

LOWER BOUNDS ON THE ROMAN AND INDEPENDENT ROMAN DOMINATION NUMBERS

Mustapha Chellali, Teresa W. Haynes, Stephen T. Hedetniemi

A Roman dominating function (RDF) on a graph G is a function $f : V(G) \rightarrow \{0, 1, 2\}$ satisfying the condition that every vertex u with $f(u) = 0$ is adjacent to at least one vertex v of G for which $f(v) = 2$. The weight of a Roman dominating function is the sum $f(V) = \sum_{v \in V} f(v)$, and the minimum weight of a Roman dominating function f is the Roman domination number $\gamma_R(G)$. An RDF f is called an independent Roman dominating function (IRDF) if the set of vertices assigned positive values under f is independent. The independent Roman domination number $i_R(G)$ is the minimum weight of an IRDF on G . We show that for every nontrivial connected graph G with maximum degree Δ , $\gamma_R(G) \geq \frac{\Delta+1}{\Delta} \gamma(G)$ and $i_R(G) \geq i(G) + \gamma(G)/\Delta$, where $\gamma(G)$ and $i(G)$ are, respectively, the domination and independent domination numbers of G . Moreover, we characterize the connected graphs attaining each lower bound. We give an additional lower bound for $\gamma_R(G)$ and compare our two new bounds on $\gamma_R(G)$ with some known lower bounds.

1. INTRODUCTION

Inspired by the strategies for defending the Roman Empire presented in the work of REVELLE and ROSING [15] and STEWART [16], COCKAYNE et al. [5] introduced Roman domination in 2004. This introductory paper sparked much interest in Roman domination and to date around 100 papers have been published on the topic. We present new lower bounds on the Roman domination and independent Roman domination numbers of graphs, and we characterize the graphs attaining two of these bounds. We first give some terminology.

2010 Mathematics Subject Classification. 05C69.

Keywords and Phrases. Roman domination, independent domination, total domination, *ve*-domination

Let $G = (V, E)$ be a graph having order $n = |V|$ vertices. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u \mid uv \in E\}$. Vertices $u \in N(v)$ are called the *neighbors* of v . The *degree* of a vertex $u \in V$ is $\deg(u) = |N(u)|$. A vertex with exactly one neighbor is called a *leaf*. The *closed neighborhood* of a vertex v is the set $N[v] = N(v) \cup \{v\}$. For a graph G , we denote by $\gamma(G)$ the domination number, $i(G)$ the independent domination number, and $\Delta(G)$ the maximum degree of G .

For a graph G , let $f : V(G) \rightarrow \{0, 1, 2\}$ be a function, and let (V_0, V_1, V_2) be the ordered partition of $V = V(G)$ defined by f , where $V_i = \{v \in V : f(v) = i\}$ for $i = 0, 1, 2$. There is a 1-1 correspondence between the functions $f : V \rightarrow \{0, 1, 2\}$ and the ordered partitions (V_0, V_1, V_2) of V , so we will write $f = (V_0, V_1, V_2)$.

A *Roman dominating function*, abbreviated RDF, on a graph G is a function $f : V \rightarrow \{0, 1, 2\}$ satisfying the condition that every vertex u with $f(u) = 0$ is adjacent to at least one vertex v of G for which $f(v) = 2$. The *weight* of an RDF is $w(f) = f(V) = \sum_{u \in V} f(u)$, and the minimum weight $w(f)$ over all such functions

is the *Roman domination number* $\gamma_R(G)$. A function $f = (V_0, V_1, V_2)$ is called a $\gamma_R(G)$ -*function* if it is an RDF on G and $f(V) = \gamma_R(G)$. It is shown in [5] that for every graph G , $\gamma_R(G) \geq \gamma(G)$ with equality if and only if G is an empty graph.

