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ON TREES WITH MAXIMUM ALGEBRAIC CONNECTIVITY

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In this paper, trees with fixed diameter and any number of vertices are investigated. A subclass of trees with diameter $2k$ is introduced, the *diameter path trees* (*dp*-trees). Two subclasses of *dp*-trees are defined in which we characterize the elements that maximize the algebraic connectivity. Also, it is proved that if any tree maximizes the algebraic connectivity over all trees with diameter $2k$ then it is a *dp*-tree. For such trees, a bound for the degrees of their vertices is given. In the case of the odd diameter, $2k-1$, we show that P_{2k} is the only tree that maximizes the algebraic connectivity.

1. INTRODUCTION

Let $G = (V, E)$ be a simple undirected graph on n vertices. The Laplacian matrix of G is the $n \times n$ matrix $L(G) = D(G) - A(G)$ where $A(G)$ is the adjacency matrix and $D(G)$ is the diagonal matrix of vertex degrees.

Let $\mu_1(G) \geq \cdots \geq \mu_{n-1}(G) \geq \mu_n(G) = 0$ be the Laplacian eigenvalues of G. Fiedler [**9**] proved that G is a connected graph if and only if $\mu_{n-1}(G) > 0$. This eigenvalue is called the *algebraic connectivity* of G, denoted by $a(G)$, and it has received much attention, mainly with respect to application on trees as it can be seen in [**7**], [**8**], [**9**], [**10**], [**11**], [**12**], [**13**], [**23**],[**25**], [**26**], [**27**], [**28**], [**29**], and [**30**]. In 2015, an interesting paper regarding the maximization of the algebraic connectivity for some classes of graphs became available, see [**18**]. Among several results and conjectures, that paper gives an upper bound to the algebraic connectivity of a tree as a function of its number of vertices and its maximal degree. One can find a complete review on ordering of trees via algebraic connectivity in [**2**]. Also, a

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general survey (up to 2006) on the algebraic connectivity of graphs can be found in [**1**].

This paper approaches trees with fixed diameter and any number of vertices. In Section 2, some basic results are revisited. In Section 3, the concept of *diameter path tree*, *dp*-tree, is introduced and, from this concept, a partition of the set of trees with even diameter is given. In Section 4, we present a subclass of *dp*-trees in which each tree maximizes the algebraic connectivity among all trees with diameter at least $2k$. Also, another subclass of *dp*-trees, the *perfect rose trees*, is defined. A short discussion on center, centroid and characteristic vertex is done and a characterization of Type I and Type II of these trees is determined. Also, we give an infinite number of Type I trees with two centroids as an answer to the question proposed in 1987 by Merris [**22**]. Section 5 is devoted to two general results: the first one presents a condition on the vertex degrees of a *dp*-tree so that it does not maximize the algebraic connectivity of trees in its class. The second result shows that the path P_{2k} is only one tree with maximum algebraic connectivity over all trees with diameter $2k-1$.

2. BASIC RESULTS

Let $T=(V,E)$ be a tree with n vertices and diameter d. In this section we review the result given of Cvetković et al. [**5**] who prove that the algebraic connectivity of trees with diameter d is maximized by the algebraic connectivity of the path P_{d+1}. The section follows with the definition of the *Fiedler vector* which allows us to classify the trees in Type I and Type II, the important classes of trees introduced by Anderson and Morley [**3, 4**] and revisited by Fiedler [**10**] and Grone and Merris [**12**]. In addition, the notion of the bottleneck matrices, as studied by Kirkland [**16, 17**], is presented in order to use it in the next sections. Finally, we present two results due to Fallat and Kirkland [**7**] that characterize the tree (unless isomorphism) which maximizes the algebraic connectivity of trees with fixed diameter and fixed number of vertices.

Proposition 1. [**5**] *Let T be a tree with diameter d. Then*

$$a(T) \leq 2\left(1-\cos\frac{\pi}{d+1}\right) = a(P_{d+1}),$$

where $a(P_{d+1})$ denotes the algebraic connectivity with diameter d.

