

## LOWER BOUNDS ON THE ROMAN AND INDEPENDENT ROMAN DOMINATION NUMBERS

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A Roman dominating function (RDF) on a graph  $G$  is a function  $f : V(G) \rightarrow \{0, 1, 2\}$  satisfying the condition that every vertex  $u$  with  $f(u) = 0$  is adjacent to at least one vertex  $v$  of  $G$  for which  $f(v) = 2$ . The weight of a Roman dominating function is the sum  $f(V) = \sum_{v \in V} f(v)$ , and the minimum weight of a Roman dominating function  $f$  is the Roman domination number  $\gamma_R(G)$ . An RDF  $f$  is called an independent Roman dominating function (IRDF) if the set of vertices assigned positive values under  $f$  is independent. The independent Roman domination number  $i_R(G)$  is the minimum weight of an IRDF on  $G$ . We show that for every nontrivial connected graph  $G$  with maximum degree  $\Delta$ ,  $\gamma_R(G) \geq \frac{\Delta+1}{\Delta} \gamma(G)$  and  $i_R(G) \geq i(G) + \gamma(G)/\Delta$ , where  $\gamma(G)$  and  $i(G)$  are, respectively, the domination and independent domination numbers of  $G$ . Moreover, we characterize the connected graphs attaining each lower bound. We give an additional lower bound for  $\gamma_R(G)$  and compare our two new bounds on  $\gamma_R(G)$  with some known lower bounds.

### 1. INTRODUCTION

Inspired by the strategies for defending the Roman Empire presented in the work of REVELLE and ROSING [15] and STEWART [16], COCKAYNE et al. [5] introduced Roman domination in 2004. This introductory paper sparked much interest in Roman domination and to date around 100 papers have been published on the topic. We present new lower bounds on the Roman domination and independent Roman domination numbers of graphs, and we characterize the graphs attaining two of these bounds. We first give some terminology.

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Let  $G = (V, E)$  be a graph having order  $n = |V|$  vertices. The *open neighborhood* of a vertex  $v \in V$  is the set  $N(v) = \{u \mid uv \in E\}$ . Vertices  $u \in N(v)$  are called the *neighbors* of  $v$ . The *degree* of a vertex  $u \in V$  is  $\deg(u) = |N(u)|$ . A vertex with exactly one neighbor is called a *leaf*. The *closed neighborhood* of a vertex  $v$  is the set  $N[v] = N(v) \cup \{v\}$ . For a graph  $G$ , we denote by  $\gamma(G)$  the domination number,  $i(G)$  the independent domination number, and  $\Delta(G)$  the maximum degree of  $G$ .

For a graph  $G$ , let  $f : V(G) \rightarrow \{0, 1, 2\}$  be a function, and let  $(V_0, V_1, V_2)$  be the ordered partition of  $V = V(G)$  defined by  $f$ , where  $V_i = \{v \in V : f(v) = i\}$  for  $i = 0, 1, 2$ . There is a 1-1 correspondence between the functions  $f : V \rightarrow \{0, 1, 2\}$  and the ordered partitions  $(V_0, V_1, V_2)$  of  $V$ , so we will write  $f = (V_0, V_1, V_2)$ .

A *Roman dominating function*, abbreviated RDF, on a graph  $G$  is a function  $f : V \rightarrow \{0, 1, 2\}$  satisfying the condition that every vertex  $u$  with  $f(u) = 0$  is adjacent to at least one vertex  $v$  of  $G$  for which  $f(v) = 2$ . The *weight* of an RDF is  $w(f) = f(V) = \sum_{u \in V} f(u)$ , and the minimum weight  $w(f)$  over all such functions

is the *Roman domination number*  $\gamma_R(G)$ . A function  $f = (V_0, V_1, V_2)$  is called a  $\gamma_R(G)$ -*function* if it is an RDF on  $G$  and  $f(V) = \gamma_R(G)$ . It is shown in [5] that for every graph  $G$ ,  $\gamma_R(G) \geq \gamma(G)$  with equality if and only if  $G$  is an empty graph.

