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HARMONIOUS COLORING OF UNIFORM HYPERGRAPHS

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A *harmonious coloring* of a k-uniform hypergraph H is a vertex coloring such that no two vertices in the same edge share the same color, and each k-element subset of colors appears on at most one edge. The *harmonious number* $h(H)$ is the least number of colors needed for such a coloring. We prove that k-uniform hypergraphs of bounded maximum degree Δ satisfy $h(H) = O(\sqrt[k]{k!m})$, where m is the number of edges in H which is best possible up to a multiplicative constant. Moreover, for every fixed Δ, this constant tends to 1 with $k \to \infty$. We use a novel method, called *entropy compression*, that emerged from the algorithmic version of the Lovász Local Lemma due to Moser and Tardos.

1. INTRODUCTION

Let $k \geq 2$ be a fixed integer. A *k-uniform hypergraph* $H = (V, \mathcal{E})$ consists of the set of vertices V and a family $\mathcal{E}$ of k-element subsets of V, called the edges of H. A *rainbow coloring* of H is a mapping $c : V \to \{1, 2, \ldots, r\}$ in which no two vertices in the same edge have the same color. In this way every edge $P \in \mathcal{E}$ is assigned with a k-element subset of colors $c(P) \subseteq \{1, 2, \ldots, r\}$. Naturally the case $k = 2$ corresponds to the classical vertex coloring of a simple graph.

A rainbow coloring c of a hypergraph H is called *harmonious* if $c(P) \neq c(Q)$ for every pair of distinct edges $P, Q \in \mathcal{E}$ (see Figure 1).

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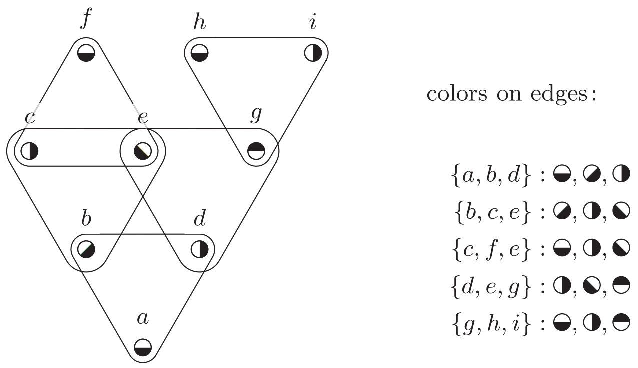

Figure 1. An example of harmonious coloring of a 3-uniform hypergraph

Let $h(H)$ be the least number of colors needed for a harmonious coloring of H. This notion arose as a natural generalization of harmonious coloring of graphs – an intensively studied topic with many deep results and challenging open problems (see [**3**], [**12**], [**14**], [**11**], [**13**], [**15**], [**20**], [**22**], [**21**]). The purpose of the present paper is to extend these investigations to more general combinatorial structures.

Let H be a k-uniform hypergraph with m edges. Let $Q_k(m)$ be the least positive integer r satisfying the inequality $\binom{r}{k} \geq m$. Clearly, the number of different k-element subsets of colors in any harmonious coloring of H must be at least as large as the number of edges of H. Hence, the inequality

$$h(H) \geq Q_k(m)$$

must hold for every k-uniform hypergraph H with m edges. This shows an $\Omega(\sqrt[k]{k!m})$ lower bound on $h(H)$ for fixed k. However, consider a k-uniform *sunflower* $S_{k,m}$ consisting of m edges whose common intersection has size $k-1$. It is easy to see that $h(S_{k,m}) = m + k - 1$. This shows that additional restrictions are needed in order to get the bounds closer to the asymptotic lower bound. Our main result provides such an upper bound for k-uniform hypergraphs with bounded maximum degree.

Theorem 1.1. *Every k-uniform hypergraph H with m edges and maximum degree Δ satisfies*

$$h(H) \leq \frac{k}{k-1}\sqrt[k]{\Delta(k-1)k!m} + f(k,\Delta),$$

where $f(k,\Delta) = 1 + \Delta^2 + (k-1)\Delta + \sum_{i=2}^{k-1} \frac{i}{i-1}\sqrt[i]{(i-1)i\frac{(k-1)\Delta^2}{k-i}}$.