Theorem 1. [**10**] *If $\mathbf{f}$ is an eigenvector associated with $a(T)$ then exactly one of the following two cases occurs:*

1. *(A) No entry of $\mathbf{f}$ is 0. In this case, there is a unique pair of vertices v_i and v_j such that v_i and v_j are adjacent with $f_i>0$ and $f_j<0$. Moreover, the entries of $\mathbf{f}$ are increasing along any path in T which starts at v_i and does not contain v_j and the entries of $\mathbf{f}$ are decreasing along any path in T which starts at v_j and does not contain v_i.*

2. (B) *Some entry of* $\mathbf{f}$ *is* 0. *In this case, the subgraph induced by the vertices corresponding to zeros in* $\mathbf{f}$ *is a connected subgraph. Moreover, there is a unique vertex* v_r *such that* $f_r = 0$ *and* v_r *is adjacent to a vertex* v_s *with* $f_s \neq 0$. *The entries of* $\mathbf{f}$ *are either increasing, decreasing or identically* 0 *along any path in* T *starting at* v_r.

The tree T is said to be *Type II* if (A) holds and the vertices v_i and v_j are called the *characteristic vertices* of T. The edge defined by v_i and v_j is called the *characteristic edge* of T. The tree T is said to be *Type I* if (B) holds and the vertex v_r is called the *characteristic vertex* of T.

The vector $\mathbf{f} = [f_1, f_2, \dots, f_{n-1}, f_n]^T$ associated with $a(T)$ such that each coordinate corresponds to a label of each vertex of T is known as the *characteristic valuation* as the *Fiedler vector* of the tree.

Let T be a tree and v a vertex of T. The removal of a vertex v of T results in a forest with two or more connected components; each one of these components is called *branches* at v. The *length* of a path is the number of its edges. Given a vertex v, $r(v)$ denotes the length of the longest path starting at v. A *center* of a tree T is a vertex c such that $r(c) = \min_{v \in V} r(v)$. The diameter of T is $\operatorname{diam}(T) = 2r(c)$. The maximum number of edges in any branch at v in T is the *weight of* v, $w(v)$. A vertex t is a *centroid* of T if $w(t) = \min_{v \in V} w(v)$.

Let $\rho(A)$ be the spectral radius of a matrix A. For nonnegative matrices A and $B, A << B$ means that there exists a permutation matrix P such that PAP^t is entrywise dominated by a principal submatrix of B whenever the order of A is strictly less than the order of B or entrywise dominated by B with strict inequality in at least one entry whenever A and B have the same order. Moreover, from the Perron-Frobenius Theory for nonnegative matrices, if A and B are positive and $A << B$ then $\rho(A) < \rho(B)$. For more details see [**14**].

Let v be a vertex of T. Let L_v be the principal submatrix of the Laplacian matrix $L(T)$ obtained by deleting the v-row and v-column from $L(T)$. The inverse of L_v, that is L_v^{-1}, is known as the *bottleneck matrix of* T *at* v.

Lemma 1. [**17**] *Let* v *be a vertex in a tree* T. *The* (i,j)*-entry of* L_v^{-1} *is equal to the number of edges of* T *which are on both the path from vertex* v_i *to vertex* v *and the path from vertex* v_j *to vertex* v.

Clearly the (i,j) entry of L_v^{-1} is positive if and only if the vertices v_i and v_j are in the same branch of T at the vertex v. Then, there is a labelling of the vertices of T such that L_v^{-1} is similar to a block diagonal matrix in which the number of diagonal blocks is the degree of the vertex v. Each block of L_v^{-1} corresponds to a branch at v and it is called the *bottleneck of matrix* of the branch. The Perron root of the bottleneck matrix of a branch is called the *Perron root of the branch.* The *Perron value* is the greatest eigenvalue among all the greatest eigenvalues to each block of L_v^{-1}. A branch is called a *Perron branch* if its Perron root is the Perron value which is equal to the spectral radius of L_v^{-1}.

The next two theorems characterize, via bottleneck matrices, Type II and Type I trees, respectively.

Theorem 2. [**17**] *Let v_i and v_j be adjacent vertices in a tree T. Then, T is a Type II tree with characteristic vertices v_i and v_j if and only if the following condition holds: There exists γ, $0 < \gamma < 1$, such that $\rho(M_1 - \gamma J) = \rho(M_2 - (1-\gamma)J)$, where M_1 is the bottleneck matrix for the branch at v_j containing v_i, and M_2 is the bottleneck matrix for the branch at v_i containing v_j. Moreover, the algebraic connectivity of T satisfies* $a(T) = \dfrac{1}{\rho(M_1 - \gamma J)} = \dfrac{1}{\rho(M_2 - (1-\gamma J))}$.

Theorem 3. [**17**] *T is a Type I tree with characteristic vertex v_r if and only if T has 2 or more Perron branches at v_r. Moreover,* $a(T) = \dfrac{1}{\rho\left(L_r^{-1}\right)}$.