COCKAYNE et al. [5] introduced the concept of independent Roman domination in graphs. An RDF  $f = (V_0, V_1, V_2)$  is called an *independent Roman dominating function*, abbreviated IRDF, if the set  $V_1 \cup V_2$  is an independent set. The *independent Roman domination number*  $i_R(G)$  is the minimum weight of an IRDF on  $G$  and an IRDF with weight  $i_R(G)$  is called an  $i_R(G)$ -*function*. Since for every IRDF,  $V_1 \cup V_2$  is an independent dominating set and  $i_R(G) = |V_1| + 2|V_2|$ , it follows that  $i(G) \leq |V_1| + |V_2| \leq i_R(G)$ . As mentioned by EBRAHIMI et al. in [6], the only graphs  $G$  such that  $i_R(G) = i(G)$  are the empty graphs.

Here we present a new lower bound on each of  $\gamma_R(G)$  and  $i_R(G)$  for all graphs  $G$  without isolated vertices. The lower bound on the Roman domination number is in terms of the domination number and maximum degree, while the lower bound on the independent Roman domination is in terms of the domination and independent domination numbers and the maximum degree. Moreover, we characterize the graphs attaining each lower bound. We also give a lower bound on  $\gamma_R(G)$  in terms of the vertex-edge domination number. Finally, we compare our two new lower bounds on the Roman domination number and existing lower bounds.

## 2. LOWER BOUNDS AND CHARACTERIZATIONS

To aid in our characterizations, we introduce a few additional definitions. A set  $S$  of vertices in a graph  $G$  is a *packing* if the vertices in  $S$  are pairwise at distance at least 3 apart in  $G$ , or equivalently, if for every vertex  $v \in V$ ,  $|N[v] \cap S| \leq 1$ . As defined in BANGE et al. [1], a dominating set  $S$  for which  $|N[v] \cap S| = 1$  for all  $v \in V$  is an *efficient dominating set*. Thus, a set  $S$  is an efficient dominating set if  $S$  is both a dominating set and a packing in  $G$ . Not every graph has an efficient dominating set, for example, the cycle  $C_5$  does not. However, as shown in [1], if a

graph  $G$  has an efficient dominating set  $S$ , then  $|S| = \gamma(G)$ , that is, every efficient dominating set is a minimum dominating set.

We need to recall the well-known lower bound on the domination number due to WALIKAR et al. [17] stating that for every graph  $G$ ,  $\gamma(G) \geq n/(1 + \Delta(G))$ . One can easily show that the only connected graphs attaining this lower bound are those graphs having efficient dominating sets  $S$  in which every vertex of  $S$  has maximum degree. Let us denote such a class of extremal connected graphs by  $\mathcal{G}$ . Note that a path  $P_4$  has a unique efficient dominating set (namely, the two leaves), but the path  $P_4$  does not belong to  $\mathcal{G}$ . The corona  $H^*$  of a graph  $H$  is the graph obtained from  $H$  by appending a vertex of degree 1 to each vertex of  $H$ . Let  $\mathcal{F}$  be the family of graphs  $G$  such that  $G$  is the cycle  $C_4$  or the corona  $H^*$ , where  $H$  is a connected graph in  $\mathcal{G}$ .

We will use the following result established by PAYAN and XUONG [12] (see also FINK, JACOBSON, KINCH and ROBERTS [7]).

