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THE RELATIONSHIP BETWEEN SEQUENTIAL FRACTIONAL DIFFERENCES AND CONVEXITY

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We consider the relationship between the sign of the sequential fractional difference $\Delta_{2+a-\mu}^{\nu}\Delta_{a}^{\mu}f(t)$, $t \in \mathbb{N}_{4+a-\mu-\nu}$, and the convexity of the map $f : \mathbb{N}_{a} \to \mathbb{R}$ in the case where $\mu \in (1, 2)$, $\nu \in (1, 2)$, and $\mu + \nu \in (2, 3)$. In particular, we demonstrate that there exist dissimilarities between the sequential case studied here and the non-sequential case, which has been previously studied. In addition, we describe a fundamental inequality that $\Delta^{2}f(t)$ must satisfy whenever $\Delta_{2+a-\mu}^{\nu}\Delta_{a}^{\mu}f(t) \geq 0$. This inequality also reveals some dissimilarities between the sequential and non-sequential settings. We provide some numerical examples to illustrate the results. Finally, in addition to the case $\mu \in (1, 2)$, $\nu \in (1, 2)$, and $\mu + \nu \in (2, 3)$, we also consider results in both the case $\mu \in (0, 1)$, $\nu \in (1, 2)$, and $\mu + \nu \in (2, 3)$ as well as the case $\mu \in (1, 2)$, $\nu \in (0, 1)$, and $\mu + \nu \in (2, 3)$.

1. INTRODUCTION

In the case of integer-order forward (delta) differences the commutativity of such operators is a triviality. That is to say, for a given function $f : \mathbb{N}_a \to \mathbb{R}$ we have that

$$\Delta(\Delta f)(t) = \Delta^2 f(t);$$

note that here and throughout we use the standard notation

$$\mathbb{N}_a := \{a, a+1, a+2, \dots\},\$$

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for a given number $a \in \mathbb{R}$. In addition, we also define the set $\mathbb{N}_{r_1}^{r_2}$, for any numbers $r_1 \leq r_2$ satisfying $r_2 - r_1 \in \mathbb{Z}$, by

$$\mathbb{N}_{r_1}^{r_2} := \{r_1, r_1 + 1, \dots, r_2\}.$$

In the fractional setting, by considerable contrast, the above commutativity property is well-known to fail. Recall that one definition of the fractional difference $f \mapsto \Delta_a^{\nu} f$ is, for a given function $f : \mathbb{N}_a \to \mathbb{R}$,

$$\Delta_a^{\nu} y(t) := \frac{1}{\Gamma(-\nu)} \sum_{s=a}^{t+\nu} (t-s-1)^{-\nu-1} f(s),$$

for $t \in \mathbb{N}_{a+N-\nu}$, where $N-1 < \nu < N$ for $N \in \mathbb{N}$. Especially due to the fact that the fractional difference is an inherently nonlocal operator, many fundamental, even trivial properties of the integer-order difference fail to hold for the corresponding fractional operator. For example, we instead can only obtain the following theorem – see [**30**, **31**].

Theorem 1.1. Let $f : \mathbb{N}_a \to \mathbb{R}$ be given and suppose $\nu, \mu > 0$, with $N - 1 < \nu \leq N$ and $M - 1 < \mu \leq M$, where $M, N \in \mathbb{N}_1$. Then for $t \in \mathbb{N}_{a+M-\mu+N-\nu}$ it holds that

(1)
$$\Delta_{a+M-\mu}^{\nu}\Delta_{a}^{\mu}f(t) = \Delta_{a}^{\nu+\mu}f(t) - \sum_{j=0}^{M-1} h_{-\nu-M+j}(t-M+\mu,a)\Delta_{a}^{j-M+\mu}f(a+M-\mu),$$

where $N - 1 < \nu < N$. If $\nu = N$, then (1) simplifies to

$$\Delta_{a+M-\mu}^{\nu}\Delta_a^{\mu}f(t) = \Delta_a^{\nu+\mu}f(t), \quad t \in \mathbb{N}_{a+M-\mu}$$

Thus, the fractional difference operator is, in general, noncommutative. And this means that a finite sequence of fractional differences, such as $\Delta_{2+a-\mu}^{\nu}\Delta_a^{\mu}f(t)$, for some choices of μ and ν , can be considered as a possible separate entity from a single fractional difference, a distinction that is rendered void in the integer-order setting due to the commutativity property mentioned above. This naturally gives rise to the question of what, if any, properties sequential fractional differences possess that are distinct from the non-sequential case. We note that sequential fractional differences were first considered, within the context of fractional boundary value problem theory, in a work by GOODRICH [25].

There has recently been substantial progress made in understanding the qualitative properties of discrete fractional operators. This is a very nontrivial program due to the inherent nonlocal nature of the fractional difference, a fact that causes great difficulty when trying to equip the operator with some reasonable geometrical meaning. DAHAL and GOODRICH [16, 17] and then later GOODRICH [27] were the first to provide some results regarding the relationship between the sign of $\Delta_a^{\nu} y(t)$ and the monotonicity, convexity, and concavity of the map y. More recently BAOGUO, et al. [12, 13] and JIA, et al. [32, 33, 34] have provided substantial and interesting generalizations of the original works by DAHAL and GOODRICH. In the case where $0 < \nu < 1$ a recent paper by ATICI and UYANIK [11] provides some additional significant contributions regarding the connection between the fractional difference and the monotonicity of functions. More generally, there has been a growing and broadening interest in the discrete fractional calculus over the past 10 years or so, beginning with the initial investigations of ATICI and ELOE [5, 6, 7, 8, 9], and continuing in a variety of directions such as operational properties of fractional differences [1, 2, 3, 4, 20, 31, 42], Laplace transforms [14], fractional boundary value problems [15, 21, 22, 24, 26, 28, 38, 39, 40], extensions to other time scales such as $q^{\mathbb{Z}}$ [14, 18, 19, 23], asymptotic behavior of solutions to fractional initial value problems [35, 36, 37], chaotic dynamics of fractional-order dynamical systems [41], and applications to modeling in the biological sciences [10]; one may also consult the book by GOODRICH and PETERSON [30] for a broad overview of these and other related topics.

In the present paper we hope to clarify further the connection between the fractional difference operator and convexity. In particular, we shall couch this study in the context of sequential fractional differences, a viewpoint that does not seem to have been considered thus far in the existing literature. In particular, we first consider

Case - I:
$$\mu \in (1, 2), \nu \in (1, 2), \text{ and } \mu + \nu \in (2, 3),$$

and then, under these assumptions, derive a fundamental inequality that the function $\Delta^2 f(t)$ must satisfy for any function $f : \mathbb{N}_a \to \mathbb{R}$ satisfying the sequential fractional difference inequality

(2)
$$\Delta_{2+a-\mu}^{\nu}\Delta_{a}^{\mu}f(t) \ge 0, t \in \mathbb{N}_{4+a-\mu-\nu}.$$

The fundamental inequality we obtain is analogous to a similar inequality obtained by JIA, et al. [33]. Using our inequality we are able to establish a connection between the convexity of f and the sign $\Delta_{2+a-\mu}^{\nu}\Delta_a^{\mu}f(t)$. We will demonstrate, in particular, that in the sequential setting the relationships obtained are more complicated than in the non-sequential setting.