In [**7**], for a given n and a fixed diameter d, Fallat and Kirkland characterize trees with n vertices that maximize the algebraic connectivity over all such trees.

Lemma 2. [**7**] *For a fixed n and a fixed even d, the tree with n vertices and diameter d which maximizes the algebraic connectivity over such all trees is that constructed by taking a path on vertices $1, 2, \ldots, d+1$ and adding $n-d-1$ pendant vertex to the vertex $\dfrac{d}{2}+1$ of the path.*

Lemma 3. [**7**] *For a fixed n and a fixed odd d, the tree with n vertices and diameter d which maximizes the algebraic connectivity over such all trees is that constructed by taking a path on vertices $1, 2, \ldots, d+1$ and adding $n-d-1$ pendant vertex to the vertex $\dfrac{d+1}{2}$ of the path.*

3. DIAMETER PATH TREES

Let k be a positive integer. Let $\mathcal{D}_k$ be the class of trees with diameter $2k$. Each $T \in \mathcal{D}_k$ has only one center c which lies on the center of each path with length $2k$. If one of these paths is such that it has two branches at c isomorphic to P_k, we call it a *diameter path* and denote P_{diam}. Every tree $T \in \mathcal{D}_k$ which contains a P_{diam} is called a *diameter path tree*, or simply, *dp-tree*. We denote $\mathcal{D}_k^+$ the class of all *dp*-trees and $\mathcal{D}_k^-$, the remaining trees in D_k. Obviously, $\mathcal{D}_k = \mathcal{D}_k^+ \uplus \mathcal{D}_k^-$, where $\uplus$ means the disjoint union of sets.

As examples, the trees given by Lemma 2 are *dp*-trees but those given by Lemma 3 are not. Also, Figure 1 displays a *dp*-tree and Figure 2 displays a non *dp*-tree.

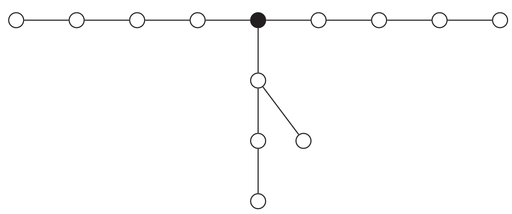

Figure 1. T_1 is a *dp*-tree.

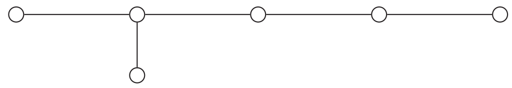

Figure 2. T_2 is not a dp-tree.

It is well known that algebraic connectivity of P_{2k+1} is $a(P_{2k+1}) = 2\Big(1 - \cos\dfrac{\pi}{2k+1}\Big)$. From now on we denote $a(P_{2k+1}) = a_k$. For all $i, 1 \leq i \leq 2k+1$, let us label v_i the vertices of P_{2k+1} (from left to right).

Remark I. For $k \geq 2$, the path P_{2k+1} is a Type I tree and v_{k+1} is its characteristic vertex. Of course it is also the center and the centroid of the tree, $c = t = v_{k+1}$. Both Perron branches at c in P_{2k+1} are isomorphic to P_k whose bottleneck matrix is

$$B = \begin{pmatrix} k & k-1 & k-2 & \cdots & 2 & 1 \\ k-1 & k-1 & k-2 & \cdots & 2 & 1 \\ k-2 & k-2 & k-3 & \cdots & 2 & 1 \\ \vdots & \vdots & \vdots & \cdots & 2 & 1 \\ 2 & 2 & 2 & \cdots & 2 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}$$

and, according to Theorem 3, its spectrum radius is $\rho(B) = \dfrac{1}{a_k}$.

Theorem 4. *For $k \geq 2$, if $T \in \mathcal{D}_k^-$ then $a(T) < a_k$.*

Proof. Let H be a tree formed by the labelled path P_{2k+1} plus a pendant edge (u, v_i) for some $1 \leq i \leq k$. Then H has only two branches at v_{k+1}. One of them, B_1 is isomorphic to P_k and, the other B_2, is the subtree of H induced by the vertices u and $v_1, \ldots, v_k$. Since $d(v_i) = 3$, for some $i = 2, \ldots, k$, H does not have any diameter path. So, $H \in \mathcal{D}_k^-$. Moreover, the bottleneck matrix B_1, denoted by M_1, is equal to the matrix B given in Remark I. So, its spectrum radius is $\rho(M_1) = \rho(B) = \dfrac{1}{a_k}$. The bottleneck matrix of B_2 is

