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INCIDENCE GRAPHS CONSTRUCTED FROM t-DESIGNS

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Let $\mathcal{D}$ be a non-trivial simple t-design. For $1 \leq s \leq t$, we generalize the concept of the incidence graph of $\mathcal{D}$ and construct a new bipartite regular graph Γ. We obtain that the edge-transitivity of the graph Γ is equivalent to the s-flag-transitivity of the design $\mathcal{D}$. We then, for $s = 2$, classify the semisymmetric graphs among the graphs Γ constructed from biplanes and triplanes. Finally, we study the connectedness and the energy of incidence graphs. Several open problems are proposed, one of which asks whether the incidence graphs have large vertex-connectivity.

1. INTRODUCTION

A combinatorial incidence structure $\mathcal{D}$ is called a t-(v, k, λ_t) *design* if $\mathcal{D}$ consists of a pair $(\mathcal{P}, \mathcal{B})$ where, $\mathcal{P}$ is a set of v elements, and $\mathcal{B}$ is a collection of b proper subsets of $\mathcal{P}$, each of size k, such that any t elements of $\mathcal{P}$ are contained in exactly λ_t elements of $\mathcal{B}$. Each element in $\mathcal{P}$ is called a *point* and each k-subset in $\mathcal{B}$ is called a *block*. We also call $\mathcal{D}$ a t*-design* if we do not wish to make reference to the other parameters. For any $\alpha \in \mathcal{P}$ and $B \in \mathcal{B}$, the pair (α, B) is called a *flag* if $\alpha \in B$. $\mathcal{D}$ is *trivial* if $t = k$. $\mathcal{D}$ is *simple* if every k-subset appears at most once in $\mathcal{B}$. $\mathcal{D}$ is a *sharp* t-design if $\mathcal{D}$ is a t-design but not a $(t+1)$-design. $\mathcal{D}$ is called a *symmetric design* if the number of points is equal to the number of blocks. Furthermore, for $t = 2$, if $\lambda_2 = 2$, then $\mathcal{D}$ is called a *biplane*, and if $\lambda_2 = 3$, then $\mathcal{D}$ is called a *triplane*.

For a point p, the *derived design* $\mathcal{D}_p$ has point-set $\mathcal{P} \setminus \{p\}$ and block-set $\{B \setminus \{p\} : B \in \mathcal{B} \text{ with } p \in B\}$. It is well known that $\mathcal{D}_p$ is a $(t-1)$-$(v-1, k-1, \lambda_t)$

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design. More generally, for $q < t$, a *q-derived design* of $\mathcal{D}$ is a design $\mathcal{D}'$ that can be obtained from $\mathcal{D}$ by successively obtaining derived designs, by deleting q points of $\mathcal{P}$, one at a time. Hence, $\mathcal{D}'$ is a $(t-q)$-$(v-q, k-q, \lambda_t)$ design.

If $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a 2-(v, k, λ_2) design, then its *complementary design*, denoted by $\overline{\mathcal{D}}$, is the design with the same point-set $\mathcal{P}$, and block-set $\overline{\mathcal{B}} = \{\mathcal{P} \setminus B : B \in \mathcal{B}\}$. Indeed, it is well known that, by an inclusion-exclusion argument, $\overline{\mathcal{D}}$ is a 2-$(v, v-k, b-2r+\lambda_2)$ design.

An *automorphism* of $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a permutation σ on $\mathcal{P}$ such that $\overline{\sigma}(B) \in \mathcal{B}$ for all $B \in \mathcal{B}$, where $\overline{\sigma}(B) = \{\sigma(\alpha) : \alpha \in B\}$. Note that $\overline{\sigma}$ is a permutation of $\mathcal{B}$, and is called the *induced permutation* of σ on $\mathcal{B}$. The set of all automorphisms of $\mathcal{D}$ forms a group under composition, and is called the *automorphism group* of $\mathcal{D}$, denoted by $\mathsf{Aut}(\mathcal{D})$. For a subgroup $G \leq \mathsf{Aut}(\mathcal{D})$, $\mathcal{D}$ is called a *point-transitive design* (resp. *block-transitive, flag-transitive*) if G is transitive on the point-set (resp. block-set, flag-set).