COCKAYNE et al. [5] introduced the concept of independent Roman domination in graphs. An RDF $f = (V_0, V_1, V_2)$ is called an *independent Roman dominating function*, abbreviated IRDF, if the set $V_1 \cup V_2$ is an independent set. The *independent Roman domination number* $i_R(G)$ is the minimum weight of an IRDF on G and an IRDF with weight $i_R(G)$ is called an $i_R(G)$ -*function*. Since for every IRDF, $V_1 \cup V_2$ is an independent dominating set and $i_R(G) = |V_1| + 2|V_2|$, it follows that $i(G) \leq |V_1| + |V_2| \leq i_R(G)$. As mentioned by EBRAHIMI et al. in [6], the only graphs G such that $i_R(G) = i(G)$ are the empty graphs.

Here we present a new lower bound on each of $\gamma_R(G)$ and $i_R(G)$ for all graphs G without isolated vertices. The lower bound on the Roman domination number is in terms of the domination number and maximum degree, while the lower bound on the independent Roman domination is in terms of the domination and independent domination numbers and the maximum degree. Moreover, we characterize the graphs attaining each lower bound. We also give a lower bound on $\gamma_R(G)$ in terms of the vertex-edge domination number. Finally, we compare our two new lower bounds on the Roman domination number and existing lower bounds.

2. LOWER BOUNDS AND CHARACTERIZATIONS

To aid in our characterizations, we introduce a few additional definitions. A set S of vertices in a graph G is a *packing* if the vertices in S are pairwise at distance at least 3 apart in G , or equivalently, if for every vertex $v \in V$, $|N[v] \cap S| \leq 1$. As defined in BANGE et al. [1], a dominating set S for which $|N[v] \cap S| = 1$ for all $v \in V$ is an *efficient dominating set*. Thus, a set S is an efficient dominating set if S is both a dominating set and a packing in G . Not every graph has an efficient dominating set, for example, the cycle C_5 does not. However, as shown in [1], if a

graph G has an efficient dominating set S , then $|S| = \gamma(G)$, that is, every efficient dominating set is a minimum dominating set.

We need to recall the well-known lower bound on the domination number due to WALIKAR et al. [17] stating that for every graph G , $\gamma(G) \geq n/(1 + \Delta(G))$. One can easily show that the only connected graphs attaining this lower bound are those graphs having efficient dominating sets S in which every vertex of S has maximum degree. Let us denote such a class of extremal connected graphs by \mathcal{G} . Note that a path P_4 has a unique efficient dominating set (namely, the two leaves), but the path P_4 does not belong to \mathcal{G} . The corona H^* of a graph H is the graph obtained from H by appending a vertex of degree 1 to each vertex of H . Let \mathcal{F} be the family of graphs G such that G is the cycle C_4 or the corona H^* , where H is a connected graph in \mathcal{G} .

We will use the following result established by PAYAN and XUONG [12] (see also FINK, JACOBSON, KINCH and ROBERTS [7]).

Theorem 1 (PAYAN, XUONG [12]). *Let G be a graph of even order n without isolated vertices. Then $\gamma(G) = n/2$ if and only if each component of G is either a cycle C_4 or the corona of a connected graph.*

Theorem 2. *Let G be a nontrivial, connected graph with maximum degree Δ . Then $\gamma_R(G) \geq \frac{\Delta+1}{\Delta} \gamma(G)$, with equality if and only if $G \in \mathcal{F}$.*

Proof. Among all $\gamma_R(G)$ -functions, let $f = (V_0, V_1, V_2)$ be $\gamma_R(G)$ -function on G with $|V_2|$ as large as possible. Note that if a vertex v in V_2 has no private neighbor with respect to V_2 in V_0 , that is, a vertex $w \in V_0$ with $N(w) \cap V_2 = \{v\}$, then reassigning v a 1 instead of a 2 produces an RDF with smaller weight than $\gamma_R(G)$, a contradiction. Hence, every vertex in V_2 has a private neighbor with respect to V_2 in V_0 . Moreover, no vertex v in V_1 has a neighbor in V_2 , else v can be reassigned 0, again producing an RDF with smaller weight than $\gamma_R(G)$, a contradiction. By our choice of f , V_1 is independent. To see this, suppose that there are adjacent vertices u and v in V_1 . But then $f' = (V'_0 = V_0 \cup \{v\}, V'_1 = V_1 - \{u, v\}, V'_2 = V_2 \cup \{u\})$, where $f'(u) = 2$, $f'(v) = 0$, and $f'(x) = f(x)$ for all $x \in V - \{u, v\}$, is a $\gamma_R(G)$ -function having $|V'_2| > |V_2|$, contradicting our choice of f .