$$M_2 = \begin{pmatrix} M_1 & (k-i)\mathbf{1} \\ (k-i)\mathbf{1}^t & k-i+1 \end{pmatrix},$$

where $\mathbf{1}$ is the column vector of 1's and $\mathbf{1}^t$ is its transposed vector. From Cauchy Theorem [**14**], the eigenvalues of M_2 interlace with the eigenvalues of M_1. Consequently, $\rho(M_1) \leq \rho(M_2)$. However, $k > 1$ then $\rho(M_2) \neq \rho(M_1)$. Since, $\rho(M_1) = \dfrac{1}{a_k}$ and $\rho(M_2) = \dfrac{1}{a(H)}$,

$$a(H) < a_k. \tag{1}$$

Now, let $T \in \mathcal{D}_k^-$. Since T does not have any diameter path, every branch B at c in T has at least a subtree isomorphic to H. However, it is well known that

$$a(T) \leq a(H). \tag{2}$$

Hence, from (2) and (1), the result follows.

4. TWO SUBCLASSES OF D_k^+

We begin this section with a subclass of *dp*-trees in which every tree in is Type I and it maximizes the algebraic connectivity among all trees with diameter at least $2k$.

4.1. $\mathcal{C}_k$ trees

We define $\mathcal{C}_k$ as the class of trees with diameter $2k$ such that each branch at c has at most k vertices. Figure 3 displays a tree that belongs to $\mathcal{C}_4$.

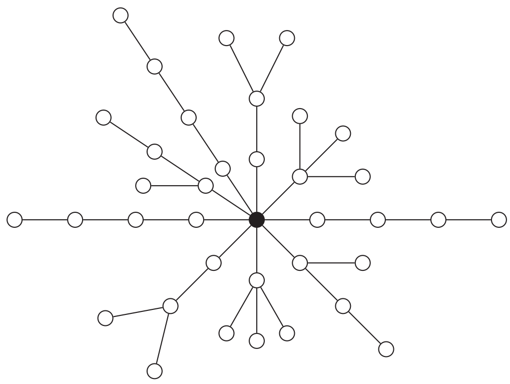

Figure 3. A tree of $\mathcal{C}_4$.

Lemma 4. *For $k \geq 2$, $\mathcal{C}_k \subset \mathcal{D}_k^+$.*

Proof. For $k \geq 2$, let $T \in \mathcal{C}_k$. Then, every branch at c has at most k vertices. Since $\operatorname{diam}(T) = 2k$, there are at least two branches at c in T with length $k - 1$. Each one of these branches is isomorphic to P_k. So, $T \in \mathcal{D}_k^+$. $\square$

The next theorem shows that every tree in $\mathcal{C}_k$ satisfies the upper bound to the algebraic connectivity given by Theorem 1.

Theorem 5. *Every tree $T \in \mathcal{C}_k$ is a Type I tree where the characteristic vertex and the centroid lies on the center of T. Besides, $\forall T \in \mathcal{C}_k$, $a(T) = a_k$ and so, it maximizes the algebraic connectivity of all trees with diameter at least $2k$.*

Proof. Let $T \in \mathcal{C}_k$. Then, T has at least one P_{diam} as an induced subtree. Let the vertices of P_{diam} be labelled as before. The center of T is $c = v_{k+1}$ and, except to leaves v_1 and v_{2k+1} and the center v_{k+1}, every vertex v of P_{diam} is such that $d(v) = 2$ in T. Each branch of P_{diam} at c has $k - 1$ edges and since any other branch at c in T (if there one) has at most k vertices, it has at most $k - 1$ edges. Consequently, the weight of the center is $w(v_{k+1}) = k-1$ and for every $v \in T, v \neq c$, $w(v) \geq k$. So, $w(c) < w(v)$ and v_{k+1} is the unique centroid of T which lies on c.

Since T is a *dp*-tree, T has at least two branches at c, B_1 and B_2 isomorphic to P_k, equal to the branches of P_{diam} at v_{k+1}. Hence, their respective bottleneck matrices $M_1 = M_2$, are equal to B given in Remark I.