The proof uses the *entropy compression* argument – a novel technique that emerged from the celebrated algorithmic version of the Lovász Local Lemma (see [**4**]), due to Moser and Tardos [**23**] (see [**17**], [**25**]). It is worth noting that this method gives a stronger actual assertion, namely that the bound from the theorem holds also for the *list* version of harmonious coloring (where each vertex has its own pre-assigned list of available colors). It can be also checked that direct application of the Lovász Local Lemma (in the asymmetric version) gives a much worse bound for $h(H)$ than the one in Theorem 1.1. Indeed, by counting probabilities we obtained expression of order $O(\Delta m)$ while in the theorem we have $O(\sqrt[k]{\Delta m})$.

A basic idea is surprisingly simple and general. Suppose we are given a combinatorial structure S and we want to color its elements by some fixed set of colors Γ so as to avoid certain conflicts. To do that we pick a long sequence of colors C and color the elements of S one by one (in some fixed order) using consecutive terms of C. Whenever a conflict appears we erase colors of some recently colored elements to get back to a proper partial coloring. Then we start again with coloring elements of S with further colors indicated by the sequence C. The process stops if either the sequence C has ended or if the whole structure has been successfully colored. Assume that the former case occurs for all sequences C (of some fixed length N). During the execution of the procedure we register information on all erasures in a special table T so that when the process stops we are able to reconstruct the entire sequence C from the table T and the final partial coloring c. This implies that the total number of color sequences C of length N is equal to the total number of possible outcomes (T, c). However it may happen that the latter number is bounded from above by some function which is strictly smaller than $|\Gamma|^N$ for large enough N. This contradiction means that the procedure had to stop before C has ended for at least one sequence C, giving a desired conflict-free coloring of the whole structure S. (For recent applications of entropy compression to diverse coloring problems see **[10]**, **[16]**, **[18]**, **[19]**).

In the next section we will give a detailed proof of Theorem 1.1 along these lines. The last section contains final remarks and some open problems.

2. PROOF OF THE MAIN RESULT

Our plan of the proof is simple. We start with a detailed description of the greedy coloring algorithm in the first subsection. In the second subsection we prove that the sets of inputs and outputs are of the same size (assuming that none of the inputs gave a complete harmonious coloring). Finally, in the third subsection we derive a contradiction by showing that the number of outputs is actually bounded by a value that is strictly smaller than the number of inputs (provided that the number of colors is as asserted).

2.1. Description of the Algorithm

Let $H = (V, \mathcal{E})$ be a k-uniform hypergraph with n vertices, m edges, and maximum degree Δ. We first describe a greedy procedure that finds a desired harmonious coloring of H provided the number of colors is as stated in the theorem.

Assume that the sets V and $\mathcal{E}$ are linearly ordered. These two orders induce the natural numbering of elements of any subset X of V or $\mathcal{E}$. Hence, every element

$e \in X$ has its respective number $n_X(e) \in \{1, 2, \ldots, |X|\}$ indicating its position in the set X determined by the initial linear order.

Let Γ be the set of colors. A *partial harmonious coloring* of H is any coloring $c : U \to \Gamma$ of a subset $U \subseteq V$ satisfying the following conditions:

- If $u, v \in U$ are in a common edge, then $c(u) \neq c(v)$.
- If $(P \cup Q) \setminus (P \cap Q) \subseteq U$, then $c(P \setminus Q) \neq c(Q \setminus P)$, for every pair of different edges $P, Q \in \mathcal{E}$.

The second condition is clearly necessary if we want an extension of a partial coloring c to a complete harmonious coloring of H. Actually, we distinguish three types of conflicts that may occur when one tries to make such an extension:

C1 (*Rainbow conflict*) There is an edge $P \in \mathcal{E}$ and two different vertices $u, v \in P$ with $c(u) = c(v)$ (see Figure 2).

