-2]) \\ &\cup (\cup_{i=(n+2)/2}^n [8mi-4mn-6m+2, 8mi-4mn-4m-1]) \\ &\cup (\cup_{i=1}^{n/2} [4mn-8mi+6m+2, 4mn-8mi+8m-1]) \\ &\cup (\cup_{i=(n+2)/2}^n [8mi-4mn-8m+1, 8mi-4mn-6m-2]) \\ &= (\cup_{i=0}^{\lceil (n-2)/2 \rceil} (8mi + [4m+1, 6m-2])) \cup (\cup_{i=0}^{\lceil (n-2)/2 \rceil} (8mi + [2m+2, 4m-1])) \\ &\cup (\cup_{i=0}^{\lceil (n-2)/2 \rceil} (8mi + [6m+2, 8m-1])) \cup (\cup_{i=0}^{\lceil (n-2)/2 \rceil} (8mi + [1, 2m-2])) \\ &= [1, 4mn] \setminus B. \end{aligned}$$

Therefore, we have $B \cup (\cup_{i=1}^{2n} \langle DA_i \rangle) = [1, 4mn]$ and $B \cap (\cup_{i=1}^{2n} \langle DA_i \rangle) = \emptyset$. This completes the proof. \square

For an integer i satisfying $1 \leq i \leq b$, let P_a^i be i th path of $P_{a,b}$, and the successive vertices of P_a^i be $x, x_{i,1}, x_{i,2}, \dots, x_{i,a-1}, y$, where x and y are two common end vertices of b internally disjoint paths each of length a . We use the symbol P_{a-2}^i to denote the subpath of P_a^i with consecutive vertices $x_{i,1}, x_{i,2}, \dots, x_{i,a-1}$.

EXAMPLE 2.4. Figure 1 displays a graph $P_{m,4}$ and Figure 2 shows a graceful labeling of $P_{4,2}$.

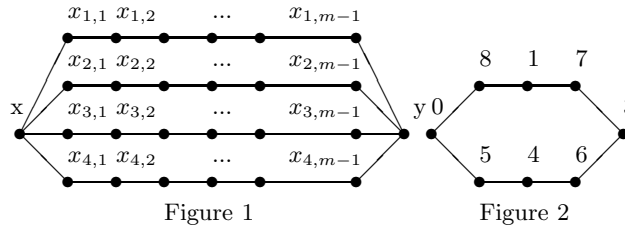


Figure 1

Figure 2

Theorem 2.5. *Graph $P_{2m,2n}$ is graceful for any positive integers m and n .*

Proof. It is easy to see that the graph $P_{2m,2n}$ has $4mn$ edges and $2n(2m-1)+2$ vertices. Let $a_{i,j}, i \in [1, 2n], j \in [1, 2m-1]$ be the same as in Lemma 2.3. We define a function g on the point set $V(P_{2m,2n})$ as follows: $g(x)=0, g(y)=m+1, g(x_{i,j})=a_{i,j}$ if $i \in [1, 2n]$ and $j \in [1, 2m-1]$.

From Lemma 2.3, we obtain that the g is an injection from $V(P_{2m,2n})$ to $[0, 4mn]$. By the above definition, we obtain the following results.

$$\begin{aligned} E_1 &= \{g^*(xx_{2i,1}) \mid i \in [1, n]\} = \{4m(n-i) + 2m + 1 \mid i \in [1, n]\} \\ &= [2m + 1, 4mn - 2m + 1]_{4m}, \\ E_2 &= \{g^*(xx_{2i-1,1}) \mid i \in [1, n]\} = \{4m(n-i+1) \mid i \in [1, n]\} = [4m, 4mn]_{4m}, \\ E_3 &= \{g^*(yx_{2i-1,2m-1}) \mid i \in [1, n]\} = \{4m(n-i) + 2m \mid i \in [1, n]\} \\ &= [2m, 4mn - 2m]_{4m}, \\ E_4 &= \{g^*(yx_{2i,2m-1}) \mid i \in [1, n]\} = \{4m(n-i) + 2m - 1 \mid i \in [1, n]\} \\ &= [2m - 1, 4mn - 2m - 1]_{4m}. \end{aligned}$$

$g^*(E(P_{2m-2}^i)) = \langle DA_i \rangle$ for $i \in [1, 2n]$. Let $B = E_1 \cup E_2 \cup E_3 \cup E_4$. Applying Lemma 2.3, we obtain $g^*(E(P_{2m,2n})) = \langle DA \rangle \cup B = [1, 4mn]$. This implies that g^* is a bijection from $E(P_{2m,2n})$ onto $[1, 4mn]$. \square

REFERENCES

1. G. S. BLOOM, S. W. GOLOMB: *Applications of numbered undirected graphs*. Proceedings of the Institute of Electrical and Electronics Engineering **65** (1977), 562–570.
2. L. DAI, Z. WANG, Z. XIE: *The gracefulness of $P_{3,4m}$* . Journal of National University of Defense Technology **5** (2005), 126–128.
3. J.A. GALLIAN: *A dynamic survey of graph labeling*. The Electronic Journal of Combinatorics (2009) DS6 1–219.
4. K. KATHIRESAN: *Two classes of graceful graph*. Ars Combinatoria **55** (2000), 129–132.
5. A. ROSA: *On certain labeling of the vertices of a graph*. Theory of Graphs (International Symposium, Rome, July 1966), Gordon and Breach, N. Y. and Dunod Paris (1967), 349–355.
6. Q. RONG, Y. YANG: *A class of graceful graphs*. Journal of Guangxi Teachers College **3**(2002), 1–4.
7. C. SEKAR: *"Studies in graph theory."* Ph. D. Thesis, Madurai Kamaraj University, 2002.
8. Y. YANG, Q. RONG, X. XU: *A class of graceful graphs*. Journal of Mathematical Research and Exposition **3** (2004), 520–524.

Department of Mathematics,
Hebei Normal University,
Shijiazhuang 050016,
P. R. China
E-mail: zhiheliang@163.com

(Received May 8, 2009)
(Revised November 2, 2009)