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A SIMPLE NECESSARY AND SUFFICIENT CONDITION FOR THE ENRICHMENT OF THE CROUZEIX-RAVIART ELEMENT

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We provide a simple condition, which is both necessary and sufficient, that guarantees the existence of an enriched Crouzeix-Raviart element. Our main result shows that the latter can be easily expressed in terms of the approximation error in a multivariate generalized trapezoidal type cubature formula. Furthermore, we derive simple explicit formulas for its associated basis functions, and then prove how to use them to characterize all admissible added degrees of freedom, that generate well defined enriched Crouzeix-Raviart elements. We also show that the approximation error using the proposed enriched element can be written as the error of the (non-enriched) Crouzeix-Raviart element plus a perturbation that depends on the enrichment function. Finally, we estimate the approximation error in L^2 norm, with explicit constants in both two and three dimensions. A complement to this result is also given for any dimension.

1. OUR MOTIVATION AND RELATED WORK

The Crouzeix-Raviart element [**7**] is the simplest and possibly the most common non-conforming finite element. It is also known as a simplicial P_1 nonconforming element. It uses piecewise linear polynomials, but unlike the classical conforming P_1 or Courant element [**6**], its degrees of freedom are the average of the function over the facets rather than the values at the vertices. The most promising features of this element are its simplicity, the low degree, and the fact that it is

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able to relax the strong continuity requirement along element interfaces. Many improvements, generalizations, extensions and more references of this element may be found, for instance, in [**3**]. On the non degenerate simplex K in $\mathbb{R}^d$ of a triangulation of a given polytope (that is, subdivided into a collection of simplices), the d-dimensional Crouzeix-Raviart element, $(K, P_1(K), \Sigma_K^{\mathrm{CR}})$ is defined by

1. $P_1(K)$ is the space of polynomials of degree 1 (linear) on K;
2. The Crouzeix-Raviart degrees of freedom are

$$\Sigma_K^{\mathrm{CR}} := \left\{L_i^{\mathrm{CR}}, i = 1, \ldots, d+1\right\}, \tag{1}$$

where

$$L_i^{\mathrm{CR}}(f) \quad = \quad \frac{1}{|F_i|}\int_{F_i} f \,\mathrm{d}\sigma, \;\; i = 1, \ldots, d+1, \tag{2}$$

where $F_1, \ldots, F_{d+1}$ are $d+1$ facets of K. We choose the special enumeration of all facets as follows: for each $j = 1, \ldots, d+1$, F_j is a subset of the hyperplane $\lambda_j(\boldsymbol{x}) = 0$. Here, $\lambda_j, j = 1, \ldots, d+1$ are the barycentric coordinates with respect to the simplex K. Recall that the barycentric coordinates $\lambda_j, j = 1, \ldots, d+1$ are affine functions on K. Throughout the paper, we denote by $|K|$ the d-dimensional Lebesgue measure of K, and by $|F|$ the $(d-1)$-dimensional Lebesgue measure of F, a part of a hyperplane in $\mathbb{R}^d$. The Crouzeix-Raviart basis functions have simple explicit expressions in terms of the barycentric coordinates, see [**8**, section 1.2.6],

$$p_i^{\mathrm{CR}}(\boldsymbol{x}) = d\left(\frac{1}{d} - \lambda_i(\boldsymbol{x})\right), i = 1, \ldots, d+1. \tag{3}$$

In many applications, it may be useful to augment the number of the Crouzeix-Raviart basis functions by adding appropriate functions. Here, we would like to enrich the space $P_1(K)$ with a given enrichment function e^{enr}, as follows:

$$\mathcal{F}_K^{\mathrm{enr}} := P_1(K) \oplus \{e^{\mathrm{enr}}\}. \tag{4}$$

Therefore, starting from the Crouzeix-Raviart local element $\left(K, P_1(K), \Sigma_K^{\mathrm{CR}}\right)$, we construct a modified nonconforming enriched triplet

$$(K, \Sigma_K^{\mathrm{enr}}, \mathcal{F}_K^{\mathrm{enr}}). \tag{5}$$

Here and throughout this paper the set of degrees of freedom Σ_K^{CR} is enriched as follows:

$$\Sigma_K^{\mathrm{enr}} := \Sigma_K^{\mathrm{CR}} \cup \left\{L_{d+2}^{\mathrm{CR}}\right\} = \left\{L_i^{\mathrm{CR}}, i = 1, \ldots, d+2\right\}$$

where L_{d+2}^{CR} has the following general form:

$$L_{d+2}^{\mathrm{CR}}(f) = \frac{\alpha}{|K|}\int_K f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \frac{\beta}{d+1}\sum_{i=1}^{d+1} f(\boldsymbol{v}_i) + (1-\alpha-\beta) f(\boldsymbol{c}), \tag{6}$$

for fixed real numbers α, β and in which $\boldsymbol{v}_i, i = 1 \ldots, d+1$, $\boldsymbol{c}$ are, respectively, the vertices and the center of gravity of K. We will also consider the case when

the Crouzeix-Raviart degrees of freedom is enriched by arbitrary linear functionals, which are more general than those given by (6).

This local enrichment approximation raises the following question: What is a necessary and sufficient condition, which will permit us to select in the enriched triplet $(K, \Sigma_K^{\mathrm{enr}}, \mathcal{F}_K^{\mathrm{enr}})$ the enrichment function e^{enr}, and the added degree of freedom L_{d+2}^{CR}, in such a way that the latter still remains a well-defined finite element?

Our interest in the topic of this paper arose from the fact that, there exist many publications which use certain particular functions for the enrichment of the Crouzeix-Raviart element, see, for instance, [**1, 3, 8, 12, 11, 13, 14**], but none of them has provided a necessary and sufficient condition required for ensuring the existence of the enriched element (5). In what follows, if there is no ambiguity, we will omit the words "with respect to the enrichment function e^{enr} and the degree of freedom L_{d+2}^{CR}".