The classification of combinatorial designs is a problem of great interest. In particular, the block-transitive and flag-transitive designs have been well investigated by many scholars [**3, 4, 15, 16, 17, 19, 20, 22**]. For the case when $\mathcal{D}$ is a projective plane of order n (and hence $\mathcal{D}$ is a symmetric design), Kantor [**17**] proved that if $G \leq \mathsf{Aut}(\mathcal{D})$ and $\mathcal{D}$ is G-flag-transitive, then either $\mathcal{D}$ is Desarguesian and $G \trianglerighteq PSL(3,n)$, or G is a Frobenius group of odd order $(n^2+n+1)(n+1)$ and n^2+n+1 is a prime. Inspired by this work, Regueiro [**26, 27, 28, 29**] completely classified the biplanes with flag-transitive automorphism groups apart from those admitting a 1-dimensional affine group. Subsequently, Zhou et al. [**5, 32, 33, 34, 35**] completely classified the triplanes under the same conditions. In this paper, by using their results, we construct some specific graphs.

For a graph Γ, an *automorphism* σ is a bijection on $V(\Gamma)$ such that $\{u, v\} \in E(\Gamma)$ if and only if $\{u, v\}^\sigma \in E(\Gamma)$. The set of all automorphisms of Γ forms a group under composition, and is called the *full automorphism group* of Γ, denoted by $\mathsf{Aut}(\Gamma)$. Γ is *vertex-transitive* (resp. *edge-transitive*) if $\mathsf{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$ (resp. $E(\Gamma)$). A regular graph Γ is said to be *semisymmetric* if Γ is edge-transitive but not vertex-transitive. Folkman [**7**] initiated the concept of semisymmetric graphs, and he constructed several infinite families of such graphs and proposed several fascinating problems. Subsequently, the classification of semisymmetric graphs became an attractive problem. Many researchers [**6, 14, 23, 24**] have found and constructed many semisymmetric graphs.

Let Γ be a simple graph of order n. The *energy* of Γ, first introduced by Gutman [**8, 10**], is defined as

$$\mathcal{E}(\Gamma) = \sum_{i=1}^{n} |\mu_i|,$$

where $\mu_1, \mu_2, \ldots, \mu_n$ are the eigenvalues of the adjacency matrix of Γ. This invariant has been widely studied in the past two decades. In theoretical chemistry, the energy of a given molecular graph is closely related the total π-electron energy of the molecule represented by that graph. For more details on the function $\mathcal{E}(\Gamma)$, the reader is referred to [**12, 13, 18, 21**] and the references therein.

As far as we know, very little is known about the study of graphs obtained from designs, and vice versa. In [**31**], the authors constructed a bipartite regular graph and investigated its graphic properties, including the enumeration of 4-cycles, the energy, and the semisymmetric property. One of the main results in [**31**] is that all the graphs constructed from non-symmetric flag-transitive 2-$(v, k, 1)$ designs are semisymmetric. The case of symmetric flag-transitive designs are not considered there as the graphs constructed in [**31**] are generally not semisymmetric. In this paper, we consider the symmetric designs that were not considered in [**31**]. We shall generalize the concept of the incidence graph of a design, and one of our main results (Theorem 8) will concern the equivalence of the edge-transitivity of this generalized incidence graph and a generalized version of flag-transitivity of the design.

This paper is organized as follows. In Section 2, we will gather several known results from design theory which we will require later, and we will also introduce some incidence graphs of designs. In Section 3, we will prove Theorem 8, and in Section 4, we will use Theorem 8 to study semisymmetric graphs in relation to biplanes and triplanes. In Sections 5 and 6, we will study the connectedness and the energy of some incidence graphs of designs. For terminologies in graph theory and design theory not mentioned here, the reader is referred to [**1, 2**].

2. PRELIMINARIES AND INCIDENCE GRAPHS

Throughout this paper, unless otherwise stated, all designs are simple, non-trivial and sharp, all groups are finite, and all graphs are simple, finite and undirected. We shall denote the binomial coefficient $\binom{n}{m}$ by $[n]_m$.

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a t-(v, k, λ_t) design, where the number of blocks in $\mathcal{B}$ is b, and let $1 \leq s \leq t$. We always assume that t, s, v, k, b and λ_t are positive integers and $v > k > t$. We recall the following two well known results in design theory.