In addition to the case identified above, we then repeat the above program in

<u>Case - II:</u> $\mu \in (0, 1), \nu \in (1, 2), \text{ and } \mu + \nu \in (2, 3).$

Under the preceding assumptions on the parameters μ and ν , the fundamental inequality is proved under the slightly altered assumption

(3)
$$\Delta_{1+a-\mu}^{\nu}\Delta_{a}^{\mu}f(t) \ge 0, t \in \mathbb{N}_{3+a-\mu-\nu},$$

where the sequential difference is now defined on $\mathbb{N}_{3+a-\mu-\nu}$ instead of $\mathbb{N}_{4+a-\mu-\nu}$. It turns out that these two cases do *not* produce identical results. In fact, the latter case (i.e., where $0 < \mu < 1$) is somewhat simpler to characterize, as we shall show. All in all, these observations reinforce the fact that the sequentialized difference has dissimilarities not only with regard to the the non-sequential difference, but even more fundamentally within various sequential differences with different choices of the underlying parameters μ and ν .

We conclude our study with

Case - III:
$$\mu \in (1, 2), \nu \in (0, 1), \text{ and } \mu + \nu \in (2, 3).$$

It turns out that there is very little difference between Cases I and III. Consequently, we do not spend much time discussing this final case.

2. CONVEXITY-TYPE RESULTS FOR SEQUENTIAL FRACTIONAL DIFFERENCES

2.1. Preliminary Definitions and Lemmata

We begin by recalling a few definitions fundamental in the discrete fractional calculus. The interested reader may consult the text by GOODRICH and PETERSON [30] for a thorough overview of the discrete fractional calculus.

Definition 2.2. We put

$$t^{\underline{\nu}} := \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)},$$

for any t and ν for which the right-hand side is defined. We also appeal to the convention that if $t + 1 - \nu$ is a pole of the Gamma function and t + 1 is not a pole, then $t^{\underline{\nu}} = 0$.

Definition 2.3. The ν -th fractional sum, $\nu > 0$, of a function $f : \mathbb{N}_a \to \mathbb{R}$, where $a \in \mathbb{R}$ is given, is

$$\Delta_a^{-\nu} f(t) := \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\nu-1} f(s),$$

for $t \in \mathbb{N}_{a+\nu}$. We also define the ν -th fractional difference of f, for $\nu > 0$, by

$$\Delta_a^{\nu} f(t) := \Delta^N \Delta_a^{\nu - N} f(t),$$

where $t \in \mathbb{N}_{a-\nu+N}$ and $N \in \mathbb{N}_1$ is the unique number satisfying $N-1 < \nu \leq N$.

Definition 2.4. The ν -th fractional Taylor monomial based at s is the map $(t, s) \mapsto h_{\nu}(t, s)$ defined by

$$h_{\nu}(t,s) := \frac{(t-s)^{\underline{\nu}}}{\Gamma(\nu+1)},$$

whenever the right-hand side is defined.

We next recall the following fundamental inequality, which was discovered by JIA, ERBE, and PETERSON [33].

Lemma 2.5. Assume that $f : \mathbb{N}_a \to \mathbb{R}$ satisfies $\Delta_a^{\nu} f(a+3-\nu+k) \ge 0$ for each $k \in \mathbb{N}_0$, where $2 < \nu < 3$. Then it holds that

$$\Delta^2 f(a+k+1) \ge -h_{-\nu}(a+3-\nu+k,a)f(a) - h_{-\nu+1}(a+3-\nu+k,a)\Delta f(a) -\sum_{i=0}^k h_{-\nu+1}(a+3-\nu+k,a+i+1)\Delta^2 f(a+i)$$

for each $k \in \mathbb{N}_0$, where

 $h_{-\nu+1}(a-3-\nu+k, a+i+1) < 0$

for each $i \in \mathbb{N}_0^k$. In addition each of the inequalities

$$h_{-\nu}(a+3-\nu+k,a) > 0$$
 and $h_{-\nu+1}(a+3-\nu+k,a) < 0$

holds.

Notice that Lemma 2.5 provides a lower bound on $\Delta^2 f(t)$ for $t \in \mathbb{N}_{a+1}$, whenever it holds that $\Delta_a^{\nu} f(t) \geq 0$ for $t \in \mathbb{N}_{a+3-\nu}$. This inequality is fundamental in the investigation of the convexity or concavity of f, for instance, since it thus allows us to gain control over the second-difference of f whenever we know that the fractional difference is nonnegative.

2.2. Case I. $1<\mu<2, 1<\nu<2, 2<\mu+\nu<3$

Throughout this subsection we make the following assumptions, though we state these in the various results for convenience.

- $1 < \mu < 2$
- $1 < \nu < 2$
- $2 < \mu + \nu < 3$

We now present an analogue of Lemma 2.5 for the sequential case for the choices of μ and ν in this subsection. As a comparison of Lemma 2.6 to Lemma 2.5 reveals, by considering the sequential-type difference an additional layer of complexity is added, and, in particular, the two fundamental inequalities are not identical.

Lemma 2.6. Assume that $1 < \mu < 2$, $1 < \nu < 2$, and $2 < \mu + \nu < 3$. Suppose that $\Delta_{2+a-\mu}^{\nu} \Delta_a^{\mu} f(t) \ge 0$, for each $t \in \mathbb{N}_{4+a-\mu-\nu}$. Then it holds that

$$\begin{split} &\Delta^2 f(a+k+2) \\ &\geq -h_{-\mu-\nu}(4+a-\mu-\nu+k,a)f(a) - h_{-\mu-\nu+1}(4+a-\mu-\nu+k,a)\Delta f(a) \\ &\quad -\sum_{s=a}^{a+1+k} h_{-\mu-\nu+1}(4+a-\mu-\nu+k,s+1)\Delta^2 f(s) \\ &\quad + \left[h_{-\nu-2}(2+a-\nu+k,a) + h_{-\nu-1}(2+a-\nu+k,a)(1-\mu)\right]f(a) \\ &\quad + h_{-\nu-1}(2+a-\nu+k,a)f(a+1), \end{split}$$

for each $k \in \mathbb{N}_0$.

Proof. Our proof is similar in character to that of Lemma 2.5 as provided by JIA, et al. in [**33**]. But due to the sequential difference, we require some estimates that do not arise in the non-sequential setting studied in [**33**].