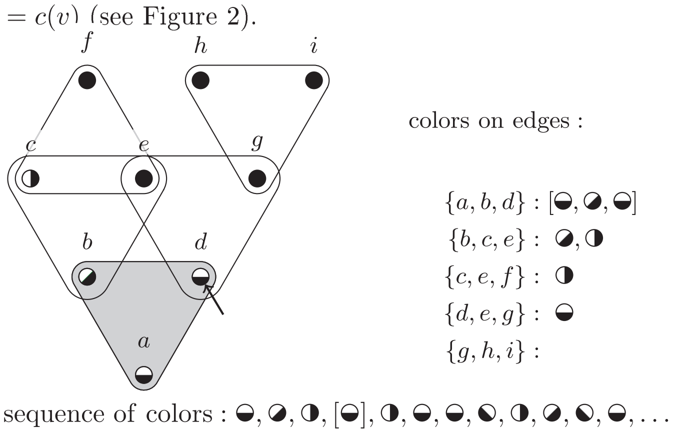

Figure 2. Rainbow conflict in the fourth step of the Algorithm

C2 (*Harmonious conflict on intersecting edges*) There are two different edges P, Q with $P \cap Q \neq \emptyset$ and all vertices from $(P \cup Q) \setminus (P \cap Q)$ already colored, such that $c(P \setminus Q) = c(Q \setminus P)$ (see Figure 3).

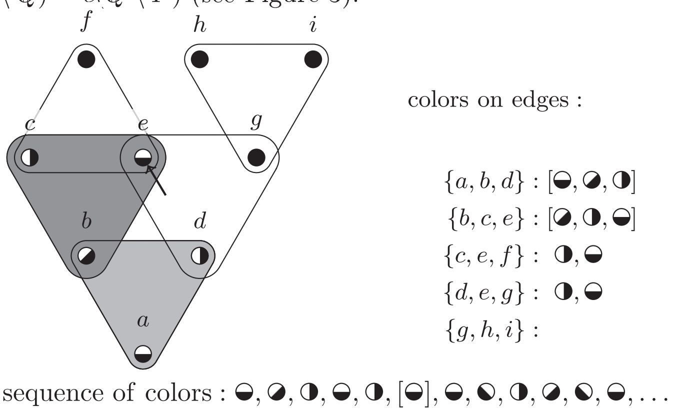

Figure 3. Harmonious conflict on intersecting edges in the sixth step of the Algorithm

C3 (*Harmonious conflict on disjoint edges*) There are two different edges P, Q with $P \cap Q = \emptyset$ and all vertices from $P \cup Q$ already colored, such that $c(P) = c(Q)$ (see Figure 4).

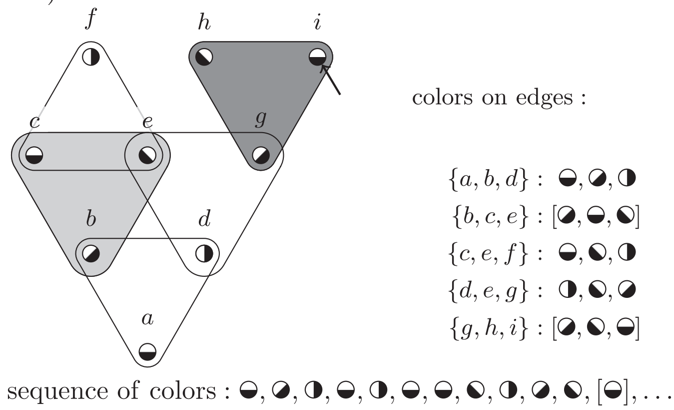

Figure 4. Harmonious conflict on disjoint edges in the twelveth step of the Algorithm

Let N be a positive integer, and let $C = (c_1, c_2, \ldots, c_N)$ be an arbitrary sequence of colors $c_i \in \Gamma$. Each step of our algorithm consists of three stages: coloring, correcting, and writing. After completing each step, the current partial coloring of H will be harmonious. Also some information will be stored in a special $1 \times N$ table T whose entries are initially empty. We now describe every particular step. Let $\mathcal{E}(v)$ denote the set of edges containing vertex v. Denote also by S_i the group of permutations of any set of size i. Suppose now that the currently executed step has number $r \geq 1$.

1. Pick the first uncolored vertex of V, say v, and assign color c_r to it.
2. Check if the new partial coloring is harmonious, and if so, place the sign $+$ in $T(r)$ (the r-th entry of table T).
3. Otherwise, correct the partial coloring by erasing colors from some vertices accordingly to the type of a conflict (if more than one conflict arose after coloring v, choose any one of them):

   (a) Suppose that conflict C1 occurs on a triple (P, u, v). Erase the color from vertex v, and write a pair of numbers (a, b) in the r-th entry of table T, where $a = n_{P \setminus \{v\}}(u)$ and $b = n_{\mathcal{E}(v)}(P)$.