Lemma 1. [**1**] *Let $\mathcal{D}$ be a t-(v, k, λ_t) design. Then $\mathcal{D}$ is also an s-(v, k, λ_s) design, where $1 \leq s \leq t$ and*

$$\lambda_s = \lambda_t \frac{\binom{v-s}{t-s}}{\binom{k-s}{t-s}} = \lambda_t \frac{[v-s]_{t-s}}{[k-s]_{t-s}}. \tag{1}$$

Lemma 2. [**1**] *If $\mathcal{D}$ is a symmetric 2-(v, k, λ_2) design, then any two blocks intersect in exactly λ_2 points.*

We remark that Lemma 2 is not true for non-symmetric 2-designs. Indeed, for general t-designs, the size of the intersection of any two blocks is difficult to determine.

Let $\mathcal{P}^{(s)}$ denote the family of all s-subsets of $\mathcal{P}$. The ordered pair (N, B) is called an *s-flag* of $\mathcal{D}$ if $B \in \mathcal{B}$, and $N \in \mathcal{P}^{(s)}$ is an s-subset of $\mathcal{P}$ contained in B. $\mathcal{D}$ is *s-flag-transitive* if $\mathsf{Aut}(\mathcal{D})$ acts transitively on the s-flag-set of $\mathcal{D}$. In particular, a 1-flag is just a flag.

The *s-incidence graph* of $\mathcal{D}$, denoted by $IG_s(\mathcal{D})$, is the bipartite graph whose bipartition classes are $\mathcal{P}^{(s)}$ and $\mathcal{B}$. For $N \in \mathcal{P}^{(s)}$ and $B \in \mathcal{B}$, we have $\{N, B\} \in E(IG_s(\mathcal{D}))$ if and only if $N \subset B$, in other words, (N, B) is an s-flag. In particular, when $s = 1$, we write $IG_1(\mathcal{D}) = IG(\mathcal{D})$, and call this the *incidence graph* of $\mathcal{D}$.

Lemma 3. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a t-(v, k, λ_t) design and $1 \le s \le t$. Then the s-incidence graph $IG_s(\mathcal{D})$ is biregular, where every vertex of the class $\mathcal{P}^{(s)}$ has degree $\lambda_s = \lambda_t \dfrac{[v-s]_{t-s}}{[k-s]_{t-s}}$, and every vertex of the class $\mathcal{B}$ has degree $[k]_s$.*

Proof. Note that by Lemma 1, since $\mathcal{D}$ is also an s-(v, k, λ_s) design, each s-subset of $\mathcal{P}$ is contained in λ_s blocks of $\mathcal{B}$. Also, each block of $\mathcal{B}$ contains $[k]_s$ s-subsets of $\mathcal{P}$. □

Next, we shall construct another type of incidence graph. Let $1 \le s \le t$. By using Lemma 3 to count the edges of $IG_s(\mathcal{D})$ in two ways, we arrive at the following equation:

$$\binom{v}{s}\lambda_s = b\binom{k}{s}, \quad \text{which is} \quad [v]_s\lambda_s = b[k]_s. \tag{2}$$

Let $\mathcal{N} = \mathcal{P}^{(s)}$, so that the cardinality of $\mathcal{N}$ is $[v]_s$. Let $\mathcal{N}^1, \mathcal{N}^2, \dots, \mathcal{N}^b$ be b copies of $\mathcal{N}$, and $\mathcal{B}^1, \mathcal{B}^2, \dots, \mathcal{B}^{[v]_s}$ be $[v]_s$ copies of $\mathcal{B}$. Let $I_1 = \{1, \dots, b\}$ and $I_2 = \{1, \dots, [v]_s\}$. We denote by $\mathsf{IG}_{[v]_s,b}(\mathcal{D}) = (V, E)$ the graph whose vertex set V is the union of all the $\mathcal{N}^i$ and $\mathcal{B}^j$, where $i \in I_1$ and $j \in I_2$. For $N \in \mathcal{N}^i$ and $B \in \mathcal{B}^j$, we have $\{N, B\} \in E$ if and only if (N, B) is an s-flag. In other words, for any pair $(\mathcal{N}^i, \mathcal{B}^j)$, we add a copy of the bipartite graph $IG_s(\mathcal{D})$ between $\mathcal{N}^i$ and $\mathcal{B}^j$. We will generally write $\Gamma = \mathsf{IG}_{[v]_s,b}(\mathcal{D})$ for the rest of the paper when the values of the parameters are understood. In the next proposition, we show that Γ is a balanced bipartite (i.e., the bipartition classes have equal size) and regular graph, for any $1 \le s \le t$.