The paper is organized in the following way. Section 2 contains the statement of our main results (see Theorem 2.2 as well as its generalized version Theorem 2.3). We establish that the enriched Crouzeix-Raviart triplet $(K, \Sigma_K^{\mathrm{enr}}, \mathcal{F}_K^{\mathrm{enr}})$ constitutes a finite element if and only if a generalized trapezoidal type cubature formula for the enrichment function e^{enr} has a nonzero approximation error. This required condition is easy to check and handle for many enrichment functions. Quite surprisingly, this very simple characterization does not seem to appear in the literature. In Section 3, we derive explicit expressions for the computation of the basis functions, and then show how to use them to characterize the set of all possible added degrees of freedom for a given enrichment function, that guarantee the existence of a well-defined enriched Crouzeix-Raviart elements (we call them admissible degrees of freedom). Then the error analysis is presented in Section 4. We show that the approximation error using the proposed element can be expressed in terms of the error of the (non-enriched) Crouzeix-Raviart element plus a perturbation that depends on the enrichment function. We derive explicit bounds on the constants in error estimates of this enriched element for both two and three dimensions. A complement to this result is also given for any dimension. Finally, we end the paper with some remarks about the possibility of extending the results to any conforming or non-conforming P_1 element.

2. THE MAIN RESULT AND ITS GENERALIZED VERSION

This section establishes a link between the fact that the enriched Crouzeix-Raviart triplet $(K, \Sigma_K^{\mathrm{enr}}, \mathcal{F}_K^{\mathrm{enr}})$ constitutes a finite element and the non vanishing of the approximation error of a certain multivariate cubature formulas of generalized trapezoidal type. First, we define the integration formula

$$L_{d+2}^{\mathrm{CR}}(f) = \frac{1}{d+1} \sum_{i=1}^{d+1} \frac{1}{|F_i|} \int_{F_i} f \, \mathrm{d}\sigma + E_d^{\mathrm{tr}}(f). \tag{7}$$

It is natural to call it a multivariate generalized trapezoidal cubature formula, since for the one-dimensional case, $d = 1$, and when $\alpha = 1, \beta = 0$, it coincides with the

well-known trapezoidal rule, which we may formulate on the interval $[a, b]$ as:

$$\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t = \frac{1}{2}\left(f(a)+f(b)\right) + E_1^{\mathrm{tr}}(f). \tag{8}$$

As in the univariate case, one interesting property of the generalized trapezoidal cubature formula (7), which is fundamental to our work is the following simple but key observation:

Lemma 2.1. *The approximation error of the generalized trapezoidal cubature formula E_d^{tr} vanishes for all affine functions.*

Proof. The following basic properties of barycentric coordinates with respect to the simplex K, which will be needed, are taken from [**8**], they can also be found or easily derived from [**5, 9, 10**].

(i) The barycentric coordinates span the space of affine functions $P_1(K)$;

(ii) $L_{d+2}^{\mathrm{CR}}(\lambda_i) = \dfrac{1}{d+1}$, $(i = 1, \ldots, d+1)$;

(iii) $L_i^{\mathrm{CR}}(\lambda_j) = \dfrac{1}{d}(1-\delta_{ij})$, $(i, j = 1, \ldots, d+1)$, where δ_{ij} stands for the Kronecker delta function.

Property (i) implies that it suffices to prove that the error E_d^{tr} vanishes on λ_i for all $i = 1, \ldots, d+1$. But then it follows from properties (ii) and (iii) that

$$\begin{aligned} E_d^{\mathrm{tr}}(\lambda_i) &:= L_{d+2}^{\mathrm{CR}}(\lambda_i) - \frac{1}{d+1}\sum_{j=1}^{d+1} L_j^{\mathrm{CR}}(\lambda_i) \\ &= \frac{1}{d+1} - \frac{1}{d+1}\sum_{j=1}^{d+1}\frac{1}{d}\left(1-\delta_{ij}\right) = 0. \end{aligned}$$

Therefore, the required statement holds. □

We are now in position to state our main result. At this stage we only assume that the enrichment function $e^{\mathrm{enr}} \in L^1(K)$.

Theorem 2.2. *Assume that we are given an enrichment integrable function e^{enr}. Then, the triplet $(K, \Sigma_K^{\mathrm{enr}}, \mathcal{F}_K^{\mathrm{enr}})$ constitutes a finite element if and only if*

$$E_d^{\mathrm{tr}}(e^{\mathrm{enr}}) \neq 0\,. \tag{9}$$

Proof. The proof is divided in two main parts. The first focuses on the sufficiency and the second one the necessity.
Sufficiency: Let us assume that we have a function $f \in \mathcal{F}_K^{\mathrm{enr}}$ such that

$$L_j^{\mathrm{CR}}(f) = 0 \qquad (j = 1, \ldots, d+2). \tag{10}$$

We only need to show that f is identically zero on K. To this end, we first observe that each $f \in \mathcal{F}_K^{\mathrm{enr}}$ may be decomposed into an affine function $p \in P_1(K)$ and into an enriched part αe^{enr}, as:

$$f = p + \alpha e^{\mathrm{enr}},$$

where α is some real number. Clearly the approximation error E_d^{tr} belongs to $\mathrm{span}(\Sigma_K^{\mathrm{enr}})$, then it follows from (10)

$$0 = E_d^{\mathrm{tr}}(f) = E_d^{\mathrm{tr}}(p) + \alpha E_d^{\mathrm{tr}}(e^{\mathrm{enr}}) = \alpha E_d^{\mathrm{tr}}(e^{\mathrm{enr}}),$$

where in the last equality we have used Lemma 2.1 and linearity of E_d^{tr}. Therefore, since $E_d^{\mathrm{tr}}(e^{\mathrm{enr}}) \neq 0$ then α should be equal to zero. This means that f must be an affine function. But then the unisolvence of $P_1(K)$ with respect to Σ_K^{CR} implies that $f = 0$. Hence we have shown that $\mathcal{F}_K^{\mathrm{enr}}$ is Σ_K^{enr} unisolvent, and so $(K, \Sigma_K^{\mathrm{enr}}, \mathcal{F}_K^{\mathrm{enr}})$ is a finite element.
Necessity: In order to establish the necessary condition, let us assume the contrary, that is, $E_d^{\mathrm{tr}}(e^{\mathrm{enr}}) = 0$. Then thanks to linearity of E_d^{tr} and Lemma 2.1, we can deduce that it vanishes on the whole space $\mathcal{F}_K^{\mathrm{enr}}$, that is

$$E_d^{\mathrm{tr}}(f) = 0, \text{ for all } f \in \mathcal{F}_K^{\mathrm{enr}}. \tag{11}$$