   Let A denote the set of all possible such pairs (a, b). Clearly, we have

$$|A| = (k-1)\Delta.$$

   (b) Suppose that conflict C2 occurs on a pair of intersecting edges (P, Q) after coloring vertex $v \in Q \setminus P$. Assume also that $|Q \setminus P| = i$ for some

$i \in \{1, 2, \ldots, k-1\}$. Then erase colors from all vertices of $Q \setminus P$ and write a triple (a, b, π) in the r-th entry of table T, where $a = n_{\mathcal{E}(v)}(Q)$, $b = n_X(P)$ where

$$X = \{E \in \mathcal{E} : v \notin E \text{ and } |E \cap Q| = |P \cap Q| = k - i\},$$

and $\pi \in S_i$ is a permutation representing the unique color-preserving bijection between sets $P \setminus Q$ and $Q \setminus P$.

Let A_i denote the set of all possible such triples (a, b, π). Since the set X has at most $\left\lfloor \frac{(k-1)\Delta}{k-i} \right\rfloor$ elements, we have

$$|A_i| \leq i!\Delta \left\lfloor \frac{(k-1)\Delta}{k-i} \right\rfloor,$$

for all $i = 1,2,3, \ldots, k-1$.

(c) Suppose that conflict C3 occurs on a pair of disjoint edges (P, Q) after coloring vertex $v \in Q$. Then, we erase colors from all vertices of Q and place a triple (a, b, π) in the r-th entry of table T, where $a = n_{\mathcal{E}(v)}(Q)$, $b = n_{\mathcal{E}}(P)$, and $\pi \in S_k$ is a permutation representing the unique color-preserving bijection between sets P and Q.

Let A_k denote the set of all possible such triples (a, b, π). Clearly, we have

$$|A_k| \leq \Delta m k!.$$

The above algorithm stops when either a complete harmonious coloring of H has been reached, or if the sequence of colors C has ended. In the latter case the output of the algorithm is a pair (T, c), where T is the resulting table and c is a partial harmonious coloring on at most $n-1$ vertices of H.

Suppose now that for any input sequence of colors C the Algorithm does not produce a complete harmonious coloring of H. Let $\mathcal{I} = \Gamma^N$ denote the set of all possible inputs, and let $\mathcal{O}$ denote the set of all possible outputs (T, c). Our plan is to prove that these two sets are of the same cardinality. Then we will derive a contradiction by showing that the size of the set $\mathcal{O}$ is strictly smaller than $|\Gamma|^N$ for sufficiently large N.

2.2. Equicardinality of the sets $\mathcal{I}$ and $\mathcal{O}$

It is clear that each color sequence $C \in \mathcal{I}$ uniquely determines the resulting pair (T, c). We are going to prove that the converse also holds: for any given output pair (T, c) there is only one input sequence C producing that pair.

Let $X^{(r)}$ denote the set of uncolored vertices after step r of the Algorithm, with $X^{(0)} = V$ by convention.

Lemma 2.2. *The set $X^{(r)}$ is uniquely determined by $X^{(r-1)}$ and $T(r)$, for every $r = 1, 2, \ldots, N$.*

Proof. Let v be the smallest vertex in $X^{(r-1)}$. One of the following four cases occurs:

1. $T(r) = +$. This means that no erasures were made in step r. Hence, $X^{(r)} = X^{(r-1)} \setminus \{v\}$.

2. $T(r) = (a, b) \in A$. This means that conflict C1 occurred after coloring v, and only that color was erased in step r. Hence, $X^{(r)} = X^{(r-1)}$.

3. $T(r) = (a, b, \pi) \in A_i$, with $i \leq k-1$. First notice that the index i is determined by the length of permutation π. Hence, we know that conflict C2 occurred after coloring v on some pair of edges (P, Q) with $v \in Q \setminus P$. This pair (P, Q) is uniquely determined by the pair of numbers (a, b). Indeed, Q is indicated as the a-th element of the set $\mathcal{E}(v)$, and then P can be detected as the b-th edge among those not containing v and whose intersection with Q has size $k - i$. We also know that only colors from the set $Q \setminus P$ were erased. Thus $X^{(r)} = X^{(r-1)} \cup (Q \setminus P)$.

4. $T(r) = (a, b, \pi) \in A_k$. This means that conflict C3 occurred on some pair of disjoint edges (P, Q) with $v \in Q$. The edge Q is indicated by number a. We also know that only colors from Q were erased. Thus $X^{(r)} = X^{(r-1)} \cup Q$.