Since G is nontrivial, it follows that $V_2 \neq \emptyset$. Hence, $V_0 \neq \emptyset$, and clearly, $\Delta|V_2| \geq |V_0|$, since every vertex in V_0 must have a neighbor in V_2 . Note that since G is connected, every vertex in V_1 has a neighbor in V_0 , implying that V_0 is a dominating set of G . Since each of the sets V_0 and $V_1 \cup V_2$ dominates G , we obtain that $\gamma(G) \leq |V_1| + |V_2|$ and $\gamma(G)/\Delta \leq |V_0|/\Delta \leq |V_2|$. Therefore, $\gamma(G) + \gamma(G)/\Delta \leq |V_1| + |V_2| + |V_2| = \gamma_R(G)$, and the bound follows.

For the characterization, assume that $\gamma_R(G) = \frac{\Delta+1}{\Delta} \gamma(G)$. Choosing f as before, we have that V_1 is an independent set and no vertex in V_1 has a neighbor in V_2 . We deduce from our previous argument that $\Delta|V_2| = |V_0|$ and V_2 is a packing set for which each vertex of V_2 has degree Δ . Moreover, $\gamma(G) + \gamma(G)/\Delta = |V_1| + |V_2| + |V_2| = \gamma_R(G)$, so $\gamma(G) = |V_1| + |V_2| = |V_2|/\Delta = |V_0|$. Hence, $|V_0| = |V_1 \cup V_2| = |V|/2$. By Theorem 1, G is a cycle C_4 or the corona H^* of a connected graph H . Suppose

that G is the corona H^* . First observe that every vertex x of V_1 is a leaf, for otherwise the unique leaf neighbor of x , say y , belongs to V_0 and y has no neighbor in V_2 , contradicting the fact that f is an RDF. Since V_2 is a packing, it follows that V_2 is an efficient dominating set of the connected subgraph induced by $V(H^*) - V_1$, implying that V_2 is also an efficient dominating set of H . Therefore, H belongs to \mathcal{G} , and so, $G \in \mathcal{F}$.

Conversely, suppose that $G \in \mathcal{F}$. Clearly, if $G = C_4$, then $\gamma_R(G) = 3 = \frac{\Delta+1}{\Delta} \gamma(G)$. Now assume that $G = H^*$, and let D be an efficient dominating set of H . Then $|D| = |V(H)| / (\Delta(H) + 1)$. Let h be an RDF on G defined by assigning 2 to every vertex of D , 1 to every leaf of G not adjacent to a vertex of D , and 0 to every vertex with a neighbor in D . Then

$$\gamma_R(G) \leq w(h) = 2|D| + (|V(H)| - |D|) = |V(H)| + |D| = \frac{\Delta(H)+2}{\Delta(H)+1} |V(H)|.$$

Using the facts that $\Delta(G) = \Delta(H) + 1$ and $\gamma(G) = |V(H)|$, we obtain $\gamma_R(G) \leq \frac{\Delta(G)+1}{\Delta(G)} \gamma(G)$. The equality follows from our lower bound. \square

To characterize the graphs attaining our second bound, we will make use of the following result due to RAUTENBACH and VOLKMANN [14].

Theorem 3 (RAUTENBACH, VOLKMANN [14]). *Let G be a graph of order n with no isolated vertices. Then $\gamma(G) + i(G) = n$ if and only if every component of G is either a cycle C_4 or the corona of some connected graph.*

Theorem 4. *Let G be a nontrivial connected graph with maximum degree Δ . Then $i_R(G) \geq i(G) + \gamma(G)/\Delta$, with equality if and only if $G \in \mathcal{F}$.*

Proof. Let $f = (V_0, V_1, V_2)$ be an $i_R(G)$ -function. Since $V_1 \cup V_2$ is an independent set of G and G is a nontrivial connected graph, it follows that $V_0 \neq \emptyset$ and every vertex in $V_1 \cup V_2$ has a neighbor in V_0 , that is, V_0 is a dominating set of G . Thus, $\gamma(G) \leq |V_0|$. Further since every vertex in V_0 has a neighbor in V_2 , $\Delta |V_2| \geq |V_0|$ and $V_1 \cup V_2$ is an independent dominating set of G . Hence, $i(G) \leq |V_1| + |V_2|$ and $\gamma(G)/\Delta \leq |V_0|/\Delta \leq |V_2|$. Therefore, $i(G) + \gamma(G)/\Delta \leq |V_1| + |V_2| + |V_2| = i_R(G)$, and the bound follows.