But, $E_d^{\mathrm{tr}} \in \mathrm{span}\left\{L_i^{\mathrm{CR}}, i = 1, \ldots, d+2\right\}$ then since $(K, \Sigma_K^{\mathrm{enr}}, \mathcal{F}_K^{\mathrm{enr}})$ is assumed to be a finite element, equality (11) contradicts the fact $L_i^{\mathrm{CR}}, i = 1, \ldots, d+2$ are linearly independent. This completes the proof of Theorem 2.2. □

An attractive feature of Theorem 2.2 is that in order to prove that the enriched triplet $(K, \Sigma_K^{\mathrm{enr}}, \mathcal{F}_K^{\mathrm{enr}})$ defines a finite element, it is enough to show that the approximation error E_d^{tr} is non-zero for e^{enr}. This simple condition is easily verified in many concrete situations.
In another direction, we may ask if there exists another cubature formula such that Theorem 2.2 continues to hold. The multivariate generalized cubature formula defined in (7) can be characterized as the one which its approximation error E_d^{tr} must also satisfy:

(i) It vanishes for all affine functions;

(ii) It belongs to span $\left\{L_i^{\mathrm{CR}}, i = 1, \ldots, d+2\right\}$, where the coefficient associated to L_{d+2}^{CR} is equal to 1.

We now additionally show that amongst all cubature formulas satisfying (i) and (ii), it is the one which characterizes the existence of the enriched Crouzeix-Raviart triplet $(K, \Sigma_K^{\mathrm{enr}}, \mathcal{F}_K^{\mathrm{enr}})$. Indeed, assume that there exist real numbers $c_i, i = 1, \ldots, d+1$ such that the approximation error

$$C_d^{\mathrm{tr}}(f) := L_{d+2}^{\mathrm{CR}}(f) - \sum_{i=1}^{d+1} c_i L_i^{\mathrm{CR}}(f)$$

vanishes on $P_1(K)$. Then, since for each $i, i = 1, \ldots, d+1, p_i^{\mathrm{CR}} \in P_1(K)$, we can easily see that

$$0 = C_d^{\mathrm{tr}}(p_i^{\mathrm{CR}}) = \frac{1}{d+1} - c_i.$$

But this implies that $C_d^{\mathrm{tr}} = E_d^{\mathrm{tr}}$, and therefore shows that the two cubature formulas are identical.

We would like to extend the result given in Theorem 2.2 to a certain general class of linear functionals, which are not necessary of the form given in (6). To this end, let $L^1(K)$ denote the set of all functions that are integrable on K. Assume that we are given any linear functional L_{d+2}^{ad}, which is defined on $L^1(K)$. In what follows we assume that the general added degree of freedom L_{d+2}^{ad} is normalized such that $L_{d+2}^{\mathrm{ad}}(1) = 1$. Now, we reserve the notation $E_d^{\mathrm{ad}}(f)$ to denote

$$E_d^{\mathrm{ad}}(f) \quad := \quad L_{d+2}^{\mathrm{ad}}(f) - \frac{1}{d+1} \sum_{i=1}^{d+1} \frac{1}{|F_i|} \int_{F_i} f \, \mathrm{d}\sigma, \tag{12}$$

here $E_d^{\mathrm{ad}}(f)$ can be interpreted as the error in approximating $L_{d+2}^{\mathrm{ad}}(f)$ by

$$\frac{1}{d+1} \sum_{i=1}^{d+1} \frac{1}{|F_i|} \int_{F_i} f \, \mathrm{d}\sigma.$$

We call that L_{d+2}^{ad} is admissible in Theorem 2.2, if the latter remains true when L_{d+2}^{CR} and E_d^{CR} are replaced, respectively, by L_{d+2}^{ad} and E_d^{ad}, which are defined as in (12).

In order to extend Theorem 2.2 to a general class of admissible linear functionals, a natural question to ask is: Under what conditions subject to which L_{d+2}^{ad} can be admissible in Theorem 2.2?

With a slight change of notation, here and in the remainder of this paper Σ_K^{enr} is also used to denote the set of degrees of freedom

$$\Sigma_K^{\mathrm{enr}} := \left\{ L_i^{\mathrm{CR}}, i = 1, \ldots, d+1 \right\} \cup \left\{ L_{d+2}^{\mathrm{ad}} \right\}.$$

According to this notation, the following result is a natural generalized version of Theorem 2.2.

Theorem 2.3. *Let L_{d+2}^{ad} be a linear functional defined on $L^1(K)$ satisfying the following conditions*

$$L_{d+2}^{\mathrm{ad}}(\lambda_i) = \frac{1}{d+1}, \ i = 1, \ldots, d+1. \tag{13}$$

Assume that we are given an enrichment function $e^{\mathrm{enr}} \in L^1(K)$. Then, the triplet $(K, \Sigma_K^{\mathrm{enr}}, \mathcal{F}_K^{\mathrm{enr}})$ constitutes a finite element if and only if

$$E_d^{\mathrm{ad}}(e^{\mathrm{enr}}) \neq 0\,. \tag{14}$$

Proof. Indeed, just observe that conditions (13) imply that the error E_d^{ad}, in approximating L_{d+2}^{ad} by $\frac{1}{d+1}\sum_{i=1}^{d+1}\frac{1}{|F_i|}\int_{F_i} f\,\mathrm{d}\sigma$, vanishes on $P_1(K)$. The rest of the proof follows similarly as in Theorem 2.2. □

Let $\boldsymbol{c}$ denote the center of gravity of the simplex K. We introduce the integration formula

$$L_{d+2}^{\mathrm{ad}}(f) = f(\boldsymbol{c}) + R_d^{\mathrm{cg}}(f), \tag{15}$$

and call it the generalized center-of-gravity cubature formula. Clearly in the one-dimensional case, if $L_{d+2}^{\mathrm{ad}}(f) = \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t$ then (15) reduces to the midpoint rule. Conditions (13) are essential in the proof of Theorem 2.3, since they imply that the approximation error E_d^{ad} vanishes on $P_1(K)$. Later we will see that these conditions are, in certain sense, necessary in Theorem 2.3. But first, the following result shows that these two properties are equivalent. Other three possible equivalence formulations of the same property are also given.