The proof of the lemma is complete. □

Let $c^{(r)}$ denote the partial harmonious coloring after the rth step of the Algorithm, with $c^{(0)}$ denoting the empty coloring by convention. Denote also by $U^{(r)}$ the set of colored vertices after completing step r, with $U^{(0)} = \emptyset$. So, $U^{(t)} = V \setminus X^{(t)}$. In the next lemma we prove that we are able to reconstruct all partial colorings $c^{(r)}$ and all terms c_r of the input color sequence C from the final partial coloring $c = c^{(N)}$ and the table T.

Lemma 2.3. *The partial harmonious coloring* $c^{(r-1)}$ *and the color* c_r *are uniquely determined by* $c^{(r)}$, $T(r)$, *and* $X^{(r-1)}$, *for every* $r = 1, 2, \ldots, N$.

Proof. Let v be the smallest vertex in the set $X^{(r-1)}$. This vertex was colored in step r by color c_r which we want to recover. First we find the set $U^{(r-1)} = V \setminus X^{(r-1)}$, which is the domain of the partial coloring $c^{(r-1)}$. The set $U^{(r)}$ of all vertices colored after step r is given by the partial coloring $c^{(r)}$. This determines the partial coloring $c^{(r-1)}$ on the set $U^{(r-1)} \cap U^{(r)}$. To find the other missing colors we need to consider just two cases depending on whether a conflict in step r occurred or not.

1. Suppose that no conflict occurred after coloring v in step r. Then $U^{(r)} = U^{(r-1)} \cup \{v\}$ and the color c_r is determined by the partial coloring $c^{(r)}$. Also the partial coloring $c^{(r-1)}$ is derived form $c^{(r)}$ by neglecting vertex v.

2. Suppose that some conflict occurred in step r. Then $U^{(r-1)} = U^{(r)} \cup E$, where E is the subset (possibly empty) of those vertices in $U^{(r-1)}$ whose colors were erased. We need to recover these colors together with the erased color of vertex v. This is possible by using the information written in $T(r)$. Indeed, using hints encoded there we may find a subset $Y \subseteq U^{(r-1)} \setminus E$ whose color pattern was the same as the color pattern of $E \cup \{v\}$. Then we extract colors of all vertices in $E \cup \{v\}$ using the color- preserving bijection determined by the permutation π, if necessary.

This completes the proof of the lemma. □

Let us summarize the established properties in the following corollary.

Corollary 2.4. *Given the output pair (T, c) of the Algorithm it is possible to uniquely reconstruct the input color sequence C. Hence, the two sets $\mathcal{I}$ and $\mathcal{O}$ have the same cardinality.*

Proof. By Lemma 2.2 we may derive sets $X^{(r)}$ for all $r = 1, 2, \ldots, N$. Then, using Lemma 2.3, we may recover all terms of the sequence C by backward induction.□

2.3. Bounding the size of $\mathcal{O}$

Let (T, c) be the output produced by the Algorithm from the color sequence C. We first estimate the number of possible tables T.

Let p denote the number of occurrences of the sign + in the table T. Let t_1 denote the number of entries of T occupied by the elements of the set $A \cup A_1$, and let t_i denote the number of entries in T filled with the elements of the set A_i, for $i = 2, 3, \ldots, k$. Finally, let $a_1 = |A \cup A_1|$, and $a_i = |A_i|$ for all $i = 2, 3, \ldots, k$. So, we have

$$a_1 \leq (k-1)\Delta + \Delta^2, a_k \leq \Delta m k!,$$

and

$$a_i \leq i!\Delta \left\lfloor \frac{(k-1)\Delta}{k-i} \right\rfloor,$$

for $i = 2, 3, \ldots, k-1$.

Lemma 2.5. *The total number of possible sequences T is bounded by*

$$\sum_{s=N-n+1}^{N} \sum_{t_1+2t_2+\cdots+kt_k=s} \binom{N}{p, t_1, t_2, \ldots, t_k} a_1^{t_1} \cdot a_2^{t_2} \cdot \ldots \cdot a_k^{t_k}.$$

Proof. The numbers $p, t_1, \ldots, t_k$ clearly sum up to N:

$$p + t_1 + t_2 + \cdots + t_k = N.$$

Notice that during execution of the Algorithm, every term c_r of the input sequence C was used only once to color some vertex v. Moreover, it was at most once erased from v. Hence, to every term $T(r)$ occupied with an element from A_i $(i \geq 2)$ there corresponds a unique set of exactly $i-1$ earlier terms of T filled with the sign $+$. Furthermore, these sets are pairwise disjoint. Since the total number of colored vertices after execution of the Algorithm is at most $n-1$, we obtain the following constraints:

$$N-n+1 \le t_1+2t_2+3t_3+\cdots+kt_k \le N.$$

Consequently, the total number of possible sequences T is bounded by the expression stated in the lemma. □

To obtain the aforementioned bound on the set $\mathcal{O}$ we will need the following two technical lemmas.