Now assume that $i_R(G) = i(G) + \gamma(G)/\Delta$. Then we must have $i(G) = |V_1| + |V_2|$ and $\gamma(G)/\Delta = |V_0|/\Delta = |V_2|$. Hence, $V_1 \cup V_2$ is a minimum independent dominating set and V_0 is a minimum dominating set of G . It follows that $\gamma(G) + i(G) = |V_1| + |V_2| + |V_0| = n$. By Theorem 3, $G = C_4$ or G is the corona of a connected graph H . Assume that $G = H^*$. Observe that since $\Delta |V_2| = |V_0|$, V_2 is a packing set in which every vertex has degree Δ . On the other hand, if some vertex of V_1 has degree at least two, then its leaf neighbor belongs to V_0 . But then such a leaf has no neighbor in V_2 , a contradiction. We deduce that every vertex of V_1 is a leaf, and so, V_2 is an efficient dominating set of $G[V(H^*) - V_1]$, and hence of the graph H . Therefore, H belongs to \mathcal{G} , and so, $G \in \mathcal{F}$.

Conversely, suppose that G is a cycle C_4 or the corona of a connected graph H , where $H \in \mathcal{G}$. Clearly, if $G = C_4$, then $i_R(G) = 3 = i(G) + \gamma(G)/\Delta$. Assume now that $G = H^*$, and let D be an efficient dominating set of H . Then $|D| = |V(H)|/(\Delta(H) + 1)$. Let h be an IRDF on G defined by assigning 2 to every vertex of D , 1 to every leaf of G not adjacent to a vertex of D , and 0 to every vertex adjacent to D . Then

$$i_R(G) \leq w(h) = 2|D| + (|V(H)| - |D|) = |V(H)| + |D|.$$

Using the facts that $\gamma(G) = i(G) = |V(H)|$, $\Delta(G) = \Delta(H) + 1$ and $|D| = |V(H)|/(\Delta(H) + 1)$, we obtain $i_R(G) \leq i(G) + \gamma(G)/\Delta(G)$. The equality follows from the lower bound. \square

We conclude this section with another lower bound on the Roman domination number. A vertex v of a graph G *ve-dominates* every edge uv incident to v , as well as every edge adjacent to these incident edges, that is, a vertex v *ve-dominates* every edge incident to a vertex in $N[v]$. A set $S \subseteq V$ is a *vertex-edge dominating set* (or simply, a *ve-dominating set*) if for every edge $e \in E$, there exists a vertex $v \in S$ such that v *ve-dominates* e . The minimum cardinality of a *ve-dominating set* of G is called the *vertex-edge domination number* $\gamma_{ve}(G)$ of G . Vertex-edge domination was introduced by PETERS [13] in his 1986 PhD thesis and studied further in [4, 9]. Trivially, $\gamma_{ve}(G) \leq \gamma(G)$ holds for every graph G .

It follows from the definition of Roman domination that for any graph G ,

$$\gamma_{ve}(G) \leq \gamma(G) \leq \gamma_R(G).$$

Theorem 2 improves this upper bound for $\gamma(G)$:

$$\gamma_{ve}(G) \leq \gamma(G) \leq \frac{\Delta}{\Delta + 1} \gamma_R(G).$$

From Theorem 2 we also have:

$$\gamma_{ve}(G) \leq \frac{\Delta + 1}{\Delta} \gamma(G) \leq \gamma_R(G).$$

But this lower bound for $\gamma_R(G)$ is not as good as the following, which is observed in COCKAYNE et al. [5] for connected graphs G :

$$\gamma_{ve}(G) \leq \gamma(G) + 1 \leq \gamma_R(G).$$

But from the proof of Theorem 2, we obtain the following lower bound on the Roman domination number.