**Theorem 1** (PAYAN, XUONG [12]). *Let  $G$  be a graph of even order  $n$  without isolated vertices. Then  $\gamma(G) = n/2$  if and only if each component of  $G$  is either a cycle  $C_4$  or the corona of a connected graph.*

**Theorem 2.** *Let  $G$  be a nontrivial, connected graph with maximum degree  $\Delta$ . Then  $\gamma_R(G) \geq \frac{\Delta+1}{\Delta} \gamma(G)$ , with equality if and only if  $G \in \mathcal{F}$ .*

**Proof.** Among all  $\gamma_R(G)$ -functions, let  $f = (V_0, V_1, V_2)$  be  $\gamma_R(G)$ -function on  $G$  with  $|V_2|$  as large as possible. Note that if a vertex  $v$  in  $V_2$  has no private neighbor with respect to  $V_2$  in  $V_0$ , that is, a vertex  $w \in V_0$  with  $N(w) \cap V_2 = \{v\}$ , then reassigning  $v$  a 1 instead of a 2 produces an RDF with smaller weight than  $\gamma_R(G)$ , a contradiction. Hence, every vertex in  $V_2$  has a private neighbor with respect to  $V_2$  in  $V_0$ . Moreover, no vertex  $v$  in  $V_1$  has a neighbor in  $V_2$ , else  $v$  can be reassigned 0, again producing an RDF with smaller weight than  $\gamma_R(G)$ , a contradiction. By our choice of  $f$ ,  $V_1$  is independent. To see this, suppose that there are adjacent vertices  $u$  and  $v$  in  $V_1$ . But then  $f' = (V'_0 = V_0 \cup \{v\}, V'_1 = V_1 - \{u, v\}, V'_2 = V_2 \cup \{u\})$ , where  $f'(u) = 2$ ,  $f'(v) = 0$ , and  $f'(x) = f(x)$  for all  $x \in V - \{u, v\}$ , is a  $\gamma_R(G)$ -function having  $|V'_2| > |V_2|$ , contradicting our choice of  $f$ .

Since  $G$  is nontrivial, it follows that  $V_2 \neq \emptyset$ . Hence,  $V_0 \neq \emptyset$ , and clearly,  $\Delta|V_2| \geq |V_0|$ , since every vertex in  $V_0$  must have a neighbor in  $V_2$ . Note that since  $G$  is connected, every vertex in  $V_1$  has a neighbor in  $V_0$ , implying that  $V_0$  is a dominating set of  $G$ . Since each of the sets  $V_0$  and  $V_1 \cup V_2$  dominates  $G$ , we obtain that  $\gamma(G) \leq |V_1| + |V_2|$  and  $\gamma(G)/\Delta \leq |V_0|/\Delta \leq |V_2|$ . Therefore,  $\gamma(G) + \gamma(G)/\Delta \leq |V_1| + |V_2| + |V_2| = \gamma_R(G)$ , and the bound follows.

For the characterization, assume that  $\gamma_R(G) = \frac{\Delta+1}{\Delta} \gamma(G)$ . Choosing  $f$  as before, we have that  $V_1$  is an independent set and no vertex in  $V_1$  has a neighbor in  $V_2$ . We deduce from our previous argument that  $\Delta|V_2| = |V_0|$  and  $V_2$  is a packing set for which each vertex of  $V_2$  has degree  $\Delta$ . Moreover,  $\gamma(G) + \gamma(G)/\Delta = |V_1| + |V_2| + |V_2| = \gamma_R(G)$ , so  $\gamma(G) = |V_1| + |V_2| = |V_2|/\Delta = |V_0|$ . Hence,  $|V_0| = |V_1 \cup V_2| = |V|/2$ . By Theorem 1,  $G$  is a cycle  $C_4$  or the corona  $H^*$  of a connected graph  $H$ . Suppose

that  $G$  is the corona  $H^*$ . First observe that every vertex  $x$  of  $V_1$  is a leaf, for otherwise the unique leaf neighbor of  $x$ , say  $y$ , belongs to  $V_0$  and  $y$  has no neighbor in  $V_2$ , contradicting the fact that  $f$  is an RDF. Since  $V_2$  is a packing, it follows that  $V_2$  is an efficient dominating set of the connected subgraph induced by  $V(H^*) - V_1$ , implying that  $V_2$  is also an efficient dominating set of  $H$ . Therefore,  $H$  belongs to  $\mathcal{G}$ , and so,  $G \in \mathcal{F}$ .