Lemma 2.4. *Let $L_{d+2}^{ad}, E_d^{\mathrm{ad}}$ and R_d^{cg} be respectively defined as in* (12) *and* (15). *Then, the following statements are equivalent*:

(i) *The linear functional L_{d+2}^{ad} satisfies*

$$L_{d+2}^{ad}(\lambda_i) = \frac{1}{d+1},\ i = 1,\dots,d+1. \tag{16}$$

(ii) *The approximation error E_d^{ad} vanishes for all affine functions*:

$$E_d^{\mathrm{ad}}(f) = 0,\ \textit{for all}\ f \in P_1(K). \tag{17}$$

(iii) *The approximation error R_d^{cg} vanishes for all affine functions*:

$$R_d^{\mathrm{cg}}(f) = 0,\ \textit{for all}\ f \in P_1(K). \tag{18}$$

(iv) *Let $\{b_1,\dots,b_{d+1}\}$ be any basis for $P_1(K)$ then the linear functional L_{d+2}^{ad} satisfies*

$$L_{d+2}^{ad}(b_i) = b_i(\boldsymbol{c}),\ i = 1,\dots,d+1. \tag{19}$$

(v) *The approximation error E_d^{ad} satisfies*

$$E_d^{\mathrm{ad}}(v) \quad = \quad L_{d+2}^{ad}\left(E_d^{\mathrm{CR}}[v]\right),\ (v \in L^1(K)),$$

where E_d^{CR} is the Crouzeix-Raviart approximation error:

$$E_d^{\mathrm{CR}}[v](\boldsymbol{x}) := v(\boldsymbol{x}) - \sum_{k=1}^{d+1} L_k^{CR}(v)p_k^{CR}(\boldsymbol{x}). \tag{20}$$

Proof. First let us recall that any polynomial $p \in P_1(K)$ can be expressed uniquely as

$$p(\boldsymbol{x}) = \sum_{j=1}^{d+1} p(\boldsymbol{v}_j)\lambda_j(\boldsymbol{x}),$$

then from the linearity of E_d^{ad} and L_{d+2}^{ad}, and the fact that

$$\sum_{i=1}^{d+1} \frac{1}{|F_i|} \int_{F_i} \lambda_j \,\mathrm{d}\sigma = 1, \ j = 1, \ldots, d+1, \tag{21}$$

we can easily deduce that

$$\begin{aligned} E_d^{\mathrm{ad}}(p) &:= L_{d+2}^{\mathrm{ad}}(p) - \frac{1}{d+1} \sum_{i=1}^{d+1} \frac{1}{|F_i|} \int_{F_i} p \,\mathrm{d}\sigma \\ &= \sum_{j=1}^{d+1} p(\boldsymbol{v}_j) L_{d+2}^{\mathrm{ad}}(\lambda_j) - \frac{1}{d+1} \sum_{j=1}^{d+1} p(\boldsymbol{v}_j) \sum_{i=1}^{d+1} \frac{1}{|F_i|} \int_{F_i} \lambda_j \,\mathrm{d}\sigma \\ &= \sum_{j=1}^{d+1} p(\boldsymbol{v}_j) \left(L_{d+2}^{\mathrm{ad}}(\lambda_j) - \frac{1}{d+1} \right). \end{aligned} \tag{22}$$

From the above equality, it is now obvious that (i) implies (ii). Moreover, since (22) holds for every $p \in P_1(K)$, and all the barycentric coordinates are affine functions over K, it follows that

$$\begin{aligned} E_d^{\mathrm{ad}}(\lambda_k) &= \sum_{j=1}^{d+1} \lambda_k(\boldsymbol{v}_j) \left(L_{d+2}^{\mathrm{ad}}(\lambda_j) - \frac{1}{d+1} \right) \\ &= \sum_{j=1}^{d+1} \delta_{kj} \left(L_{d+2}^{\mathrm{ad}}(\lambda_j) - \frac{1}{d+1} \right) = L_{d+2}^{\mathrm{ad}}(\lambda_k) - \frac{1}{d+1}, \ k = 1, \ldots, d+1. \end{aligned}$$

This immediately implies that if (ii) holds, then so does (i). We now show that (i) and (iii) are equivalent. We first observe that, the center of gravity $\boldsymbol{c}$ of K is the average of its vertices

$$\boldsymbol{c} = \frac{1}{d+1} \sum_{j=1}^{d+1} \boldsymbol{v}_j.$$

Therefore, for any polynomial $p \in P_1(K)$ we have

$$\begin{aligned} R_d^{\mathrm{cg}}(p) &:= L_{d+2}^{\mathrm{ad}}(p) - p(\boldsymbol{c}) = L_{d+2}^{\mathrm{ad}}(p) - \frac{1}{d+1} \sum_{j=1}^{d+1} p(\boldsymbol{v}_j) \\ &= \sum_{j=1}^{d+1} p(\boldsymbol{v}_j) L_{d+2}^{\mathrm{ad}}(\lambda_j) - \frac{1}{d+1} \sum_{j=1}^{d+1} p(\boldsymbol{v}_j) = \sum_{j=1}^{d+1} p(\boldsymbol{v}_j) \left(L_{d+2}^{\mathrm{ad}}(\lambda_j) - \frac{1}{d+1} \right). \end{aligned} \tag{23}$$

Now the rest of the proof proceeds as the corresponding part of the proof of the equivalence of (i) and (ii). The equivalence between (iii) and (iv) follows easily from the definition of R_d^{cg}, and simple algebraic manipulations. We now show (i) is equivalent to (v). The implications that (i) implies (v) follows using linearity of L_{d+2}^{ad} and the fact that $L_{d+2}^{\mathrm{ad}}(\lambda_i) = L_{d+2}^{\mathrm{ad}}(p_i^{\mathrm{CR}})$. For the reverse implication, assume that (v) holds, then since $E_d^{\mathrm{CR}}[f] = 0$ for all $f \in P_1(K)$, we get

$$E_d^{\mathrm{ad}}(f) = 0, (f \in P_1(K)).$$

We now take $f = \lambda_j$ and then use (21) to deduce that (i) holds too. This proves the lemma. □

Since L_{d+2}^{CR} satisfies conditions (13), then Theorem 2.2 appears now as an obvious corollary of the conclusions of Theorem 2.3 and Lemma 2.4.