We begin by writing

$$(4) \quad 0 \leq \Delta_{2+a-\mu}^{\nu} \Delta_{a}^{\mu} f(t) = \Delta_{a}^{\mu+\nu} f(t) - h_{-\nu-2}(t-2+\mu,a) \Delta_{a}^{-2+\mu} f(a+2-\mu) - h_{-\nu-1}(t-2+\mu,a) \Delta_{a}^{-1+\mu} f(a+2-\mu) = \frac{1}{\Gamma(-\mu-\nu)} \int_{a}^{t+\mu+\nu+1} (t-s-1) \frac{-\mu-\nu-1}{2} f(s) \Delta s - h_{-\nu-2}(t-2+\mu,a) \Delta_{a}^{-2+\mu} f(2+a-\mu) - h_{-\nu-1}(t-2+\mu,a) \Delta_{a}^{-1+\mu} f(2+a-\mu),$$

where we use the fact that (see, for example, [30])

$$\begin{aligned} \Delta_a^{\mu+\nu} f(t) &= \int_a^{t+\mu+\nu+1} h_{-\mu-\nu-1}(t,s+1) f(s) \ \Delta s \\ &= \frac{1}{\Gamma(-\mu-\nu)} \int_a^{t+\mu+\nu+1} (t-s-1)^{-\mu-\nu-1} f(s) \ \Delta s. \end{aligned}$$

Recalling both that $\mu - 2 < 0$ and $\mu - 1 > 0$, we note both that

(5)
$$\Delta_a^{-2+\mu} f(a+2-\mu) = \frac{1}{\Gamma(2-\mu)} \sum_{s=a}^a (a+1-\mu-s)^{1-\mu} f(s) = f(a)$$

and that

(6)
$$\Delta_a^{-1+\mu}f(a+2-\mu) = \frac{1}{\Gamma(1-\mu)}\sum_{s=a}^{a+1}(a+1-s-\mu)^{-\mu}f(s) = f(a+1)+(1-\mu)f(a).$$

Thus, putting (5)-(6) into (4) yields

(7)
$$0 \leq \frac{1}{\Gamma(-\mu-\nu)} \int_{a}^{t+\mu+\nu+1} (t-s-1)^{-\mu-\nu-1} f(s) \Delta s - \left[h_{-\nu-2}(t-2+\mu,a) + h_{-\nu-1}(t-2+\mu,a)(1-\mu)\right] f(a) - h_{-\nu-1}(t-2+\mu,a) f(a+1).$$

Now, using integration by parts twice we estimate

(8)
$$\int_{a}^{t+\mu+\nu+1} h_{-\mu-\nu-1}(t,s+1)f(s) \Delta s$$
$$= -h_{-\mu-\nu}(t,s)f(s)\Big|_{s=a}^{s=t+\mu+\nu+1} + \int_{a}^{t+\mu+\nu} h_{-\mu-\nu}(t,s+1)\Delta f(s) \Delta s$$
$$= h_{-\nu-\mu}(t,a)f(a) - \left[h_{-\mu-\nu+1}(t,s)\Delta f(s)\right]_{s=a}^{s=t+\mu+\nu}$$

$$\begin{split} &+ \int_{a}^{t+\mu+\nu-1} h_{-\mu-\nu+1}(t,s+1) \Delta^2 f(s) \ \Delta s \\ &= h_{-\mu-\nu}(t,a) f(a) + h_{-\mu-\nu+1}(t,a) \Delta f(a) \\ &+ \int_{a}^{t+\mu+\nu-1} h_{-\mu-\nu+1}(t,s+1) \Delta^2 f(s) \ \Delta s. \end{split}$$

Consequently, putting (8) into (7) yields the estimate

(9)
$$0 \le h_{-\mu-\nu}(t,a)f(a) + h_{-\mu-\nu+1}(t,a)\Delta f(a) + \int_{a}^{t+\mu+\nu-1} h_{-\mu-\nu+1}(t,s+1)\Delta^{2}f(s) \Delta s - \left[h_{-\nu-2}(t-2+\mu,a) + h_{-\nu-1}(t-2+\mu,a)(1-\mu)\right]f(a) - h_{-\nu-1}(t-2+\mu,a)f(a+1).$$

Recall that, by definition,

$$\int_{a}^{t+\mu+\nu-1} h_{-\mu-\nu+1}(t,s+1)\Delta^{2}f(s) \ \Delta s = \sum_{s=a}^{t+\mu+\nu-2} h_{-\mu-\nu+1}(t,s+1)\Delta^{2}f(s).$$

Note, in addition, that for $t \in \mathbb{N}_{4+a-\mu-\nu}$ we can realize the number t in the form $t = 4 + a - \mu - \nu + k$ for $k \in \mathbb{N}_0$. Then combining these two facts we see that inequality (9) can be cast in the form

(10)
$$0 \leq h_{-\mu-\nu}(4+a-\mu-\nu+k,a)f(a) + h_{-\mu-\nu+1}(4+a-\mu-\nu+k,a)\Delta f(a) + \sum_{s=a}^{a+2+k} h_{-\mu-\nu+1}(4+a-\mu-\nu+k,s+1)\Delta^2 f(s) - [h_{-\nu-2}(2+a-\nu+k,a) + h_{-\nu-1}(2+a-\nu+k,a)(1-\mu)]f(a) - h_{-\nu-1}(2+a-\nu+k,a)f(a+1),$$

for each $k \in \mathbb{N}_0$. Next notice that

(11)
$$h_{-\mu-\nu+1}(4+a-\mu-\nu+k,s+1)\Big|_{s=2+a+k} = \frac{(4+a-\mu-\nu+k-(2+a+k)-1)^{-\mu-\nu+1}}{\Gamma(-\mu-\nu+2)} = 1.$$

By putting (11) into estimate (10) and rearranging terms we deduce that

$$\begin{split} &\Delta^2 f(a+k+2) \\ &\geq -h_{-\mu-\nu}(4+a-\mu-\nu+k,a)f(a)-h_{-\mu-\nu+1}(4+a-\mu-\nu+k,a)\Delta f(a) \\ &\quad -\sum_{s=a}^{a+1+k} h_{-\mu-\nu+1}(4+a-\mu-\nu+k,s+1)\Delta^2 f(s) \end{split}$$

+
$$[h_{-\nu-2}(2+a-\nu+k,a)+h_{-\nu-1}(2+a-\nu+k,a)(1-\mu)]f(a)$$

+ $h_{-\nu-1}(2+a-\nu+k,a)f(a+1),$

for each $k \in \mathbb{N}_0$. And this completes the proof.

REMARK 2.7. A comparison of Lemma 2.6 to Lemma 2.5 reveals two, key dissimilarities.

• First of all, due to the sequential difference we pick up the extra terms

$$[h_{-\nu-2}(2+a-\nu+k,a)+h_{-\nu-1}(2+a-\nu+k,a)(1-\mu)]f(a) + h_{-\nu-1}(2+a-\nu+k,a)f(a+1).$$

This term does not occur in the non-sequential fundamental inequality.

• Second of all, the sequentialization causes the fundamental inequality to yield a lower bound for $\Delta^2 f(t)$ on the set \mathbb{N}_{a+2} . By contrast, in the non-sequential setting (again, Lemma 2.5) we see that the lower bound for $\Delta^2 f(t)$ is valid on the slightly larger set \mathbb{N}_{a+1} .