Theorem 5. *For any connected graph of order $n \geq 3$, $2\gamma_{ve}(G) \leq \gamma_R(G)$.*

Proof. As in the proof of Theorem 2, among all $\gamma_R(G)$ -functions, let $f = (V_0, V_1, V_2)$ be $\gamma_R(G)$ -function on G with $|V_2|$ as large as possible, or equivalently with $|V_1|$ as small as possible. Recall that by our choice of f , V_1 is independent.

Let uv be an edge of G . We know that the set V_2 dominates every vertex in V_0 , since f is a Roman dominating function. Thus, if u or v is in $V_0 \cup V_2$, then V_2 ve -dominates uv . Since V_1 is independent, at least one of u and v is in $V_0 \cup V_2$. It follows therefore that V_2 is a ve -dominating set of G . Hence, $\gamma_{ve}(G) \leq |V_2| \leq \frac{\gamma_R(G)}{2}$.

3. COMPARISONS OF LOWER BOUNDS

We conclude by comparing our new lower bounds on the Roman domination number with three previously known lower bounds. We show that in fact these five bounds are pairwise incomparable. Let $\gamma_t(G)$ denote the minimum cardinality of a total dominating set of G , that is, a dominating set whose induced subgraph has no isolated vertices. For a graph G with order n , no isolated vertices and maximum degree Δ , each of the following is a lower bound on $\gamma_R(G)$:

- (i) $\gamma_t(G)$ [8],
- (ii) $\frac{2n}{\Delta+1}$ [5],
- (iii) $\frac{\Delta+1}{\Delta}\gamma(G)$ (Theorem 2),
- (iv) $\left\lceil \frac{\text{diam}(G)+2}{2} \right\rceil$ [11],
- (v) $2\gamma_{ve}(G)$ (Theorem 5).

It is worth mentioning that there is a known lower bound due BERMUDO, FERNAU, and SIGARRETA [3] that is not on this list. In [3], the authors prove a very interesting relationship between Roman domination and the differential of a graph. For a set S , let $B(S)$ be the set of vertices in $V - S$ that have a neighbor in the set S . The *differential* of a set S is defined in [10] as $\partial(S) = |B(S)| - |S|$, and the maximum value of $\partial(S)$ for any subset S of V is the *differential* of G , denoted $\partial(G)$. In [3], they prove the unexpected result that $\gamma_R(G) = n - \partial(G)$. Hence, some bounds on $\gamma_R(G)$ from [3] came from known bounds on $\partial(G)$. In particular, they prove the lower bound of $n - \gamma(G)(\Delta - 1)$ from a bound on the differential first given in [2]. Since we can show that $n - \gamma(G)(\Delta - 1)$ is at most $\frac{2n}{\Delta+1}$ for any graph G , we omit this bound from our comparison.

To see that no two of these bounds are comparable, we consider the following families of graphs: the corona P_{3k}^* ; the path P_{12k} ; the graph F_k obtained from k ($k \geq 3$) cycles C_5 and a path P_k by identifying one vertex of each cycle with a vertex of the path so that no two cycles have a common vertex, and then by subdividing each edge of the path exactly twice; and the graph (broom) B_k for $k \geq 3$ formed from the star $K_{1,k}$ by subdividing one of its edges exactly $12k$ times. The following table gives values for the bounds for each of these families.

	$\gamma_t(G)$	$\frac{2n}{\Delta+1}$	$\frac{\Delta+1}{\Delta}\gamma(G)$	$\left\lceil \frac{\text{diam}(G)+2}{2} \right\rceil$	$2\gamma_{ve}(G)$
P_{3k}^*	$3k$	$3k$	$4k$	$\frac{3k+3}{2}$	$2k$
P_{12k}	$6k$	$8k$	$6k$	$6k+1$	$6k$
F_k	$3k$	$\frac{14k-4}{5}$	$\frac{5k}{2}$	$\frac{3k+3}{2}$	$4k$
B_{12k}	$6k+2$	$\frac{26k+2}{k+1}$	$4k+6$	$6k+2$	$6k+2$

From the values in the table, we deduce that the bounds are pairwise incomparable. In conclusion, we note that although our new lower bounds on the Roman domination number are incomparable to the other lower bounds, they can be arbitrarily larger than each of them.