3. DERIVATION OF SIMPLE REPRESENTATION OF THE BASIS FUNCTIONS

Throughout this section we assume that the linear functional L_{d+2}^{ad} given in (12) satisfies conditions (13). These latter obviously imply that

$$L_{d+2}^{\mathrm{ad}}(p_k^{\mathrm{CR}}) = \frac{1}{d+1}, k = 1, \ldots, d+1, \tag{24}$$

where $p_k^{\mathrm{CR}}, k = 1, \ldots, d+1$ are the basis functions of the Crouzeix-Raviart element as defined in (3). From now on we often change notation and denote L_{d+2}^{ad} by L_{d+2}^{CR}. We also assume that the enrichment function e^{enr} is given such that $E_d^{\mathrm{ad}}(e^{\mathrm{enr}}) \neq 0$. Theorem 2.3 therefore guarantees that the enriched Crouzeix-Raviart triplet $(K, \Sigma_K^{\mathrm{enr}}, \mathcal{F}_K^{\mathrm{enr}})$ is a well-defined finite element. Our first goal here is to obtain simple explicit expressions for all its associated basis functions. We recall that the latter are the (unique) set of functions $\{\varphi_i\}_{i=1}^{d+2}$ spanning $\mathcal{F}_K^{\mathrm{enr}}$, such that the Kronecker delta property is satisfied:

$$L_i^{\mathrm{CR}}(\varphi_j) = \delta_{ij}, \text{for all } 1 \leq i, j \leq d+2. \tag{25}$$

Since, we can express any basis function φ_i as the sum of a polynomial $q_i \in P_1(K)$ and the enrichment function e^{enr} multiplied by a scalar α_{d+2}^i, we then look for some coefficients $\alpha_j^i, j = 1, \ldots, d+2$, such that for $1 \leq i \leq d+2$

$$\varphi_i(\boldsymbol{x}) = q_i(\boldsymbol{x}) + \alpha_{d+2}^i\, e^{\mathrm{enr}}(\boldsymbol{x}) = \sum_{k=1}^{d+1} \alpha_k^i\, p_k^{\mathrm{CR}}(\boldsymbol{x}) + \alpha_{d+2}^i\, e^{\mathrm{enr}}(\boldsymbol{x}). \tag{26}$$

The coefficients $\alpha_j^i, j = 1, \ldots, d+2$ for each basis function φ_i are obtained by solving an $(d+2) \times (d+2)$ system of linear equations.

We begin by the calculation of the first $d+1$ basis functions $\varphi_i, i = 1, \ldots, d+1$. For each i, a direct computation, together with (24) and (25), shows that φ_i satisfies

$$\alpha_j^i + \alpha_{d+2}^i \, L_j^{\mathrm{CR}}(e^{\mathrm{enr}}) = \delta_{ij}, j = 1, \ldots, d+1,$$
$$\frac{1}{d+1} \sum_{j=1}^{d+1} \alpha_j^i + \alpha_{d+2}^i \, L_{d+2}^{\mathrm{CR}}(e^{\mathrm{enr}}) = 0.$$

Hence, a simple calculation yields

$$\alpha_j^i = \delta_{ij} + \frac{L_j^{\mathrm{CR}}(e^{\mathrm{enr}})}{(d+1)\, E_d^{\mathrm{ad}}(e^{\mathrm{enr}})}, j = 1, \ldots, d+1, \tag{27}$$

$$\alpha_{d+2}^i = -\frac{1}{(d+1) E_d^{\mathrm{ad}}(e^{\mathrm{enr}})}. \tag{28}$$

We proceed now to calculate the basis function φ_{d+2}. Again, by applying the properties of the Kronecker delta (25), it follows that φ_{d+2} satisfies the following system:

$$\alpha_j^{d+2} + \alpha_{d+2}^{d+2} \, L_j^{\mathrm{CR}}(e^{\mathrm{enr}}) = 0, \ j = 1, \ldots, d+1$$
$$\frac{1}{d+1} \sum_{j=1}^{d+1} \alpha_j^{d+2} + \alpha_{d+2}^{d+2} L_{d+2}^{\mathrm{CR}}(e^{\mathrm{enr}}) = 1.$$

An explicit computation yields

$$\alpha_j^{d+2} = -\frac{L_j^{\mathrm{CR}}(e^{\mathrm{enr}})}{E_d^{\mathrm{ad}}(e^{\mathrm{enr}})}, \ j = 1, \ldots, d+1, \tag{29}$$

$$\alpha_{d+2}^{d+2} = \frac{1}{E_d^{\mathrm{ad}}(e^{\mathrm{enr}})}. \tag{30}$$

Hence we can write the basis function φ_{d+2} as follows:

$$\varphi_{d+2}(\boldsymbol{x}) = \frac{E_d^{\mathrm{CR}}[e^{\mathrm{enr}}](\boldsymbol{x})}{E_d^{\mathrm{ad}}(e^{\mathrm{enr}})}, \tag{31}$$

where E_d^{CR} is the approximation error for the Crouzeix-Raviart element:

$$E_d^{\mathrm{CR}}[e^{\mathrm{enr}}](\boldsymbol{x}) := e^{\mathrm{enr}}(\boldsymbol{x}) - \sum_{k=1}^{d+1} L_k^{\mathrm{CR}}(e^{\mathrm{enr}}) p_k^{\mathrm{CR}}(\boldsymbol{x}). \tag{32}$$

Finally, it can readily be shown that the basis functions specified by relations (26), (27), (28), and (31) satisfy

$$\varphi_i(\boldsymbol{x}) = p_i^{\mathrm{CR}}(\boldsymbol{x}) - \frac{E_d^{\mathrm{CR}}[e^{\mathrm{enr}}](\boldsymbol{x})}{(d+1) E_d^{\mathrm{ad}}(e^{\mathrm{enr}})}, \ i = 1, \ldots, d+1. \tag{33}$$

In summary, we have established the following statement.