All in all, then, the sequentialization of the difference results in a similar but, nonetheless, altered fundamental inequality.

With the sequential analogue of Lemma 2.5 in hand, we now state and prove our convexity result for sequential differences.

Theorem 2.8. Assume that $1 < \mu < 2$, $1 < \nu < 2$, and $2 < \mu + \nu < 3$. Let $f : \mathbb{N}_a \to \mathbb{R}$ be a map satisfying the inequality

(12)
$$-h_{-\mu-\nu}(4+a-\mu-\nu+k,a)f(a) - h_{-\mu-\nu+1}(4+a-\mu-\nu+k,a)\Delta f(a) -\sum_{s=a}^{a+1} h_{-\mu-\nu+1}(4+a-\mu-\nu+k,s+1)\Delta^2 f(s) + \left[h_{-\nu-2}(2+a-\nu+k,a) + h_{-\nu-1}(2+a-\nu+k,a)(1-\mu)\right]f(a) + h_{-\nu-1}(2+a-\nu+k,a)f(a+1) \ge 0,$$

for each $k \in \mathbb{N}_0$. If, in addition, it holds that $\Delta_{2+a-\mu}^{\nu} \Delta_a^{\mu} f(t) \ge 0$, for each $t \in \mathbb{N}_{4+a-\mu-\nu}$, then $\Delta^2 f(t) \ge 0$, for each $t \in \mathbb{N}_{a+2}$.

Proof. The proof is a straightforward induction argument. The principal observation is that we may compute

(13)
$$h_{-\mu-\nu+1}(4+a-\mu-\nu+k,s+1) = \frac{(3+a-\mu-\nu+k-s)^{-\mu-\nu+1}}{\Gamma(-\mu-\nu+2)}$$
$$= \frac{\Gamma(4+a-\mu-\nu+k-s)}{\Gamma(-\mu-\nu+2)\Gamma(3+a+k-s)},$$

for each $k \in \mathbb{N}_0$ and $s \in \mathbb{N}_a^{a+1+k}$. Since $a \leq s \leq a+1+k$ and $k \geq 0$, it follows from (13) that $\Gamma(4 + a - \mu - \nu + k - s) > 0$ for all admissible k and s. Similarly, $\Gamma(3 + a + k - s) > 0$. Since $\Gamma(-\mu - \nu + 2) < 0$, we conclude that $h_{-\mu-\nu+1}(4 + a - \mu - \nu + k, s + 1) < 0$, for each admissible k and s. To complete the actual induction argument, we first note that from Lemma 2.6 (in case k = 0) we may estimate

$$\begin{split} \Delta^2 f(a+2) &\geq -h_{-\mu-\nu}(4+a-\mu-\nu,a)f(a) - h_{-\mu-\nu+1}(4+a-\mu-\nu,a)\Delta f(a) \\ &- \sum_{s=a}^{a+1} h_{-\mu-\nu+1}(4+a-\mu-\nu,s+1)\Delta^2 f(s) \\ &+ \left[h_{-\nu-2}(2+a-\nu,a) + h_{-\nu-1}(2+a-\nu,a)(1-\mu)\right]f(a) \\ &+ h_{-\nu-1}(2+a-\nu,a)f(a+1) \geq 0, \end{split}$$

invoking inequality (12). Thus, $\Delta^2 f(a+2) \ge 0$ holds. But we then can write (utilizing Lemma 2.6 once more)

$$(14) \qquad \Delta^{2} f(a+3) \\ \geq \left[-h_{-\mu-\nu}(5+a-\mu-\nu,a)f(a) - h_{-\mu-\nu+1}(5+a-\mu-\nu,a)\Delta f(a) - \sum_{s=a}^{a+1} h_{-\mu-\nu+1}(5+a-\mu-\nu,s+1)\Delta^{2} f(s) + \left[h_{-\nu-2}(3+a-\nu+,a) + h_{-\nu-1}(3+a-\nu,a)(1-\mu) \right] f(a) + h_{-\nu-1}(3+a-\nu,a)f(a+1) \right] \\ - \frac{h_{-\mu-\nu+1}(5+a-\mu-\nu,a+3)\Delta^{2} f(a+2)}{>0} \geq 0,$$

where the final term is nonnegative due to the observations in the first paragraph of this proof, whereas the quantity in the brackets is nonnegative due to inequality (12).

Finally, repeating the calculation in (14) we obtain by induction that

$$\Delta^2 f(k) \ge 0$$

for each $k \in \mathbb{N}_{a+2}$. And this completes the proof.

If we are willing to impose some additional hypotheses on the values f(k) for $k \in \mathbb{N}_0^3$, then it is possible to obtain corollaries to Theorem 2.8 that are possibly simpler to apply in practice. We state and prove next a couple representative results in this direction.

Corollary 2.9. Assume that $1 < \mu < 2$, $1 < \nu < 2$, and $2 < \mu + \nu < 3$. Suppose that $f : \mathbb{N}_a \to \mathbb{R}$ satisfies

- 1. $\Delta_{2+a-\mu}^{\nu}\Delta_a^{\mu}f(t) \ge 0$, for $t \in \mathbb{N}_{4+a-\mu-\nu}$;
- 2. $f(a) \le 0;$

- 3. $\Delta^2 f(a+1) \ge 0$; and
- 4. for each $k \in \mathbb{N}_0$

(15)
$$\frac{(2-\mu-\nu)\Gamma(4-\mu-\nu+k)}{(k+2)(k+1)!\Gamma(2-\mu-\nu)}f(a) + \left[\frac{(\mu+\nu+k)\Gamma(4-\mu-\nu+k)}{(k+2)(k+2)!\Gamma(2-\mu-\nu)} + \frac{\Gamma(3-\nu+k)}{(k+3)!\Gamma(-\nu)}\right]f(a+1) - \frac{\Gamma(4-\mu-\nu+k)}{(k+2)!\Gamma(2-\mu-\nu)}f(a+2) \ge 0.$$

Then $\Delta^2 f(t) \ge 0$ for each $t \in \mathbb{N}_{a+2}$.