Let R be any other branch in T (if there is any) such that $R \neq B_i$, $i = 1, 2$, and M_R be its bottleneck matrix. Since the order of M_R is at most k, one can easily see that $M_R << M_1$ and so, $M_R << M_2$. Thus, B_1 and B_2 are Perron branches at c in T. From Theorem 3, T is a Type I tree and $a(T) = a_k$. From Proposition 1, every $T \in \mathcal{C}_k$ satisfies the upper bound of the algebraic connectivity of the path P_{2k+1}. Besides, it is well known that $a(P_{2k+1}) < a(P_{2k})$. Consequently, $a(T)$ maximizes the algebraic connectivity of all trees with diameter at least $2k$.

4.2. Perfect rose trees

The trees called *Fiedler roses*, also known as *rose trees*, were introduced by Evans [**6**] and, more recently studied by Lefèvre [**19**].

Given k, t, ℓ natural numbers, a *rose tree* $R(k, t, \ell)$ is the graph built from the path $P_{k+t+1} = \{v_1, \ldots, v_{k+t+1}\}$ and the star $S_{1,\ell}$ by connecting the vertex v_{k+1} of the path to the center $s = v_{k+t+2}$ of the star. Let $v_{k+t+3}, , \ldots, v_n$ be the other vertices of the star. Denote $[S_{1,\ell}]$ the subgraph of $R(k, t, \ell)$ induced by $\{v_i, i = k + t + 2, \ldots, v_n\}$. If $t = k$, we say that it is a *perfect rose tree*, denoted $R(k, \ell)$. Obviously $R(k, \ell) \in \mathcal{D}_k^+$, it has $2k + \ell + 2$ vertices and the center v_{k+1} of the path is the center c of the tree. Figure 4 displays the perfect rose tree $R(4, 5)$.

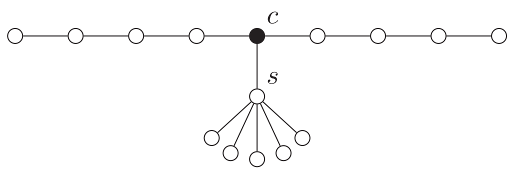

Figure 4. $R(4, 5)$ with c and s as highlighted vertices

Denote the algebraic connectivity of $R(k, \ell)$ more simply as $a_{k,\ell}$. Let $\mathbf{f} = [f_1, \ldots, f_{k+1}, .., f_{2k+1}, f_{2k+2}, \ldots, f_n]$ be the Fiedler vector of $R(k, \ell)$ where $f_i, 1 \leq i \leq n$ is the label of v_i.

The next result was implicitly proved, under distinct arguments, by Lefèvre [**19**] and Zimmermann [**32**].

Theorem 6. *Let $R(k, \ell)$ be a perfect rose tree, $k \geq 2$. The following statements are equivalent:*

(a) $R_{k,\ell}$ *is Type* I;

(b) $R(k, \ell)$ *maximizes the algebraic connectivity*;

(c) $\frac{1}{2}(\ell + 2 + \sqrt{\ell^2 + 4\ell}\,) \leq \frac{1}{a_k}$.

Proof. (a) $\Rightarrow$ (b) Let $k \geq 2$ and $R(k, \ell)$ be a Type I tree with v_{k+1} as its characteristic vertex. There are 3 branches at v_{k+1} in $R(k, \ell)$, of which, two of them are isomorphic to P_k. So, their respective bottleneck matrices are equal to B as given in Remark I such that

$$\rho(B) = \frac{1}{a_k}, \tag{3}$$

where $a_k = a(P_{2k+1}) = \dfrac{1}{2\left(1-\cos\left(\frac{\pi}{2k+1}\right)\right)}$. Then, (b) holds.

(b) $\Rightarrow$ (c) The third branch of $R(k,\ell)$ is isomorphic to $S_{1,\ell}$ with $(\ell+1)\times(\ell+1)$ bottleneck matrix,

$$M = \begin{pmatrix} I+J & \mathbf{1} \\ \mathbf{1}^t & 1 \end{pmatrix},$$

where I is the identity matrix, J is the matrix of 1's, both of them of order ℓ, $\mathbf{1}$ is the column vector of 1's and $\mathbf{1}^t$ is its transposed vector.