Lemma 2.6. *For every pair of integers $t \ge 1$ and $r \ge 2$ we have*

$$\binom{rt}{t} \le \frac{r^{rt}}{(r-1)^{(r-1)t}}.$$

Proof. We use the following well known Stirling bounds for factorials:

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n+1}} \le n! \le \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}}.$$

Substituting these inequalities to the formula for $\binom{rt}{t}$ and simplifying the resulting expression we get

$$\begin{aligned}
\binom{rt}{t} &= \frac{(rt)!}{t!\cdot((r-1)t)!}\\
&\le \frac{\sqrt{2\pi rt}\Big(\frac{rt}{e}\Big)^{rt} e^{\frac{1}{12rt}}}{\left(\sqrt{2\pi t}\Big(\frac{t}{e}\Big)^t e^{\frac{1}{12t+1}}\right)\cdot\left(\sqrt{2\pi(r-1)t}\Big(\frac{(r-1)t}{e}\Big)^{(r-1)t} e^{\frac{1}{12(r-1)t+1}}\right)}\\
&= \frac{\sqrt{2\pi rt}}{\sqrt{2\pi t}\cdot\sqrt{2\pi(r-1)t}}\cdot\frac{(rt)^{rt}}{t^t\cdot((r-1)t)^{(r-1)t}}\cdot\frac{e^{(r-1)t}\cdot e^t}{e^{rt}}\cdot\frac{e^{\frac{1}{12rt}}}{e^{\frac{1}{12t+1}}\cdot e^{\frac{1}{12(r-1)t+1}}}\\
&= \frac{\sqrt{r}}{\sqrt{2\pi(r-1)t}}\cdot\frac{r^{rt}}{(r-1)^{(r-1)t}}\cdot\exp\left(\frac{1}{12rt}-\frac{1}{12t+1}-\frac{1}{12(r-1)t+1}\right).
\end{aligned}$$

The first factor in the above product is clearly at most one. Also the last factor does not exceed one since

$$\begin{aligned}
\exp\left(\frac{1}{12rt}-\frac{1}{12t+1}-\frac{1}{12(r-1)t+1}\right) &\le \exp\left(\frac{1}{12t}\Big(\frac{1}{r}-\frac{1}{1+1}-\frac{1}{(r-1)+1}\Big)\right)\\
&= \exp\left(-\frac{1}{24t}\right) \le 1.
\end{aligned}$$

In consequence we get $\binom{rt}{t} \le \frac{r^{rt}}{(r-1)^{(r-1)t}}$ as asserted. □

Lemma 2.7. *If $p, t_1, t_2, \ldots, t_k$ are non-negative integers such that $p+t_1+\cdots+t_k = N$ and $t_1 + 2t_2 + \cdots + kt_k = s \leq N$, then*

$$\binom{N}{p, t_1, t_2, \ldots, t_k} \leq \binom{N}{s}\binom{s}{t_1, 2t_2, \ldots, kt_k} q_1^{t_1} \cdot q_2^{2t_2} \cdots q_k^{kt_k},$$

where $q_1 = 1$ and $q_i = \frac{i}{i-1}\sqrt[i]{i-1}$ for $i = 2, 3, \ldots, k$.

Proof. By the assumption on the numbers p and t_i we get

$$p = (N-s) + t_2 + 2t_3 + \cdots + (k-1)t_k.$$

This implies that

$$p! \geq (N-s)! \cdot (t_2)! \cdot (2t_3)! \cdots ((k-1)t_k)!.$$

In consequence we may write

$$\begin{aligned}\binom{N}{p, t_1, t_2, \ldots, t_k} &= \frac{N!}{p! \cdot t_1! \cdot t_2! \cdots t_k!} \\ &\leq \frac{N!}{(N-s)!s!} \cdot \frac{s!}{(t_1)! \cdot (2t_2)! \cdots (kt_k)!} \cdot \frac{(2t_2)!}{t_2! \cdot t_2!} \cdots \frac{(kt_k)!}{t_k! \cdot ((k-1)t_k)!} \\ &= \binom{N}{s}\binom{s}{t_1, 2t_2, \ldots, kt_k}\binom{2t_2}{t_2} \cdots \binom{kt_k}{t_k}.\end{aligned}$$