Proposition 4. *Let $\mathcal{D}$ be a t-(v, k, λ_t) design with b blocks, and $1 \le s \le t$. Then $\Gamma = \mathsf{IG}_{[v]_s,b}(\mathcal{D})$ is a simple, balanced bipartite, and regular graph.*

Proof. Since $IG_s(\mathcal{D})$ is simple and bipartite, by the construction of Γ, it is easy to see that Γ is also simple and bipartite.

Next, note that the bipartition classes of Γ are $V_1 = \bigcup_{i=1}^{b} \mathcal{N}^i$ and $V_2 = \bigcup_{j=1}^{[v]_s} \mathcal{B}^j$. Since $\mathcal{N}^i$ is a copy of $\mathcal{N}$, and $\mathcal{B}^j$ is a copy of $\mathcal{B}$, we have $|\mathcal{N}^i| = [v]_s$ and $|\mathcal{B}^j| = b$, and hence $|V_1| = |V_2| = [v]_s b$, and Γ is balanced.

Lastly, by Lemma 3, since there are $[v]_s$ copies of $\mathcal{B}$, the degree of every vertex in V_1 is $[v]_s\lambda_s$. Also, since there are b copies of $\mathcal{N}$, the degree of every vertex in V_2 is $b[k]_s$. Now, the degrees of the vertices from both classes are equal, since $[v]_s\lambda_s = b[k]_s$ from (2). Therefore, Γ is a $[v]_s\lambda_s$-regular graph. $\square$

Next, we recall a result about eigenvalues. Let $\mathfrak{S}(M)$ denote the *spectrum* of the square matrix M, that is, the collection of the eigenvalues of M. Let $A = (a_{ij})$ be an $m \times n$ matrix, and B be a $p \times q$ matrix. Then the *Kronecker product* of A and B, denoted by $A \otimes B$, is the matrix $(a_{ij}B)_{mp\times nq}$.

Lemma 5. [**11**] *Let A and B be two square matrices. Furthermore, let $\nu \in \mathfrak{S}(A)$ with corresponding eigenvector $\boldsymbol{x}$, and $\mu \in \mathfrak{S}(B)$ with corresponding eigenvector $\boldsymbol{y}$. Then $\nu\mu$ is an eigenvalue of $A \otimes B$ with corresponding eigenvector $\boldsymbol{x} \otimes \boldsymbol{y}$. Any eigenvalue of $A \otimes B$ arises as such a product of eigenvalues of A and B.*

Finally, the following two theorems are about the classification of flag-transitive biplanes and triplanes, respectively.

Theorem 6. [**26, 27, 28, 29**] *If $\mathcal{D}$ is a non-trivial biplane with a primitive, flag-transitive automorphism group G, then one of the following holds:*

(i) *$\mathcal{D}$ has parameters* $(16, 6, 2)$,

(ii) *$\mathcal{D}$ has parameters* $(7, 4, 2)$,

(iii) *$\mathcal{D}$ has parameters* $(11, 5, 2)$,

(iv) $G \leq A\Gamma L_1(q)$, *for some odd prime power* q.

Theorem 7. [**5**] *If $\mathcal{D}$ is a non-trivial triplane with a primitive, flag-transitive automorphism group G, then one of the following holds:*

(i) *$\mathcal{D}$ has parameters* $(45, 12, 3)$,

(ii) *$\mathcal{D}$ has parameters* $(11, 6, 3)$,

(iii) *$\mathcal{D}$ has parameters* $(15, 7, 3)$,

(iv) $G \leq A\Gamma L_1(q)$, *where q is some power of a prime p and* $p \geq 5$.

Remark. Note that there are three non-isomorphic biplanes with parameter $(16, 6, 2)$ and many non-isomorphic triplanes with parameter $(45, 12, 3)$. Only one of the non-isomorphic biplanes (resp. non-isomorphic triplanes) has a primitive, flag-transitive automorphism group. One may refer to [**25**] and [**26**] for such examples. The biplanes and triplanes with the other parameters in Theorems 6 and 7 are unique. The designs with the parameters in Theorems 6 and 7 are taken to be the unique primitive flag-transitive designs in Section 4.

3. EQUIVALENCE OF EDGE-TRANSITIVITY AND s-FLAG-TRANSITIVITY

In this section, we shall prove the following result, which considers the equivalence of edge-transitivity of the graph Γ, and s-flag-transitivity of $\mathcal{D}$, for a design $\mathcal{D}$.