Proof. We begin by noting that

(16)
$$h_{-\mu-\nu}(4+a-\mu-\nu+k,a) = \frac{(4-\mu-\nu+k)^{-\mu-\nu}}{\Gamma(-\mu-\nu+1)} = \frac{\Gamma(5-\mu-\nu+k)}{\Gamma(-\mu-\nu+1)\Gamma(5+k)} > 0,$$

for each $k \in \mathbb{N}_0$. In particular, then, since $f(a) \leq 0$, from (16) we see that

(17)
$$-h_{-\mu-\nu}(4+a-\mu-\nu+k,a)f(a) \ge 0,$$

for each $k \in \mathbb{N}_0$. Similarly, since $\Delta^2 f(a+1) \ge 0$, it follows that

$$-\sum_{s=a}^{a+1} h_{-\mu-\nu+1}(4+a-\mu-\nu+k,s+1)\Delta^2 f(s)$$

$$\geq -h_{-\mu-\nu+1}(4+a-\mu-\nu+k,a+1)\Delta^2 f(a),$$

for each $k \in \mathbb{N}_0$. Finally, we note that

$$h_{-\nu-2}(2+a-\nu+k,a) + h_{-\nu-1}(2+a-\nu+k,a)(1-\mu)$$

= $\frac{(2-\nu+k)^{-\nu-2}}{\Gamma(-\nu-1)} + \frac{(2-\nu+k)^{-\nu-1}}{\Gamma(-\nu)}(1-\mu)$
= $\underbrace{\frac{\Gamma(3-\nu+k)}{\Gamma(-\nu-1)\Gamma(5+k)}}_{<0} + \underbrace{\frac{\Gamma(3-\nu+k)}{\Gamma(-\nu)\Gamma(4+k)}}_{>0} \underbrace{(1-\mu)}_{<0} < 0,$

for each $k \in \mathbb{N}_0$, from which it follows that

(18)
$$\left[h_{-\nu-2}(2+a-\nu+k,a) + h_{-\nu-1}(2+a-\nu+k,a)(1-\mu) \right] f(a) \ge 0.$$

With the preceding calculations in mind, we can then estimate

$$-h_{-\mu-\nu}(4+a-\mu-\nu+k,a)f(a) - h_{-\mu-\nu+1}(4+a-\mu-\nu+k,a)\Delta f(a)$$

$$-\sum_{s=a}^{a+1} h_{-\mu-\nu+1}(4+a-\mu-\nu+k,s+1)\Delta^2 f(s) + \left[h_{-\nu-2}(2+a-\nu+k,a)+h_{-\nu-1}(2+a-\nu+k,a)(1-\mu)\right]f(a) + h_{-\nu-1}(2+a-\nu+k,a)f(a+1) \geq -h_{-\mu-\nu+1}(4+a-\mu-\nu+k,a)\Delta f(a) - h_{-\mu-\nu+1}(4+a-\mu-\nu+k,a+1)\Delta^2 f(a) + h_{-\nu-1}(2+a-\nu+k,a)f(a+1) \geq 0,$$

for each $k \in \mathbb{N}_0$, where the final inequality follows from inequality (15). Since the above estimate holds for each $k \in \mathbb{N}_0$ and, in addition, we are assuming that $\Delta_{2+a-\mu}^{\nu}\Delta_{a}^{\mu}f(t) \geq 0$, for each $t \in \mathbb{N}_{4+a-\mu-\nu}$, we see that the hypotheses of Theorem 2.8 are satisfied. Thus, we conclude that $\Delta^{2}f(t) \geq 0$, for each $t \in \mathbb{N}_{a+2}$, as claimed.

Corollary 2.10. Assume that $1 < \mu < 2$, $1 < \nu < 2$, and $2 < \mu + \nu < 3$. Suppose that $f : \mathbb{N}_a \to \mathbb{R}$ satisfies

- 1. $\Delta_{2+a-\mu}^{\nu}\Delta_a^{\mu}f(t) \ge 0$, for $t \in \mathbb{N}_{4+a-\mu-\nu}$;
- 2. $f(a) \le 0;$
- 3. $f(a+1) \ge 0$; and
- 4. for each $k \in \mathbb{N}_0$

$$-h_{-\mu-\nu+1}(4+a-\mu-\nu+k,a+1)\Delta^2 f(a) -h_{-\mu-\nu+1}(4+a-\mu-\nu+k,a+2)\Delta^2 f(a+1) \ge 0.$$

Then $\Delta^2 f(t) \geq 0$ for each $t \in \mathbb{N}_{a+2}$.

Proof. Similar to the proof of Corollary 2.9, we begin with some preliminary calculations. In particular, since $f(a) \leq 0$ is again assumed, we again obtain estimates (17)-(18). Note also that

$$h_{-\nu-1}(2+a-\nu+k,a) = \frac{(2-\nu+k)^{-\nu-1}}{\Gamma(-\nu)} = \frac{\Gamma(3-\nu+k)}{\Gamma(-\nu)\Gamma(4+k)} > 0$$

for each $k \in \mathbb{N}_0$. Therefore, since, by assumption, $f(a+1) \ge 0$, it follows that

$$h_{-\nu-1}(2+a-\nu+k,a)f(a+1) \ge 0,$$

for each $k \in \mathbb{N}_0$. Finally, since

$$h_{-\mu-\nu+1}(4+a-\mu-\nu+k,a) = \frac{(4-\mu-\nu+k)^{-\mu-\nu+1}}{\Gamma(-\mu-\nu+2)}$$
$$= \frac{\Gamma(5-\mu-\nu+k)}{\Gamma(-\mu-\nu+2)\Gamma(4+k)} < 0$$

it thus follows that

$$-h_{-\mu-\nu+1}(4+a-\mu-\nu+k,a)\Delta f(a) \ge 0,$$

where we have used the fact that since $f(a) \leq 0$ and $f(a+1) \geq 0$, it holds that $\Delta f(a) \geq 0$.

We thus conclude that

$$(19) - h_{-\mu-\nu}(4 + a - \mu - \nu + k, a)f(a) - h_{-\mu-\nu+1}(4 + a - \mu - \nu + k, a)\Delta f(a) - \sum_{s=a}^{a+1} h_{-\mu-\nu+1}(4 + a - \mu - \nu + k, s + 1)\Delta^2 f(s) + [h_{-\nu-2}(2 + a - \nu + k, a) + h_{-\nu-1}(2 + a - \nu + k, a)(1 - \mu)]f(a) + h_{-\nu-1}(2 + a - \nu + k, a)f(a + 1) \geq - \sum_{s=a}^{a+1} h_{-\mu-\nu+1}(4 + a - \mu - \nu + k, s + 1)\Delta^2 f(s) \geq 0,$$

for each $k \in \mathbb{N}_0$. Thus, as in the proof of Corollary 2.9 we may invoke Theorem 2.8 to deduce that $\Delta^2 f(t) \ge 0$ for each $t \in \mathbb{N}_{a+2}$, as claimed.

We conclude with an example to illustrate the application of the preceding results. Our example illustrates that a function $f : \mathbb{N}_0 \to \mathbb{R}$ may satisfy the conditions

- 1. $f(0) \le 0;$
- 2. $\Delta f(0) \ge 0;$
- 3. $\Delta^2 f(0) \ge 0$; and
- 4. $\Delta_{2-\mu}^{\nu} \Delta_0^{\mu} f(t) \ge 0$, for $t \in \mathbb{N}_{4-\mu-\nu}^{5-\mu-\nu}$

and yet satisfy

$$\Delta^2 f(1) < 0$$
 and $\Delta^2 f(2) < 0$

The significance of this is that in the non-sequential setting, it was shown by both GOODRICH [27] and JIA, et al. [33] that the conditions (1)-(3) above together with the condition $\Delta_0^{\mu} f(t) \ge 0$, for $t \in \mathbb{N}_{3-\mu}$, was sufficient to guarantee that $\Delta^2 f(t) \ge 0$ for $t \in \mathbb{N}_1$. Here, by contrast, we see that replacing the non-sequential difference with the sequential difference results in an apparently different level of control. More specifically, we **cannot** necessarily expect the function to be convex on \mathbb{N}_1 or even \mathbb{N}_2 . The sequential difference simply does not provide that degree of control over the convexity of the map f.