Acknowledgements. The second author was partially supported by the University of Johannesburg.

REFERENCES

1. D. BANGE, A. E. BARKAUSKAS, P. J. SLATER: *Efficient dominating sets in graphs*. In: Applications of Discrete Math., R. D. Ringeisen and F. S. Roberts, eds., SIAM, Philadelphia (1988), 189–199.
2. L. A. BASILIO, S. BERMUDO, J. M. SIGARRETA: *Bounds on the differential of a graph*. Util. Math., to appear.
3. S. BERMUDO, H. FERNAU, J. M. SIGARRETA: *The differential and the Roman domination number of a graph*. Appl. Anal. Discrete Math., **8** (2014), 155–171.
4. R. BOUTRIG, M. CHELLALI, T. W. HAYNES, S. T. HEDETNIEMI: *Vertex-edge domination in graphs*. Aequat. Math., in press.
5. E. J. COCKAYNE, P. M. DREYER, SR., S. M. HEDETNIEMI, S. T. HEDETNIEMI: *On Roman domination in graphs*. Discrete Math., **278** (2004), 11–22.
6. E. EBRAHIMI, N. JAFARI RAD, C. M. MYNHARDT, Y. WU: *Bounds for independent Roman domination in graphs*. J. Combin. Math. Combin. Comput., **80** (2012), 351–365.
7. J. F. FINK, M. S. JACOBSON, L. F. KINCH, J. ROBERTS: *On graphs having domination number half their order*. Period. Math. Hungar., **16** (1985), 287–293.
8. S. T. HEDETNIEMI, R. R. RUBALCABA, P. J. SLATER, M. WALSH: *Few Compare to the Great Roman Empire*. Congr. Numer., **217** (2013), 129–136.
9. J. R. LEWIS, S. T. HEDETNIEMI, T. W. HAYNES, G. H. FRICKE: *Vertex-edge domination*. Util. Math., **81** (2010), 193–213.

10. J. L. MASHBURN, T. W. HAYNES, S. M. HEDETNEMI, S. T. HEDETNEMI, P. J. SLATER: *Differentials in graphs*. Util. Math., **69** (2006), 43–54.
11. B. P. MOBARAKY, S. M. SHEIKHOESLAMI: *Bounds on Roman domination numbers of graphs*. (English Summary) Mat. Vesnik, **60** (2008), 247–253.
12. C. PAYAN, N. H. XUONG: *Domination-balanced graphs*. J. Graph Theory, **6** (1982), 23–32.
13. J. W. PETERS: *Theoretical and Algorithmic Results on Domination and Connectivity*. Ph. D. Thesis, Clemson University, 1986.
14. D. RAUTENBACH, L. VOLKMANN: *Independent domination and matchings in graphs*. Discrete Math., **259** (2002), 325–330.
15. C. S. REVELLE, K. E. ROSING: *Defendens imperium romanum: A classical problem in military strategy*. Amer. Math. Monthly, **107** (7) (2000), 585–594.
16. I. STEWART: *Defend the Roman Empire!*. Sci. Amer., **281** (6) (1999), 136–139.
17. H. B. WALIKAR: B. D. ACHARYA, E. SAMPATHKUMAR: *Recent developments in the theory of domination in graphs*. In MRI Lecture Notes in Math., Mahta Research Instit., Allahabad, Volume 1, 1979.

LAMDA-RO Laboratory,
Department of Mathematics,
University of Blida, B.P. 270, Blida
Algeria
E-mail: m_chellali@yahoo.com

(Received June 10, 2015)
(Revised October 28, 2015)

Department of Mathematics,
East Tennessee State University,
Johnson City, TN 37614
USA

and

Department of Mathematics,
University of Johannesburg
Auckland Park
South Africa
E-mail: haynes@etsu.edu

School of Computing, Clemson University
Clemson, SC 29634
USA
E-mail: hedet@clemson.edu