Conversely, suppose that  $G \in \mathcal{F}$ . Clearly, if  $G = C_4$ , then  $\gamma_R(G) = 3 = \frac{\Delta+1}{\Delta} \gamma(G)$ . Now assume that  $G = H^*$ , and let  $D$  be an efficient dominating set of  $H$ . Then  $|D| = |V(H)| / (\Delta(H) + 1)$ . Let  $h$  be an RDF on  $G$  defined by assigning 2 to every vertex of  $D$ , 1 to every leaf of  $G$  not adjacent to a vertex of  $D$ , and 0 to every vertex with a neighbor in  $D$ . Then

$$\gamma_R(G) \leq w(h) = 2|D| + (|V(H)| - |D|) = |V(H)| + |D| = \frac{\Delta(H)+2}{\Delta(H)+1} |V(H)|.$$

Using the facts that  $\Delta(G) = \Delta(H) + 1$  and  $\gamma(G) = |V(H)|$ , we obtain  $\gamma_R(G) \leq \frac{\Delta(G)+1}{\Delta(G)} \gamma(G)$ . The equality follows from our lower bound.  $\square$

To characterize the graphs attaining our second bound, we will make use of the following result due to RAUTENBACH and VOLKMANN [14].

**Theorem 3** (RAUTENBACH, VOLKMANN [14]). *Let  $G$  be a graph of order  $n$  with no isolated vertices. Then  $\gamma(G) + i(G) = n$  if and only if every component of  $G$  is either a cycle  $C_4$  or the corona of some connected graph.*

**Theorem 4.** *Let  $G$  be a nontrivial connected graph with maximum degree  $\Delta$ . Then  $i_R(G) \geq i(G) + \gamma(G)/\Delta$ , with equality if and only if  $G \in \mathcal{F}$ .*

**Proof.** Let  $f = (V_0, V_1, V_2)$  be an  $i_R(G)$ -function. Since  $V_1 \cup V_2$  is an independent set of  $G$  and  $G$  is a nontrivial connected graph, it follows that  $V_0 \neq \emptyset$  and every vertex in  $V_1 \cup V_2$  has a neighbor in  $V_0$ , that is,  $V_0$  is a dominating set of  $G$ . Thus,  $\gamma(G) \leq |V_0|$ . Further since every vertex in  $V_0$  has a neighbor in  $V_2$ ,  $\Delta |V_2| \geq |V_0|$  and  $V_1 \cup V_2$  is an independent dominating set of  $G$ . Hence,  $i(G) \leq |V_1| + |V_2|$  and  $\gamma(G)/\Delta \leq |V_0|/\Delta \leq |V_2|$ . Therefore,  $i(G) + \gamma(G)/\Delta \leq |V_1| + |V_2| + |V_2| = i_R(G)$ , and the bound follows.

Now assume that  $i_R(G) = i(G) + \gamma(G)/\Delta$ . Then we must have  $i(G) = |V_1| + |V_2|$  and  $\gamma(G)/\Delta = |V_0|/\Delta = |V_2|$ . Hence,  $V_1 \cup V_2$  is a minimum independent dominating set and  $V_0$  is a minimum dominating set of  $G$ . It follows that  $\gamma(G) + i(G) = |V_1| + |V_2| + |V_0| = n$ . By Theorem 3,  $G = C_4$  or  $G$  is the corona of a connected graph  $H$ . Assume that  $G = H^*$ . Observe that since  $\Delta |V_2| = |V_0|$ ,  $V_2$  is a packing set in which every vertex has degree  $\Delta$ . On the other hand, if some vertex of  $V_1$  has degree at least two, then its leaf neighbor belongs to  $V_0$ . But then such a leaf has no neighbor in  $V_2$ , a contradiction. We deduce that every vertex of  $V_1$  is a leaf, and so,  $V_2$  is an efficient dominating set of  $G[V(H^*) - V_1]$ , and hence of the graph  $H$ . Therefore,  $H$  belongs to  $\mathcal{G}$ , and so,  $G \in \mathcal{F}$ .