To complete the proof we need to show that $\binom{it_i}{t_i} \leq q_i^{it_i}$ for $i \geq 2$, which follows directly form Lemma 2.6 and the definition of q_i:

$$\binom{it_i}{t_i} \leq \frac{i^{it_i}}{(i-1)^{(i-1)t_i}} = \left(\frac{i}{i-1}\right)^{it_i} (i-1)^{t_i} = q_i^{it_i}. \qquad \square$$

Now we are ready to derive the final contradiction.

Lemma 2.8. *Let $R = \frac{k}{k-1}\sqrt[k]{(k-1)k!\Delta m} + f(k, \Delta)$, where $f(k, \Delta)$ is as stated in Theorem* 1.1. *Then the set of outputs $\mathcal{O}$ of the Algorithm satisfies $|\mathcal{O}| < R^N$ if N is sufficiently large.*

Proof. By Lemma 2.5 and Lemma 2.7, the total number of tables T is bounded by:

$$\begin{aligned}&\sum_{s=N-n+1}^{N} \sum_{t_1+2t_2+\cdots+kt_k=s} \binom{N}{p, t_1, t_2, \ldots, t_k} a_1^{t_1} \cdot a_2^{t_2} \cdot \ldots \cdot a_k^{t_k} \\ &\leq \sum_{s=N-n+1}^{N} \binom{N}{s} \sum_{t_1+2t_2+\cdots+kt_k=s} \binom{s}{t_1, 2t_2, \ldots, kt_k} (q_1 a_1)^{t_1} \cdot (q_2\sqrt{a_2})^{2t_2} \cdots (q_k \sqrt[k]{a_k})^{kt_k}\end{aligned}$$

$$\leq \sum_{s=N-n+1}^{N} \binom{N}{s} (q_1 a_1 + q_2 \sqrt{a_2} + \cdots + q_k \sqrt[k]{a_k})^s$$
$$< \left(\frac{1}{2} + q_1 a_1 + q_2 \sqrt{a_2} + \cdots + q_k \sqrt[k]{a_k}\right)^N,$$

where the last inequality holds for sufficiently large N. To see this put

$$q_1 a_1 + q_2 \sqrt{a_2} + \cdots + q_k \sqrt[k]{a_k} = A,$$

and notice that the penultimate expression is not greater than $N^n A^N$, which is smaller than $\left(\frac{1}{2} + A\right)^N$ when n is fixed while N is large enough. For the same reason the number of all partial colorings, which is at most $(|\Gamma| + 1)^n$, is also small. In consequence, the total number of output pairs (T, c) is bounded by

$$(|\Gamma| + 1)^n \cdot \left(\frac{1}{2} + q_1 a_1 + q_2 \sqrt{a_2} + \cdots + q_k \sqrt[k]{a_k}\right)^N$$
$$< (1 + q_1 a_1 + q_2 \sqrt{a_2} + \cdots + q_k \sqrt[k]{a_k})^N,$$

for sufficiently large N. Substituting for q_i and a_i the appropriate expressions gives

$$1 + q_1 a_1 + q_2 \sqrt{a_2} + \ldots + q_k \sqrt[k]{a_k} \leq R,$$

which completes the proof. □

In this way the proof of Theorem 1.1 is complete.

3. FINAL DISCUSSION

Let us conclude the paper with some remarks and two main conjectures. First observe that the bound obtained in Theorem 1.1 is asymptotically tight (provided that both k and m are large enough) in the following sense.

Corollary 3.9. *For every $\epsilon > 0$ and every integer Δ there exist integers k_0 and m_0 with the following property: Every k-uniform hypergraph with m edges (where $k \geq k_0$ and $m \geq m_0$) admits a harmonious coloring using at most $(1 + \epsilon) \sqrt[k]{k!m}$ colors.*

It is natural to analyze if this can be further improved.