Theorem 8. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a t-(v, k, λ_t) design with $|\mathcal{B}| = b$, and $1 \leq s \leq t$. Then, $\Gamma = \mathsf{IG}_{[v]_s,b}(\mathcal{D})$ is edge-transitive if and only if $\mathcal{D}$ is s-flag-transitive.*

To prove Theorem 8, we first prove some auxiliary results. For groups G and H, we write $H \lesssim G$ to denote that H is isomorphic to a subgroup of G. Recall that $\mathcal{N} = \mathcal{P}^{(s)}$ is the collection of all s-subsets of $\mathcal{P}$.

Lemma 9. $\mathsf{Aut}(\mathcal{D}) \lesssim \mathsf{Aut}(IG_s(\mathcal{D}))$.

Proof. Let $N = \{\alpha_1, \ldots, \alpha_s\}$ be any s-subset of $\mathcal{P}$ and $B = \{\beta_1, \ldots, \beta_k\}$ be any block of $\mathcal{B}$. $N^\sigma = \{\alpha_1^\sigma, \ldots, \alpha_s^\sigma\}$, $B^\sigma = \{\beta_1^\sigma, \ldots, \beta_k^\sigma\}$ naturally for any $\sigma \in \mathsf{Aut}(\mathcal{D})$. Therefore $N \subset B$ if and only if $N^\sigma \subset B^\sigma$. For fixed $\sigma \in \mathsf{Aut}(\mathcal{D})$, $(\sigma_s, \sigma_\mathcal{B})$ is corresponding to σ satisfying: $(\sigma_s, \sigma_\mathcal{B})(N) = N^\sigma$, $(\sigma_s, \sigma_\mathcal{B})(B) = B^\sigma$ for any $N \in \mathcal{N}$ and any $B \in \mathcal{B}$. Note that $B^\sigma \in \mathcal{B}$ when $\sigma \in \mathsf{Aut}(\mathcal{D})$. Meanwhile, N, B, N^σ, B^σ are vertices of $IG_s(\mathcal{D})$. It follows that $(\sigma_s, \sigma_\mathcal{B}) \in \mathsf{Aut}(IG_s(\mathcal{D}))$. Let $\varphi : \mathsf{Aut}(\mathcal{D}) \to \mathsf{Aut}(IG_s(\mathcal{D}))$, $\sigma \mapsto (\sigma_s, \sigma_\mathcal{B})$. It is easy to see that φ is a homomorphism. It is well known that the permutation of $\mathcal{P}$ which fixes each member of $\mathcal{N}$, must be the identity. This implies that if $(\sigma_s, \sigma_\mathcal{B})$ fixes all the vertices of $IG_s(\mathcal{D})$, then $\sigma = \mathrm{id}$, the identity of $\mathsf{Aut}(\mathcal{D})$, which means the induced action of $\mathsf{Aut}(\mathcal{D})$ on $IG_s(\mathcal{D})$ is faithful. Therefore $\mathsf{Aut}(\mathcal{D}) \cong \varphi(\mathsf{Aut}(\mathcal{D}))$.

Theorem 10. *For $\Gamma = \mathsf{IG}_{[v]_s,b}(\mathcal{D})$, we have* $\mathsf{Aut}(\mathcal{D}) \lesssim \mathsf{Aut}(\Gamma)$.

Proof. Let $K := \varphi(\mathsf{Aut}(\mathcal{D}))$, where φ is a homomorphism as stated in Lemma 9. Let

$$\theta_i : \mathcal{N} \to \mathcal{N}^i,\ N_p \mapsto N_p^i, \quad \text{for any } i \in I_1,\ p \in I_2,$$

and

$$\eta_j : \mathcal{B} \to \mathcal{B}^j,\ B_q \mapsto B_q^j, \quad \text{for any } j \in I_2,\ q \in I_1.$$

We first prove two claims. Recall that for $\sigma \in \mathsf{Aut}(\mathcal{D})$, we write σ_s and $\sigma_\mathcal{B}$ for the induced permutations of σ on $\mathcal{N}$ and $\mathcal{B}$, respectively.