The spectrum of M is known and their eigenvalues are: $1^{[\ell]}, \frac{1}{2}(\ell+2+\sqrt{\ell^2+4\ell}\,)$ and $\frac{1}{2}(\ell+2-\sqrt{\ell^2+4\ell}\,)$. Hence, the spectral radius of M is

$$\rho(M) = \frac{1}{2}(\ell+2+\sqrt{\ell^2+4\ell}). \tag{4}$$

and

$$a_{k,\ell} = \frac{1}{\max\{\rho(M),\rho(B)\}}. \tag{5}$$

From (3), $\rho(B) \le \rho(M)$ and from (4),

$$\frac{1}{2}(\ell+2+\sqrt{\ell^2+4\ell}) \le \frac{1}{a_k}. \tag{6}$$

(c) $\Rightarrow$ (a) Now, we have to suppose that the inequality (6) holds. So, from (5), we get $a_k = a_{k,\ell}$. Hence, $R(k,\ell)$ maximizes the algebraic connectivity among all such trees. Moreover, it has two Perron branches at v_{k+1} and, by Theorem 3, $R(k,\ell)$ is a Type I tree.

Remark II. Denote y_k the maximum number of leaves of the induced subtree $[S_{1,\ell}]$ in $R(k,\ell)$ such that $a_{k,\ell} = a_k$. For $k=2$, and by Theorem 2, $y_k = 1$. For $k>2$, we get $y_k = \left\lfloor \frac{(1-a_k)^2}{a_k} \right\rfloor$ via the aid of the eigenvalues location algorithm given in [**15**]. Since $k>2$, $a_k \le 0.19806$, so $y_k = \left\lfloor \frac{1}{a_k} \right\rfloor - 2$.

Corollary 1. *The perfect rose tree $R(k,\ell) \in \mathcal{D}_k^+ - \mathcal{C}_k$ is a Type I tree if and only if $k-1<\ell<y_k+1$.*

Proof. By the defintion of $\mathcal{C}_k$, $R(k,\ell) \in \mathcal{D}_k^+ - \mathcal{C}_k$ if and only if $k-1<\ell$ and, from Theorem 6, $R(k,\ell)$ is a Type I tree if and only if $\ell < y_k+1$. ☐

Since $(\mathcal{D}_k^+ - \mathcal{C}_k) \neq \emptyset$, we obtain a partition of $\mathcal{D}_k$ as $\mathcal{D}_k = \mathcal{C}_k \uplus (\mathcal{D}_k^+ - \mathcal{C}_k) \uplus \mathcal{D}_k^-$. See Figure 5.

For instance, in $\mathcal{D}_3^+$ there are only four Type I trees: $R(3,0)$, $R(3,1)$, $R(3,2)$ and $R(3,3)$ because $y_3 = 3$; all of them have $a_3 = a(P_7) = 0.19806$. Observe that $R(3,3) \notin \mathcal{C}_3$.

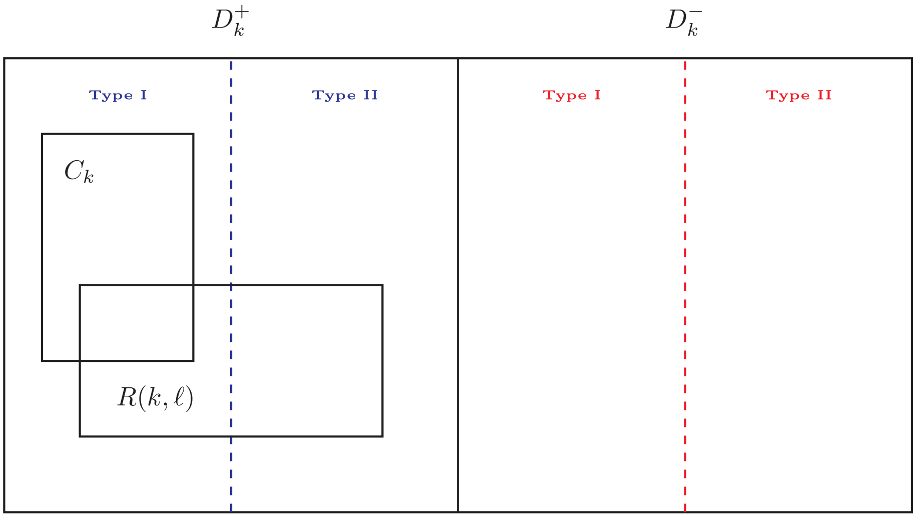

Figure 5. Partition of $D_k = C_k \biguplus (D_k^+ - C_k) \biguplus D_k^-$.