Conjecture 4. *For each $k, \Delta \geq 2$ there exists a constant $c = c(k, \Delta)$ such that every k-uniform hypergraph H with m edges and maximum degree Δ satisfies*

$$h(H) \leq \sqrt[k]{k!m} + c.$$

This problem is a natural extension of a question posed by Edwards [**13**] for simple graphs (which is still open, though confirmed for some classes of graphs (see [**12**], [**14**], [**11**], [**15**])). He proved there that

$$h(G) = (1 + o(1))\sqrt{2m}$$

holds for graphs G with bounded maximum degree. The proof uses the celebrated Pippenger-Spencer theorem on the chromatic index of uniform hypergraphs [**24**]. Even an analogue of this result for uniform hypergraphs would be much stronger than Theorem 1.1. Notice however that our proof is valid for the list version of harmonious coloring (In this setting c_i is not the color used for a current vertex v, but the index of the color on its list L_v). So, we could formulate our result in the following stronger form.

Theorem 4.1. *Let H be a k-uniform hypergraph with m edges and maximum degree Δ. Suppose that every vertex v of H is assigned with an arbitrary set of colors L_v of size at least*

$$\frac{k}{k-1}\sqrt[k]{(k-1)k!\Delta m} + f(k, \Delta).$$

Then there exists a harmonious coloring c of H such that $c(v) \in L_v$ for every vertex v.

It is known that for simple graphs the difference between the harmonious number $h(G)$ and its list analogue may be arbitrarily large (see [**1**], [**26**]). Moreover, our method yields an efficient randomized algorithm constructing the desired harmonious coloring of a given hypergraph.

Another approach would be to try to analyze Conjecture 4 for large k and small Δ. Interestingly, the first non-trivial case of $k = 3$ and $\Delta = 2$ is widely open, even for so called *linear* hypergraphs (each pair of edges intersect in at most one vertex). Indeed, this is a special case of the following intriguing problem, concerning edge colorings of graphs, introduced by Aigner, Triesch and Tuza [**2**], and studied intensively by many authors (see [**9**], [**7**], [**5**], [**6**], [**8**]).

Consider a proper edge coloring of a k-regular graph G. Let $P(v)$ denote the set of k colors of edges incident to vertex v. A coloring is called *vertex-distinguishing* if $P(u) \neq P(v)$ for every pair of distinct vertices u and v of G. Notice that such coloring of G coincides with harmonious coloring of the *dual* hypergraph $D = D(G)$ to the graph G. (The vertices of D are the edges of G, and the edges of D are maximal sets of edges incident to the same vertex in G. Notice that maximum vertex degree in every such hypergraph is 2, and every two edges of D have at most one vertex in common) – see Figure 5.

Burris and Shelp [**8**] conjecture that every k-regular graph G on n vertices has a vertex-distinguishing coloring using at most $Q_k(n) + 1$ colors. However, a weaker version with any larger additive constant is open even for 3-regular graphs. This suggests that Conjecture 4 may be very hard or perhaps not true at all. From Theorem 1.1 we get the following bound for 3-regular graphs.

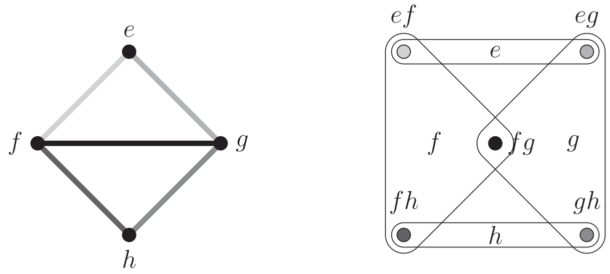

Figure 5. A vertex-distingushing coloring of Diamond

Corollary 4.2. *Every* 3*-regular graph with* n *vertices has a vertex-distinguishing coloring using at most*

$$\left(\frac{3}{2}\sqrt[3]{4}\right) \cdot Q_3(n) + c \leq 2.3812 \cdot Q_3(n) + c$$

colors, where c *is a constant.*

The second conjecture we present is inspired by another conjecture formulated by Burris and Shelp in [**8**], which was confirmed in [**9**]. It says that every graph G on n vertices (with at most one isolated vertex and no isolated edges) has a vertex-distinguishing coloring using at most $n+1$ colors. The following conjecture is a generalization of this statement to general hypergraphs.

Conjecture 5. *Every hypergraph* H *with* m *edges and maximum degree* Δ *satisfies*

$$h(H) \leq m + \Delta - 1.$$

It would be nice to verify this conjecture at least for uniform hypergraphs.

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