Proposition 3.5. *The basis functions* $\{\varphi_i\}_{i=1}^{d+2}$ *for the enriched Crouzeix-Raviart element are determined as follows*:

$$\varphi_i(\boldsymbol{x}) = \begin{cases} p_i^{\mathrm{CR}}(\boldsymbol{x}) - \dfrac{E_d^{\mathrm{CR}}[e^{\mathrm{enr}}](\boldsymbol{x})}{(d+1)E_d^{\mathrm{ad}}(e^{\mathrm{enr}})}, & \text{if} \quad i = 1, \ldots, d+1, \\ \dfrac{E_d^{\mathrm{CR}}[e^{\mathrm{enr}}](\boldsymbol{x})}{E_d^{\mathrm{ad}}(e^{\mathrm{enr}})}, & \text{if} \quad i = d+2. \end{cases} \tag{34}$$

where

$$E_d^{\mathrm{CR}}[e^{\mathrm{enr}}](\boldsymbol{x}) := e^{\mathrm{enr}}(\boldsymbol{x}) - \sum_{k=1}^{d+1} L_k^{\mathrm{CR}}(e^{\mathrm{enr}}) p_k^{\mathrm{CR}}(\boldsymbol{x}). \tag{35}$$

Arguably the simplest choice for the enrichment function e^{enr} is when it is selected such that $L_j^{\mathrm{CR}}(e^{\mathrm{enr}}) = 0, j = 1, \ldots, d+1$. This holds for any integrable function vanishing on all facets of K. In this particular situation, the approximation operator E_d^{CR} preserves the enrichment function e^{enr}, therefore if we rewrite formulas (33) and (34), then the basis functions take the following simplest forms:

Corollary 3.6. *If* $L_j^{\mathrm{CR}}(e^{\mathrm{enr}}) = 0, j = 1, \ldots, d+1$, *then, under the condition* $L_{d+2}^{\mathrm{CR}}(e^{\mathrm{enr}}) \neq 0$, *the basis functions* $\{\varphi_i\}_{i=1}^{d+2}$ *for the enriched Crouzeix-Raviart element are determined as follows*:

$$\varphi_i(\boldsymbol{x}) = \begin{cases} p_i^{\mathrm{CR}}(\boldsymbol{x}) - \dfrac{e^{\mathrm{enr}}(\boldsymbol{x})}{(d+1)L_{d+2}^{\mathrm{CR}}(e^{\mathrm{enr}})}, & \text{if} \quad i = 1, \ldots, d+1, \\ \dfrac{e^{\mathrm{enr}}(\boldsymbol{x})}{L_{d+2}^{CR}(e^{\mathrm{enr}})}, & \text{if} \quad i = d+2. \end{cases} \tag{36}$$

We continue to assume, as we did in computing the basis functions, that the general added degree of freedom L_{d+2}^{ad} is normalized such that $L_{d+2}^{\mathrm{ad}}(1) = 1$. Theorem 2.3 is proved under conditions (13), moreover Lemma 2.4 showed that these latter are in fact equivalent to (17), (18), (19) or (20). Based on identity (33) of basis functions, our aim now is to establish that the required conditions (13) are, in certain sense, necessary for the enriched triplet $(K, \Sigma_K^{\mathrm{enr}}, \mathcal{F}_K^{\mathrm{enr}})$ to be a finite element. Indeed, we have

Proposition 3.7. *Assume that we are given an enrichment function* $e^{\mathrm{enr}} \in L^1(K)$, *such that* $E_d^{\mathrm{ad}}(e^{\mathrm{enr}}) \neq 0$ *and the triplet* $(K, \Sigma_K^{\mathrm{enr}}, \mathcal{F}_K^{\mathrm{enr}})$ *constitutes a finite element with basis functions given in* (34). *Then the following identities hold for all* $i = 1, \ldots, d+1$:

$$L_{d+2}^{\mathrm{ad}}(\lambda_i) = L_{d+2}^{\mathrm{ad}}(p_i^{\mathrm{CR}}) \tag{37}$$

$$= \frac{1}{d+1}. \tag{38}$$

Proof. This is a simple consequence of Lemma 2.4 and identity (33). Indeed, from

the latter and by linearity of L^{ad}_{d+2} we get

$$
\begin{aligned}
0 &= L^{\mathrm{ad}}_{d+2}(\varphi_i) \\
&= L^{\mathrm{ad}}_{d+2}(p_i^{\mathrm{CR}}) - \frac{L^{\mathrm{ad}}_{d+2}(\varphi_{d+2})}{d+1} \\
&= L^{\mathrm{ad}}_{d+2}(p_i^{\mathrm{CR}}) - \frac{1}{d+1}, i = 1, \ldots, d+1.
\end{aligned} \tag{39}
$$

We next verify property (iv) of Lemma 2.4 for the basis for $P_1(K)$

$$\left\{p_i^{\mathrm{CR}}, i = 1, \ldots, d+1\right\},$$

which in fact is the one associated to the standard (non-enriched) Crouzeix-Raviart element. But, since $p_i^{\mathrm{CR}} = d\Big(\frac{1}{d} - \lambda_i\Big)$ then it is easy to check that

$$p_i^{\mathrm{CR}}(\boldsymbol{c}) = \frac{1}{d+1},\ i = 1, \ldots, d+1, \tag{40}$$

from which, together with (39), we can deduce

$$L^{\mathrm{ad}}_{d+2}(p_i^{\mathrm{CR}}) = p_i^{\mathrm{CR}}(\boldsymbol{c}), i = 1, \ldots, d+1.$$

This verifies property (iv) of Lemma 2.4 for the particular Crouzeix-Raviart basis function. Consequently, Lemma 2.4 applies and shows that property (i) also holds. Hence, we conclude that identities (37) and (38) hold simultaneously. This completes the proof of Proposition 3.7.