4.3. Centroids and characteristic vertices

According to Merris [**22**], some properties of the characteristic vertex (vertices) are reminiscent of similar properties of centroid(s) and/or center(s) of a tree T. Moreover, Merris, in the same paper, posed the following questions:

1. What is the relationship between the characteristic vertex (characteristic vertices) and centroid(s) or center(s) of a tree T?
2. If T is of Type I must it have be a unique centroid point?
3. If there is a unique centroid t and a unique center c, is (are) the characteristic vertex (characteristic vertices) contained in the path from t to c?

From 1987, when these questions were posed, up to now, we could not find answers to them. The next proposition gives a necessary and sufficient condition for a perfect rose tree to have two centroids. This characterization leads to Proposition 3 which provides us infinite counter-examples to the second question posed by Merris [**22**].

Proposition 2. *A perfect rose tree $R(k,\ell)$ has two centroids c and s if and only if $\ell = 2k$. Otherwise, if $\ell < 2k$, the centroid lies on the center of the tree and, if $\ell > 2k$, the centroid lies on s, the center of the star $S_{1,\ell}$.*

Proof. Let $R(k,\ell)$ be labelled as in Figure 6.

It is enough to observe the weights of the vertices of $R(k,\ell)$ as follow

$$w(v_i) = \begin{cases} \ell, & i = k+1; \\ 2k, & i = s; \\ w(v_{2k-i+2}) = 2k+\ell-i+1, & i = 1,..,k; \\ 2k+\ell-1, & \text{for the other cases.} \end{cases}$$

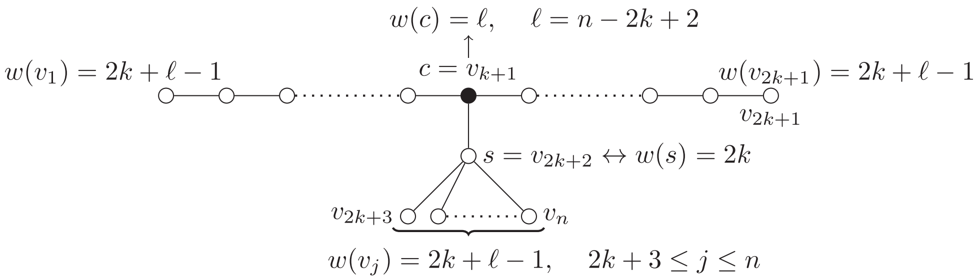

Figure 6. The weights of the vertices of $R(k,l)$.

Proposition 3. *For $k \geq 5$, every perfect rose tree $R(k,2k)$ is a Type I tree and it has two centroids.*

Proof. From Proposition 2, $R(k,2k)$ has two centroids and, from Theorems 3 and 6, $R(k,2k)$ is a Type I tree if and only if

$$2k + 2 + \sqrt{16k^2 + 8k} \leq \frac{2}{a_k}. \tag{7}$$

Hence, we can see that (7) holds if and only if $k \geq 5$.

5. TWO GENERAL RESULTS

Given k, ℓ and $r \leq k-2$, a *generalized perfect rose tree* $G(k,r,\ell)$ is the graph built from the path P_{2k+1} and the star $S_{1,\ell}$ by connecting the center of the path v_{k+1} and the center s of the star to the extremities of a path P_r, $r \geq 0$. If $r = 0$, v_{k+1} is directly connected with s. The order of $G(k,r,\ell)$ is $2k+\ell+r+2$.

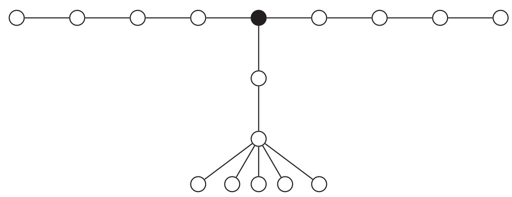

Figure 7. Generalized perfect rose tree $R(4,1,5)$

Lemma 5. *Let* $y_k = \left\lfloor \frac{1}{a_k} \right\rfloor - 2$. *If* $1 \leq r \leq k-2$ *and* $\ell = y_k$ *then,* $a(G(k,r,y_k)) < a(R(k,y_k))$.

Proof. The three branches of the generalized perfect rose at c are B_1 and B_2, both of them isomorphic to P_k, and B_3 composed by P_r and $S_{1,\ell}$. The bottleneck matrices of B_1 and B_2 are equal to B given in Remark I. The bottleneck matrix of the B_3 has order $y_k + r + 1$ and is equal to

$$M = \begin{pmatrix} X & R_r \\ R_r^t & Y \end{pmatrix},$$

where $X = [x_{ij}]$ is a matrix of order $y_k + 1$ such that

$$x_{ij} = \begin{cases} r+2, & i = j; \\ r+1, & i < j. \end{cases}$$

R_r is a matrix with $y_k + 1$ line vectors equal to $(r, r-1, r-2, \cdots, 2, 1)$ and Y is a matrix of order r equal to the bottleneck matrix B of the branch P_r given by Remark I.