Conversely, suppose that  $G$  is a cycle  $C_4$  or the corona of a connected graph  $H$ , where  $H \in \mathcal{G}$ . Clearly, if  $G = C_4$ , then  $i_R(G) = 3 = i(G) + \gamma(G)/\Delta$ . Assume now that  $G = H^*$ , and let  $D$  be an efficient dominating set of  $H$ . Then  $|D| = |V(H)|/(\Delta(H) + 1)$ . Let  $h$  be an IRDF on  $G$  defined by assigning 2 to every vertex of  $D$ , 1 to every leaf of  $G$  not adjacent to a vertex of  $D$ , and 0 to every vertex adjacent to  $D$ . Then

$$i_R(G) \leq w(h) = 2|D| + (|V(H)| - |D|) = |V(H)| + |D|.$$

Using the facts that  $\gamma(G) = i(G) = |V(H)|$ ,  $\Delta(G) = \Delta(H) + 1$  and  $|D| = |V(H)|/(\Delta(H) + 1)$ , we obtain  $i_R(G) \leq i(G) + \gamma(G)/\Delta(G)$ . The equality follows from the lower bound.  $\square$

We conclude this section with another lower bound on the Roman domination number. A vertex  $v$  of a graph  $G$  *ve-dominates* every edge  $uv$  incident to  $v$ , as well as every edge adjacent to these incident edges, that is, a vertex  $v$  *ve-dominates* every edge incident to a vertex in  $N[v]$ . A set  $S \subseteq V$  is a *vertex-edge dominating set* (or simply, a *ve-dominating set*) if for every edge  $e \in E$ , there exists a vertex  $v \in S$  such that  $v$  *ve-dominates*  $e$ . The minimum cardinality of a *ve-dominating set* of  $G$  is called the *vertex-edge domination number*  $\gamma_{ve}(G)$  of  $G$ . Vertex-edge domination was introduced by PETERS [13] in his 1986 PhD thesis and studied further in [4, 9]. Trivially,  $\gamma_{ve}(G) \leq \gamma(G)$  holds for every graph  $G$ .

It follows from the definition of Roman domination that for any graph  $G$ ,

$$\gamma_{ve}(G) \leq \gamma(G) \leq \gamma_R(G).$$

Theorem 2 improves this upper bound for  $\gamma(G)$ :

$$\gamma_{ve}(G) \leq \gamma(G) \leq \frac{\Delta}{\Delta + 1} \gamma_R(G).$$

From Theorem 2 we also have:

$$\gamma_{ve}(G) \leq \frac{\Delta + 1}{\Delta} \gamma(G) \leq \gamma_R(G).$$

But this lower bound for  $\gamma_R(G)$  is not as good as the following, which is observed in COCKAYNE et al. [5] for connected graphs  $G$ :

$$\gamma_{ve}(G) \leq \gamma(G) + 1 \leq \gamma_R(G).$$

But from the proof of Theorem 2, we obtain the following lower bound on the Roman domination number.

**Theorem 5.** *For any connected graph of order  $n \geq 3$ ,  $2\gamma_{ve}(G) \leq \gamma_R(G)$ .*

**Proof.** As in the proof of Theorem 2, among all  $\gamma_R(G)$ -functions, let  $f = (V_0, V_1, V_2)$  be  $\gamma_R(G)$ -function on  $G$  with  $|V_2|$  as large as possible, or equivalently with  $|V_1|$  as small as possible. Recall that by our choice of  $f$ ,  $V_1$  is independent.