4. ERROR BOUNDS FOR THE ENRICHED CROUZEIX-RAVIART ELEMENT

This section establishes explicit bounds on the constants in error estimates for both two and three dimensions. These estimates follow via an error representation formula and a well known explicit error estimate for the (non-enriched) Crouzeix-Raviart element, see [**16**, Lemma 5.3]. A complement to this result is also provided for any dimension. For the nonconforming enriched finite element $(K, \Sigma_K^{\mathrm{enr}}, \mathcal{F}_K^{\mathrm{enr}})$, we define the approximation operator $\Pi_d^{\mathrm{enr}} : H^1(K) \to \mathcal{F}_K^{\mathrm{enr}}$ by

$$\Pi_d^{\mathrm{enr}}[v] = \sum_{i=1}^{d+2} L_i^{\mathrm{CR}}(v)\varphi_i,$$

where $\varphi_i, i = 1, \ldots, d+2$ are the basis functions defined in Proposition 3.5. We are interested in evaluating or estimating the approximation error

$$E_d^{\mathrm{enr}}[v] := v - \Pi_d^{\mathrm{enr}}[v].$$

The following Lemma shows that the local approximation error $E_d^{\mathrm{enr}}[v]$ can be decomposed as the error of the (non-enriched) Crouzeix-Raviart element plus a perturbation that depends on the enrichment function e^{enr}.

Lemma 4.8. *Assume that we are given an enrichment function* $e^{\mathrm{enr}} \in H^1(K)$, *such that* $E_d^{\mathrm{ad}}(e^{\mathrm{enr}}) \neq 0$. *Then, for all* $v \in H^1(K)$, *the approximation error at any point* $\boldsymbol{x} \in K$ *is given by*

$$E_d^{\mathrm{enr}}[v](\boldsymbol{x}) = R_d^{\mathrm{lin}}[v](\boldsymbol{x}) + R_d^{\mathrm{enr}}[v](\boldsymbol{x}), \tag{41}$$

where

$$R_d^{\mathrm{lin}}[v](\boldsymbol{x}) := E_d^{\mathrm{CR}}[v](\boldsymbol{x}) \tag{42}$$

$$R_d^{\mathrm{enr}}[v](\boldsymbol{x}) := -\frac{E_d^{\mathrm{ad}}(v)}{E_d^{\mathrm{ad}}(e^{\mathrm{enr}})} E_d^{\mathrm{CR}}[e^{\mathrm{enr}}](\boldsymbol{x}). \tag{43}$$

Proof. It follows immediately from (33) that, for all $\boldsymbol{x} \in K$,

$$\begin{aligned}\sum_{i=1}^{d+1} L_i^{\mathrm{CR}}(v)\varphi_i(\boldsymbol{x}) &= \sum_{i=1}^{d+1} L_i^{\mathrm{CR}}(v)\left(p_i^{\mathrm{CR}}(\boldsymbol{x}) - \frac{E_d^{\mathrm{CR}}[e^{\mathrm{enr}}](\boldsymbol{x})}{(d+1)E_d^{\mathrm{ad}}(e^{\mathrm{enr}})}\right)\\ &= \sum_{i=1}^{d+1} L_i^{\mathrm{CR}}(v)p_i^{\mathrm{CR}}(\boldsymbol{x}) - \frac{E_d^{\mathrm{CR}}[e^{\mathrm{enr}}](\boldsymbol{x})}{E_d^{\mathrm{ad}}(e^{\mathrm{enr}})}\left(\frac{1}{d+1}\sum_{i=1}^{d+1} L_i^{\mathrm{CR}}(v)\right).\end{aligned}$$

From which and (34), we then easily get

$$\begin{aligned}E_d^{\mathrm{enr}}[v](\boldsymbol{x}) &:= v(\boldsymbol{x}) - \sum_{i=1}^{d+2} L_i^{\mathrm{CR}}(v)\varphi_i(\boldsymbol{x})\\ &= v(\boldsymbol{x}) - \sum_{i=1}^{d+1} L_i^{\mathrm{CR}}(v)\varphi_i(\boldsymbol{x}) - L_{d+2}^{\mathrm{CR}}(v)\varphi_{d+2}(\boldsymbol{x})\\ &= E_d^{\mathrm{CR}}[v](\boldsymbol{x}) + \frac{E_d^{\mathrm{CR}}[e^{\mathrm{enr}}](\boldsymbol{x})}{E_d^{\mathrm{ad}}(e^{\mathrm{enr}})}\left(\frac{1}{d+1}\sum_{i=1}^{d+1} L_i^{\mathrm{CR}}(v) - L_{d+2}^{\mathrm{CR}}(v)\right)\\ &= E_d^{\mathrm{CR}}[v](\boldsymbol{x}) - \frac{E_d^{\mathrm{ad}}(v)}{E_d^{\mathrm{ad}}(e^{\mathrm{enr}})} E_d^{\mathrm{CR}}[e^{\mathrm{enr}}](\boldsymbol{x}).\end{aligned}$$

Thus, $E_d^{\mathrm{enr}}[v]$ can be uniquely decomposed into its (non-enriched) part $R_d^{\mathrm{lin}}[v]$ and its enriched part $R_d^{\mathrm{enr}}[v]$. This shows that the required decomposition holds. □

We are now in the position to establish the L^2-estimate with explicit constants for the enriched approximation error E_d^{enr}. To do this we use Lemma 4.8 and an explicit error estimate due to Vohralik [**16**, Lemma 5.3]. Here we assume that $e^{\mathrm{enr}} \in H^2(K)$, and the enriched degree of freedom is fixed as follows

$$L_{d+2}^{\mathrm{CR}}(v) = \frac{1}{|K|}\int_K v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