Since $B << M$, $\rho(B) < \rho(M)$. Consequently, $a(G(k,r,y_k)) < a(R(k,y_k))$.

Theorem 7. *Let* $T \in \mathcal{D}_k$. *If* $a(T) = a_k$ *then* $T \in \mathcal{D}_k^+$ *and for every vertex* v *of* T, $v \neq c$, $d(v) < \left\lfloor \frac{1}{a_k} \right\rfloor - 2$.

Proof. Straightforward from Theorem 4, $T \in \mathcal{D}_k^+$. Let $v \neq c$ in T such that $d(v) \geq y_k$, being $y_k = \left\lfloor \frac{1}{a_k} \right\rfloor - 2$. Let $x_i, 1 \leq i \leq y_k + 1$ vertices adjacent to v in T. Choose x^* as one of vertices x_i. There is a path $\theta = [x^*, u_1, u_2, \ldots, u_r, c]$ from x_i to c with length $r \leq k-1$. The tree T has an induced subtree H spanned by v, $x_i, 1 \leq i \leq y_k + 1$ and, $u_j, 1 \leq j \leq r$. Hence, H is isomorphic to $G(k,r,y_k)$. Besides, we know that $a(T) < a(H)$. From Lemma 5, $a(H) < a(R(k,\ell)) = a_{k,y_k} = a_k$. So, $a(T) < a_k$ which is a contradiction. □

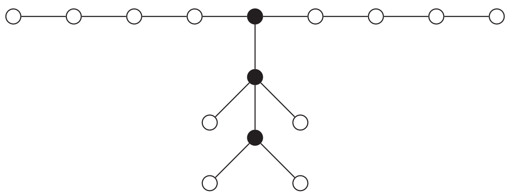

Figure 8. A counter-example to the converse for Theorem 7.

The converse of the result above does not hold. Figure 8 displays a tree T with $\Delta = 4$. In this case $y_4 = 6$ and $a(T) = 0.11856 \neq a_4 = a(P_9) = 0.12061$.

From now on, we consider trees with fixed odd diameter $2k - 1$ and the number of vertices as a variable $n \geq 2k$.

Theorem 8. *For $k \geq 2$, let T be a tree with fixed odd diameter $2k-1$ and any vertices $n \geq 2k$. Let $\mathcal{O}$ be the class of all these trees. The path P_{2k} is the unique tree on such conditions that for every tree T, $T \in \mathcal{O}, a(T) < a(P_{2k})$.*

Proof. For $k \geq 2$, let $d = 2k-1$ be fixed diameter of all trees with any vertex $n \geq 2k$. Let $\mathcal{O}_n$ be the set of all such trees with fixed n . All we need to do is apply Lemma 3, for each $n \geq 2k$. For j, a natural number, define $T_{n,j}$ to be the tree $\in \mathcal{O}_n$ with j pendent vertices adjacent to the vertex $\frac{2k}{2}$ in P_{2k}. So, we have:

Fact 1. For every n, $a(T_{n,j})$ decreasing with j, see Fallat and Kirkland [**7**];

Fact 2. From Lemma 3, the tree $T_{2k,0}$ is equal to the path P_{2k-1}. So, $a(T_{2k,0}) = a(P_{2k-1})$;

Fact 3. Let be T_n any tree with n vertices and diameter $2k-1$, implicitly in the proof of Lemma 3, in [**7**], $a(T_{n,j}) > a(T_n)$.

From these facts above, for every $T \in \mathcal{O}$, we get $a(P_{2k-1}) > a(T)$.

Final considerations. Among all trees with fixed diameter $2k-1$ and any number of vertices, Theorem 8 proves that the path P_{2k} is the only one with maximum algebraic connectivity. However, for the case of fixed even diameter, according to Lemma 2, Theorems 5 and 6, there are infinite trees with $n \geq 2k+1$ vertices and the same algebraic connectivity that P_{2k+1} which maximize the algebraic connectivity over all such trees. All these trees are *dp*-trees. Finally, Proposition 3 determines an infinite number of Type I trees with two centroids as an answer to the question proposed in 1987 by Merris, [**22**].

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