Let  $uv$  be an edge of  $G$ . We know that the set  $V_2$  dominates every vertex in  $V_0$ , since  $f$  is a Roman dominating function. Thus, if  $u$  or  $v$  is in  $V_0 \cup V_2$ , then  $V_2$   $ve$ -dominates  $uv$ . Since  $V_1$  is independent, at least one of  $u$  and  $v$  is in  $V_0 \cup V_2$ . It follows therefore that  $V_2$  is a  $ve$ -dominating set of  $G$ . Hence,  $\gamma_{ve}(G) \leq |V_2| \leq \frac{\gamma_R(G)}{2}$ .

### 3. COMPARISONS OF LOWER BOUNDS

We conclude by comparing our new lower bounds on the Roman domination number with three previously known lower bounds. We show that in fact these five bounds are pairwise incomparable. Let  $\gamma_t(G)$  denote the minimum cardinality of a total dominating set of  $G$ , that is, a dominating set whose induced subgraph has no isolated vertices. For a graph  $G$  with order  $n$ , no isolated vertices and maximum degree  $\Delta$ , each of the following is a lower bound on  $\gamma_R(G)$ :

- (i)  $\gamma_t(G)$  [8],
- (ii)  $\frac{2n}{\Delta+1}$  [5],
- (iii)  $\frac{\Delta+1}{\Delta}\gamma(G)$  (Theorem 2),
- (iv)  $\left\lceil \frac{\text{diam}(G)+2}{2} \right\rceil$  [11],
- (v)  $2\gamma_{ve}(G)$  (Theorem 5).

It is worth mentioning that there is a known lower bound due BERMUDO, FERNAU, and SIGARRETA [3] that is not on this list. In [3], the authors prove a very interesting relationship between Roman domination and the differential of a graph. For a set  $S$ , let  $B(S)$  be the set of vertices in  $V - S$  that have a neighbor in the set  $S$ . The *differential* of a set  $S$  is defined in [10] as  $\partial(S) = |B(S)| - |S|$ , and the maximum value of  $\partial(S)$  for any subset  $S$  of  $V$  is the *differential* of  $G$ , denoted  $\partial(G)$ . In [3], they prove the unexpected result that  $\gamma_R(G) = n - \partial(G)$ . Hence, some bounds on  $\gamma_R(G)$  from [3] came from known bounds on  $\partial(G)$ . In particular, they prove the lower bound of  $n - \gamma(G)(\Delta - 1)$  from a bound on the differential first given in [2]. Since we can show that  $n - \gamma(G)(\Delta - 1)$  is at most  $\frac{2n}{\Delta+1}$  for any graph  $G$ , we omit this bound from our comparison.

To see that no two of these bounds are comparable, we consider the following families of graphs: the corona  $P_{3k}^*$ ; the path  $P_{12k}$ ; the graph  $F_k$  obtained from  $k$  ( $k \geq 3$ ) cycles  $C_5$  and a path  $P_k$  by identifying one vertex of each cycle with a vertex of the path so that no two cycles have a common vertex, and then by subdividing each edge of the path exactly twice; and the graph (broom)  $B_k$  for  $k \geq 3$  formed from the star  $K_{1,k}$  by subdividing one of its edges exactly  $12k$  times. The following table gives values for the bounds for each of these families.

	$\gamma_t(G)$	$\frac{2n}{\Delta+1}$	$\frac{\Delta+1}{\Delta}\gamma(G)$	$\left\lceil \frac{\text{diam}(G)+2}{2} \right\rceil$	$2\gamma_{ve}(G)$
$P_{3k}^*$	$3k$	$3k$	$4k$	$\frac{3k+3}{2}$	$2k$
$P_{12k}$	$6k$	$8k$	$6k$	$6k+1$	$6k$
$F_k$	$3k$	$\frac{14k-4}{5}$	$\frac{5k}{2}$	$\frac{3k+3}{2}$	$4k$
$B_{12k}$	$6k+2$	$\frac{26k+2}{k+1}$	$4k+6$	$6k+2$	$6k+2$

From the values in the table, we deduce that the bounds are pairwise incomparable. In conclusion, we note that although our new lower bounds on the Roman domination number are incomparable to the other lower bounds, they can be arbitrarily larger than each of them.

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