EXAMPLE 2.11. Set $\nu = 1.25$ and $\mu = 1.25$; note that $\mu + \nu = 2.5 \in (2,3)$. Consider the function $f : \mathbb{N}_0 \to \mathbb{R}$ with values given in the following table.

ĺ	t	0	1	2	3	4	5
	f(t)	-1	-0.5	0.1	0.5	0.88	1.254

Direct calculation reveals that f(0) < 0, $\Delta f(0) > 0$, $\Delta^2 f(0) > 0$, and

 $\Delta_{0.75}^{1.25} \Delta_0^{1.25} f(t) > 0$, for $t \in \mathbb{N}_{1.5}^{2.5}$.

Yet direct calculation also immediately reveals that $\Delta^2 f(k) < 0$ for $k \in \mathbb{N}_1^3$. In other words, no matter how one defines f on \mathbb{N}_6 , it does **not** hold that $\Delta^2 f(t) > 0$ on \mathbb{N}_1 nor, for that matter, on \mathbb{N}_2 or even \mathbb{N}_3 . And so this demonstrates dissimilarities between the control that the non-sequential and sequential fractional differences possess over the convexity of f.

2.3. Case II. $0<\mu<1, 1<\nu<2, 2<\mu+\nu<3$

Throughout this subsection we make the following assumptions.

- $0 < \mu < 1$
- $1 < \nu < 2$
- $2 < \mu + \nu < 3$

Thus, the only alteration from the assumptions in the last subsection is that instead of $\mu \in (1, 2)$, here we assume that $\mu \in (0, 1)$. As we will see, the analogue of Lemma 2.6 is somewhat simpler in this case. However, it still is not the same as in the non-sequential setting. So, to begin, we derive the analogue of Lemma 2.6 in this case and then, as in the previous subsection, we derive some convexity-type results in light of the fundamental inequality.

Lemma 2.12. Assume that $0 < \mu < 1$, $1 < \nu < 2$, and $2 < \mu + \nu < 3$. Suppose that $\Delta^{\nu}_{1+a-\mu}\Delta^{\mu}_{a}f(t) \ge 0$, for each $t \in \mathbb{N}_{3+a-\mu-\nu}$. Then it holds that

(20)
$$\Delta^{2} f(a+k+1) \geq -h_{-\mu-\nu}(a+3-\mu-\nu+k,a)f(a) \\ -h_{-\mu-\nu+1}(a+3-\mu-\nu+k,a)\Delta f(a) \\ -\sum_{s=a}^{a+k} h_{-\mu-\nu+1}(a+3-\mu-\nu+k,s+1)\Delta^{2} f(s) \\ +h_{-\nu-1}(a+2-\nu+k,a)f(a),$$

for each $k \in \mathbb{N}_0$.

Proof. The proof of this lemma is very similar to the proof of Lemma 2.6; in fact, it is somewhat simpler. Therefore, we omit some of the details.

We note first that by Theorem 1.1 we obtain

(21)
$$\Delta_{1+a-\mu}^{\nu}\Delta_{a}^{\mu}f(t) = \Delta_{a}^{\mu+\nu}f(t) - h_{-\nu-1}(t-1+\mu,a)\Delta_{a}^{\mu-1}f(a+1-\mu).$$

Since we compute

(22)
$$\Delta_a^{\mu-1} f(a+1-\mu) = \frac{1}{\Gamma(1-\mu)} \sum_{s=a}^a (a-\mu-s)^{-\mu} f(s) = f(a),$$

we see that upon combining (21)-(22) we obtain the estimate

$$0 \leq \Delta_{a}^{\mu+\nu} f(t) - h_{-\nu-1}(t-1+\mu,a)f(a)$$

= $\int_{a}^{t+\mu+\nu+1} h_{-\mu-\nu-1}(t,s+1)f(s) \Delta s - h_{-\nu-1}(t-1+\mu,a)f(a)$
= $h_{-\mu-\nu}(t,a)f(a) + h_{-\mu-\nu+1}(t,a)\Delta f(a) + \sum_{s=a}^{t+\mu+\nu-2} h_{-\mu-\nu+1}(t,s+1)\Delta^{2}f(s)$
 $- h_{-\nu-1}(t-1+\mu,a)f(a).$

Similar to the proof of Lemma 2.5, since $t \in \mathbb{N}_{3+a-\mu-\nu}$, we may realize t in the form $t = a+3-\mu-\nu+k$, for $k \in \mathbb{N}_0$. Doing this then allows us to recast the above inequality in the form

$$\begin{split} \Delta^2 f(a+1+k) &\geq -h_{-\mu-\nu}(a+3-\mu-\nu+k,a)f(a) \\ &\quad -h_{-\mu-\nu+1}(a+3-\mu-\nu+k,a)\Delta f(a) \\ &\quad -\sum_{s=a}^{a+k} h_{-\mu-\nu+1}(a+3-\mu-\nu+k,s+1)\Delta^2 f(s) \\ &\quad +h_{-\nu-1}(a+2-\nu+k,a)f(a), \end{split}$$

for each $k \in \mathbb{N}_0$, as claimed. And this completes the proof.

REMARK 2.13. So, for this particular sequentialized fractional difference we see, by comparing Lemma 2.5 to Lemma 2.12, that the additional term acquired here is

$$h_{-\nu-1}(a+2-\nu+k,a)f(a).$$

We also see that that in this case the fundamental inequality, while more complicated than the non-sequential case, is somewhat simpler than the sequential case in which μ , $\nu \in (1, 2)$.

REMARK 2.14. A second important dissimilarity between the two cases studied here is that, as we previously saw, the fundamental inequality in case $\mu \in (1, 2)$ only yields potential information about the convexity of $t \mapsto f(t)$ for $t \in \mathbb{N}_{a+2}$. By contrast, Lemma 2.12 shows that in case $\mu \in (0, 1)$ we can obtain information about the convexity of $t \mapsto f(t)$ for $t \in \mathbb{N}_{a+1} \supset \mathbb{N}_{a+2}$.

Now we provide a basic convexity-type result that follows from Lemma 2.12.

Theorem 2.15. Assume that $0 < \mu < 1$, $1 < \nu < 2$, and $2 < \mu + \nu < 3$. Suppose that $\Delta^{\nu}_{1+a-\mu}\Delta^{\mu}_{a}f(t) \ge 0$, for each $t \in \mathbb{N}_{3+a-\mu-\nu}$. In addition, suppose that

(23)
$$\begin{bmatrix} -h_{-\mu-\nu}(a+3-\mu-\nu+k,a)+h_{-\nu-1}(a+2-\nu+k,a) \end{bmatrix} f(a) -h_{-\mu-\nu+1}(a+3-\mu-\nu+k,a)\Delta f(a) -h_{-\mu-\nu+1}(a+3-\mu-\nu+k,a+1)\Delta^2 f(a) \ge 0$$

for each $k \in \mathbb{N}_0$. Then $\Delta^2 f(t) \ge 0$ for each $t \in \mathbb{N}_{a+1}$.