Note that, in the present situation, for all $v \in H^1(K)$, we get (by definition) that the mean average of $E_d^{\mathrm{enr}}[v]$ vanishes, that is,

$$L_{d+2}^{\mathrm{CR}}(E_d^{\mathrm{enr}}[v]) = 0.$$

Theorem 4.9. *For any $v \in H^1(K)$, the following explicit error estimate holds.*

$$\|E_d^{\mathrm{enr}}[v]\|_{L^2(K)} \leq \left(1 + \frac{\|E_d^{\mathrm{CR}}[e^{\mathrm{enr}}]\|_{L^2(K)}}{|L_{d+2}^{\mathrm{ad}}(E_d^{\mathrm{CR}}[e^{\mathrm{enr}}])|\sqrt{|K|}}\right) c_d h\, |v|_{H^1(K)}\,, \tag{44}$$

where

$$c_d = \begin{cases} \sqrt{6}, & \text{if} \quad d = 2, \\ 3, & \text{if} \quad d = 3. \end{cases} \tag{45}$$

Proof. From Lemma 4.8, we have, by the triangle inequality,

$$\|E_d^{\mathrm{enr}}[v]\|_{L^2(K)} \leq \left\|E_d^{\mathrm{CR}}[v]\right\|_{L^2(K)} + \frac{\left\|E_d^{\mathrm{CR}}[e^{\mathrm{enr}}]\right\|_{L^2(K)}}{\left|E_d^{\mathrm{ad}}(e^{\mathrm{enr}})\right|}\left|E_d^{\mathrm{ad}}(v)\right|. \tag{46}$$

Now, an elementary calculation yields

$$\begin{aligned} E_d^{\mathrm{ad}}(v) &:= L_{d+2}^{\mathrm{ad}}(v) - \frac{1}{d+1}\sum_{i=1}^{d+1}\frac{1}{|F_i|}\int_{F_i} v\,\mathrm{d}\sigma \\ &= L_{d+2}^{\mathrm{ad}}\left(v - \sum_{i=1}^{d+1} p_i^{\mathrm{CR}}\frac{1}{|F_i|}\int_{F_i} v\,\mathrm{d}\sigma\right) = L_{d+2}^{\mathrm{ad}}\left(E_d^{\mathrm{CR}}[v]\right), \end{aligned}$$

then, using the Cauchy-Schwarz inequality, we can write

$$\left|E_d^{\mathrm{ad}}(v)\right| \leq \frac{\left\|E_d^{\mathrm{CR}}[v]\right\|_{L^2(K)}}{\sqrt{|K|}},$$

whence, together with (46), we get

$$\|E_d^{\mathrm{enr}}[v]\|_{L^2(K)} \leq \left(1 + \frac{\|E_d^{\mathrm{CR}}[e^{\mathrm{enr}}]\|_{L^2(K)}}{|E_d^{\mathrm{ad}}(e^{\mathrm{enr}})|\sqrt{|K|}}\right)\|E_d^{\mathrm{CR}}[v]\|_{L^2(K)}. \tag{47}$$

Observing that

$$E_d^{\mathrm{ad}}(e^{\mathrm{enr}}) = L_{d+2}^{\mathrm{ad}}\left(E_d^{\mathrm{CR}}[e^{\mathrm{enr}}]\right),$$

and using (47), we then deduce

$$\|E_d^{\mathrm{enr}}[v]\|_{L^2(K)} \leq \left(1 + \frac{\|E_d^{\mathrm{CR}}[e^{\mathrm{enr}}]\|_{L^2(K)}}{|L_{d+2}^{\mathrm{ad}}(E_d^{\mathrm{CR}}[e^{\mathrm{enr}}])|\sqrt{|K|}}\right)\left\|E_d^{\mathrm{CR}}[v]\right\|_{L^2(K)}. \tag{48}$$

But, we already know from [**16**, Lemma 5.3] that

$$\left\|E_d^{\mathrm{CR}}[v]\right\|_{L^2(K)} \leq c_d h\, |v|_{H^1(K)}\,,$$

where c_d is given by (45), then this together with (48) gives the desired estimate. □

In what follows, h denotes the diameter of K, that is the supremum of the lengths of all line segments contained in K. As a complement to Theorem 4.9, for all $d \geq 1$, we have an alternative approach to estimate the approximation error $\|E_d^{\mathrm{enr}}[v]\|_{L^2(K)}$.

Theorem 4.10. *For any $v \in H^{1+\sigma}(K)$, with $0 < \sigma \leq 1$, the following error estimate holds.*

$$\|E_d^{\mathrm{enr}}[v]\|_{L^2(K)} \leq ch^{1+\sigma} |v|_{H^{1+\sigma}(K)}. \tag{49}$$

Proof. Recall that the Poincaré inequality, which can be found in [**2, 15**], states that: there exists a universal constant c such that for any $f \in H^1(K)$, there holds

$$\|f - f_K\|_{L^2(K)} \leq ch \|\nabla f\|_{L^2(K)}, \tag{50}$$

where f_K stands for the mean average of f over K. The best constant within the class of convex domains is $c = 1/\pi$, see [**15, 2**]. Now fix $v \in H^{1+\sigma}(K)$, with $0 < \sigma \leq 1$. Since the mean average of $E_d^{\mathrm{enr}}[v]$ vanishes and $E_d^{\mathrm{enr}}[v]$ belongs to $H^1(K)$, then we can apply the Poincaré inequality to $E_d^{\mathrm{enr}}[v]$ to obtain

$$\|E_d^{\mathrm{enr}}[v]\|_{L^2(K)} \leq ch \|\nabla E_d^{\mathrm{enr}}[v]\|_{L^2(K)}. \tag{51}$$

Furthermore, since E_d^{enr} vanishes on $P_1(K)$, then the desired result follows by using standard results from operator interpolation theory involving local polynomial preserving property (in the present case the local affine functions), see, for instance, [**5, 8**]. □

Finally, we conclude this paper with the following remarks. Simple inspection shows that the error estimates, provided by our Theorems 4.9 and 4.10, remain true if K is any convex polytope in $\mathbb{R}^d$. Here, the underlying principle is that one triangulates the polytope into simplices (under shape regularity assumption on the triangulation, see [**16**]) and then use these error estimates over all simplices, we then may get by summing that the latter also holds globally. On the other hand, although presented within the enriched Crouzeix-Raviart context, the same enrichment idea can also be extended to any conforming or non-conforming P_1 element. The extension to the general case involving multiple enrichment functions is a work in progress.

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