Proof. Essentially we proceed exactly as in the proof of Theorem 2.8. First notice that

(24)
$$-h_{-\mu-\nu+1}(a+3-\mu-\nu+k,a+1) = -\frac{(2-\mu-\nu+k)^{-\mu-\nu+1}}{\Gamma(-\mu-\nu+2)}$$
$$= -\frac{\Gamma(3-\mu-\nu+k)}{\Gamma(-\mu-\nu+2)\Gamma(2+k)} > 0$$

for each $k \in \mathbb{N}_0$. Next suppose that $\Delta^2 f(k) \ge 0$ for each $k \in \mathbb{N}_{a+1}^{a+1+k_0}$ for some $k_0 \in \mathbb{N}_0$. Then by both Lemma 2.12 and inequality (24) we estimate

$$(25) \quad \Delta^{2} f(a+k_{0}+2) \geq -h_{-\mu-\nu}(a+4-\mu-\nu+k_{0},a)f(a) \\ -h_{-\mu-\nu+1}(a+4-\mu-\nu+k_{0},a)\Delta f(a) \\ -\sum_{s=a}^{a+k_{0}+1} h_{-\mu-\nu+1}(a+4-\mu-\nu+k_{0},s+1)\Delta^{2} f(s) \\ +h_{-\nu-1}(a+3-\nu+k_{0},a)f(a) \\ \geq -h_{-\mu-\nu}(a+4-\mu-\nu+k_{0},a)f(a) \\ -h_{-\mu-\nu+1}(a+4-\mu-\nu+k_{0},a+1)\Delta^{2} f(a) \\ -h_{-\mu-\nu+1}(a+3-\nu+k_{0},a)f(a). \end{aligned}$$

But since the right-hand side of (25) is nonnegative for each $k \in \mathbb{N}_0$ by assumption (23), it follows that $\Delta^2(a + k_0 + 2) \ge 0$. By the arbitrariness of $k \in \mathbb{N}_0$ together with the principle of strong induction the proof is complete.

REMARK 2.16. As in Remark 2.14, we note that Theorem 2.15 allows us to conclude that $\Delta^2 f(t) \geq 0$ on \mathbb{N}_{a+1} , whereas Theorem 2.8 allows us to conclude only that $\Delta^2 f(t) \geq 0$ on \mathbb{N}_{a+2} . This is another dissimilarity between the setting in which μ , $\nu \in (1, 2)$ and in which $\mu \in (0, 1)$ and $\nu \in (1, 2)$.

If one prefers, by rephrasing inequality (23), one may obtain the following corollary to Theorem 2.15. We state the corollary without proof since it follows by simplifying and rewriting inequality (23).

Corollary 2.17. Assume that $0 < \mu < 1$, $1 < \nu < 2$, and $2 < \mu + \nu < 3$. Suppose that $\Delta_{1+a-\mu}^{\nu} \Delta_a^{\mu} f(t) \ge 0$, for each $t \in \mathbb{N}_{3+a-\mu-\nu}$. In addition, suppose that

(26)
$$\frac{1}{\mu+\nu-1}f(a+2) - \frac{\mu+\nu+k+1}{(\mu+\nu-1)(k+2)}f(a+1) + \left[\frac{\mu+\nu}{(k+2)(k+3)} + \frac{\Gamma(3-\nu+k)\Gamma(-\mu-\nu+1)}{(k+2)(k+3)\Gamma(-\nu)\Gamma(3-\mu-\nu+k)}\right]f(a) \ge 0,$$

for each $k \in \mathbb{N}_0$. Then $\Delta^2 f(t) \ge 0$ for each $t \in \mathbb{N}_{a+1}$.

REMARK 2.18. One may note that inequality (26) in Corollary 2.17 is very similar to inequality given in [29, Theorem 2]. In particular, the term

$$\frac{\Gamma(3-\nu+k)\Gamma(-\mu-\nu+1)}{(k+2)(k+3)\Gamma(-\nu)\Gamma(3-\mu-\nu+k)}f(a)$$

is the additional term acquired by using a sequentialized difference rather than, as in [29], a non-sequential difference. It is interesting to note, in this light, that

(27)
$$\frac{\Gamma(3-\nu+k)\Gamma(-\mu-\nu+1)}{(k+2)(k+3)\Gamma(-\nu)\Gamma(3-\mu-\nu+k)} > 0,$$

for each $k \in \mathbb{N}_0$. This is of note since, at the same time,

(28)
$$\frac{\mu + \nu}{(k+2)(k+3)} > 0$$

for each $k \in \mathbb{N}_0$. Due to (27)–(28) it follows that if f(a) > 0, then

(29)
$$\left[\frac{\mu + \nu}{(k+2)(k+3)} + \frac{\Gamma(3-\nu+k)\Gamma(-\mu-\nu+1)}{(k+2)(k+3)\Gamma(-\nu)\Gamma(3-\mu-\nu+k)} \right] f(a)$$

$$> \frac{\mu + \nu}{(k+2)(k+3)} f(a) > 0,$$

whereas if f(a) < 0, then

(30)
$$\left[\frac{\mu + \nu}{(k+2)(k+3)} + \frac{\Gamma(3-\nu+k)\Gamma(-\mu-\nu+1)}{(k+2)(k+3)\Gamma(-\nu)\Gamma(3-\mu-\nu+k)} \right] f(a) < \frac{\mu + \nu}{(k+2)(k+3)} f(a) < 0.$$

All in all, (26) and (29)–(30) seem to suggest that in case f(a) > 0 it should be easier for (26) to be verified, whereas in case f(a) < 0 it should be more difficult for (26) to be verified. In other words, it seems that when f(a) > 0 acquiring convexity may be slightly "easier" in the *sequential* than non-sequential setting, whereas when f(a) < 0 it may be slight "easier" in the *non-sequential* than sequential setting. Investigating these relationships could be an interesting avenue for future investigation.

As in the previous subsection we continue with a corollary that specializes both Theorem 2.15 and Corollary 2.17 in case we are willing to make assumptions about the sign of f(k) for $k \in \mathbb{N}_0^2$. In practice, this may be simpler to apply than either Theorem 2.15 or Corollary 2.17 directly.

Corollary 2.19. Assume that $0 < \mu < 1$, $1 < \nu < 2$, and $2 < \mu + \nu < 3$. Suppose that $\Delta_{1+a-\mu}^{\nu} \Delta_a^{\mu} f(t) \ge 0$ for each $t \in \mathbb{N}_{3+a-\mu-\nu}$. In addition, suppose that

1. f(a) > 0; and

2.
$$f(a+2) \ge \frac{\mu+\nu+k+1}{k+2}f(a+1)$$
, for each $k \in \mathbb{N}_0$.

Then $\Delta^2 f(t) \ge 0$ for each $t \in \mathbb{N}_{a+1}$.

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Proof. We simply observe that under conditions (1)–(2) inequality (26) holds for each $k \in \mathbb{N}_0$. By Corollary 2.17 we conclude that $\Delta^2 f(t) \ge 0$ for each $t \in \mathbb{N}_{a+1}$, as claimed. And this completes the proof.

REMARK 2.20. This particular corollary is not exhaustive, and it is possible to write down additional similar corollaries that follow from Theorem 2.15. But we omit such results here – see [29, Corollaries 1–3] for these types of results in the non-sequential setting.

We conclude this subsection and the paper with a second example and a concluding remark. Similar to Example 2.11 this example demonstrates that when $0 < \mu < 1$ and $1 < \nu < 2$ there apparently remain dissimilarities between the sequential and non-sequential settings insofar as convexity is concerned.

EXAMPLE 2.21. Set $\nu = 1.5$ and $\mu = 0.51$; note that $\mu + \nu = 2.01 \in (2,3)$. Consider the function $f : \mathbb{N}_0 \to \mathbb{R}$ with values given in the following table.

t	0	1	2	3	4
f(t)	-0.5	0	0.6	1.19259	1.793517

Direct calculation reveals that f(0) < 0, $\Delta f(0) > 0$, $\Delta^2 f(0) > 0$, and

$$\Delta_{0.49}^{1.5} \Delta_0^{0.51} f(t) > 0$$
, for $t \in \mathbb{N}_{0.99}^{1.99}$

Yet direct calculation also immediately reveals that $\Delta^2 f(1) < 0$. In other words, in a manner very similar to Example 2.11, no matter how one defines f on \mathbb{N}_5 , it does **not** hold that $\Delta^2 f(t) > 0$ on all of \mathbb{N}_1 . So, once again, by comparing this example to the results on convexity for non-sequential differences (e.g., **[27, 33]**), we observe dissimilarities between the sequential and non-sequential settings.

REMARK 2.22. It is easy to see that if one defines f as in Example 2.21, then $\Delta^2 f(2) > 0$. In fact, if one demands that $\Delta_{0.49}^{1.5} \Delta_0^{0.51} f(2.99) \ge 0$ hold, then one is forced to define f(5) in such a way that $\Delta^2 f(3) > 0$ holds. In particular, one must select $f(5) \ge 2.399712113$. Inductively, then, one may conclude that $\Delta^2 f(t)$ eventually is forced to be positive in this example, but it is **not** necessarily always positive as Example 2.21 demonstrates.

2.4. Case III. $1 < \mu < 2, 0 < \nu < 1, 2 < \mu + \nu < 3$

Throughout this subsection we make the following assumptions.

- $1 < \mu < 2$
- $0 < \nu < 1$
- $2 < \mu + \nu < 3$

It turns out that this case is very similar to the results obtained in Case I. As such, we do not provide many details here.

To begin we obtain the following lemma, which is the analogue of Lemma 2.6. Note that essentially the only dissimilarity between Lemma 2.6 and Lemma 2.23 is that the domains are slightly different.

Lemma 2.23. Assume that $1 < \mu < 2$, $0 < \nu < 1$, and $2 < \mu + \nu < 3$. Suppose that $\Delta_{2+a-\mu}^{\nu} \Delta_a^{\mu} f(t) \ge 0$, for each $t \in \mathbb{N}_{3+a-\mu-\nu}$. Then it holds that

$$\begin{split} &\Delta^2 f(a+k+1) \\ &\geq -h_{-\mu-\nu}(3+a-\mu-\nu+k,a)f(a) - h_{-\mu-\nu+1}(3+a-\mu-\nu+k,a)\Delta f(a) \\ &-\sum_{s=a}^{a+k} h_{-\mu-\nu+1}(3+a-\mu-\nu+k,s+1)\Delta^2 f(s) \\ &+ \left[h_{-\nu-2}(1+a-\nu+k,a) + h_{-\nu-1}(1+a-\nu+k,a)(1-\mu)\right]f(a) \\ &+ h_{-\nu-1}(1+a-\nu+k,a)f(a+1), \end{split}$$

for each $k \in \mathbb{N}_0$.

Proof. Since the proof, save but for essentially trivial alterations, is nearly identical to the proof of Lemma 2.6, we omit it. \Box

Once we have Lemma 2.23 in hand, we can provide the following convexitytype result. This serves as the analogue of Theorem 2.8. Furthermore, just as in the preceding subsections we can provide several corollaries that follow from Theorem 2.24. However, since they are essentially like those presented in Subsection 2.2, we omit their formal statements here.

Theorem 2.24. Assume that $1 < \mu < 2$, $0 < \nu < 1$, and $2 < \mu + \nu < 3$. Let $f : \mathbb{N}_a \to \mathbb{R}$ be a map satisfying the inequality

$$-h_{-\mu-\nu}(3+a-\mu-\nu+k,a)f(a) - h_{-\mu-\nu+1}(3+a-\mu-\nu+k,a)\Delta f(a) -h_{-\mu-\nu+1}(3+a-\mu-\nu+k,a+1)\Delta^2 f(a) + [h_{-\nu-2}(1+a-\nu+k,a) + h_{-\nu-1}(1+a-\nu+k,a)(1-\mu)]f(a) + h_{-\nu-1}(1+a-\nu+k,a)f(a+1) \ge 0,$$

for each $k \in \mathbb{N}_0$. If, in addition, it holds that $\Delta_{2+a-\mu}^{\nu} \Delta_a^{\mu} f(t) \ge 0$, for each $t \in \mathbb{N}_{3+a-\mu-\nu}$, then $\Delta^2 f(t) \ge 0$, for each $t \in \mathbb{N}_{a+1}$.

REMARK 2.25. Notice that Theorem 2.24 is valid on the set \mathbb{N}_{a+1} , which is similar to the results of Subsection 2.3 but *dissimilar* to the results of Subsection 2.2. In particular, this suggests that requiring $1 < \mu < 2$ results in slightly different convexity-type results depending upon whether $0 < \nu < 1$ (as in this subsection) or $1 < \nu < 2$ (as in Subsection 2.2).

We conclude this section and the paper with the following remarks.

REMARK 2.26. Although we have elected to state our results in this paper couched in the convexity-type setting, it is, of course, possible to "reverse" these and thus recast them in terms of concavity-type results. Since this procedure is similar, for example, to that outlined in [27] or [30], we omit the formal statement of these types of results.

REMARK 2.27. It would be interesting to investigate to what degree the results presented herein are sharp (i.e., to what extent, if any, they can be improved). This, we believe, would be an interesting line for future research. Acknowledgments. The author would like to thank the anonymous referee for his or her thoughtful remarks, which led to an improved presentation of this paper.

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