Claim 1. *$\theta_i(\sigma_s\sigma'_s) = \theta_i(\sigma_s)\theta_i(\sigma'_s)$ for any $\sigma, \sigma' \in \mathsf{Aut}(\mathcal{D})$. A similar statement holds for η_j.*

Claim 2. *Let $\tau := \theta_1(\sigma_s) \cdots \theta_b(\sigma_s)$, $\overline{\tau} := \eta_1(\sigma_\mathcal{B}) \cdots \eta_{[v]_s}(\sigma_\mathcal{B})$ for $\sigma \in \mathsf{Aut}(\mathcal{D})$. Then $(\tau, \overline{\tau}) \in \mathsf{Aut}(\mathsf{IG}_{[v]_s,b}(\mathcal{D}))$.*

Proof of Claim 1. Note that θ_i induces a permutation $\theta_i(\sigma_s)$ on $\mathcal{N}^i$ which is isomorphic to σ_s on $\mathcal{N}$, and η_j induces a permutation $\eta_j(\sigma_\mathcal{B})$ on $\mathcal{B}^j$ which is isomorphic to $\sigma_\mathcal{B}$ on $\mathcal{B}$. It is obvious that the following two diagrams are commutative for each $\sigma \in \mathsf{Aut}(\mathcal{D})$.

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{\theta_i} & \mathcal{N}^i \\ \downarrow{\scriptstyle \sigma_s} & & \downarrow{\scriptstyle \theta_i(\sigma_s)} \\ \mathcal{N} & \xrightarrow{\theta_i} & \mathcal{N}^i \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{B} & \xrightarrow{\eta_j} & \mathcal{B}^j \\ \downarrow{\scriptstyle \sigma_{\mathcal{B}}} & & \downarrow{\scriptstyle \eta_j(\sigma_{\mathcal{B}})} \\ \mathcal{B} & \xrightarrow{\eta_j} & \mathcal{B}^j \end{array}$$

It follows that $\theta_i(\sigma_s(N_p)) = \theta_i(\sigma_s)(N_p^i)$ for any $N_p \in \mathcal{N}$ from the first diagram. This implies that

$$\begin{aligned} \theta_i(\sigma_s\sigma_s')(N_p^i) &= \theta_i(\sigma_s\sigma_s'(N_p)) = \theta_i(\sigma_s)((\sigma_s'(N_p))^i) \\ &= \theta_i(\sigma_s)(\theta_i(\sigma_s'(N_p))) = \theta_i(\sigma_s)\theta_i(\sigma_s')(N_p^i). \end{aligned}$$

Since N_p is arbitrary, we have $\theta_i(\sigma_s\sigma_s') = \theta_i(\sigma_s)\theta_i(\sigma_s')$. Similarly, we have $\eta_j(\sigma_{\mathcal{B}}\sigma_{\mathcal{B}}') = \eta_j(\sigma_{\mathcal{B}})\eta_j(\sigma_{\mathcal{B}}')$ from the second diagram. This proves Claim 1.

Proof of Claim 2. For any $\{N_p^i, B_q^j\} \in E(\Gamma)$, we have $\{N_p, B_q\} \in E(IG_s(\mathcal{D}))$. Suppose that $\sigma_s(N_p) = N_{p'}$, $\sigma_{\mathcal{B}}(B_q) = B_{q'}$. We have $\{N_p^i, B_q^j\}^{(\tau,\overline{\tau})} = \{\tau(N_p^i), \overline{\tau}(B_q^j)\}$. Furthermore,

$$\begin{aligned} &\tau(N_p^i) = \theta_1(\sigma_s)\theta_2(\sigma_s)\cdots\theta_b(\sigma_s)(N_p^i) = \theta_i(\sigma_s)(N_p^i) = \theta_i(\sigma_s(N_p)) = \theta_i(N_{p'}) = N_{p'}^i, \\ &\overline{\tau}(B_q^j) = \eta_1(\sigma_{\mathcal{B}})\eta_2(\sigma_{\mathcal{B}})\cdots\eta_{[v]_s}(\sigma_{\mathcal{B}})(B_q^j) = \eta_j(\sigma_{\mathcal{B}})(B_q^j) = \eta_j(\sigma_{\mathcal{B}}(B_q)) = \eta_j(B_{q'}) = B_{q'}^j. \end{aligned}$$

Since $\sigma \in \mathsf{Aut}(\mathcal{D})$ and $\mathsf{Aut}(\mathcal{D}) \lesssim \mathsf{Aut}(IG_s(\mathcal{D}))$ from Lemma 9, we have $\{N_{p'}, B_{q'}\} \in E(IG_s(\mathcal{D}))$, and thus $\{N_{p'}^i, B_{q'}^j\} \in E(\Gamma)$. This implies that $(\tau, \overline{\tau})$ is an automorphism of Γ, and Claim 2 holds.

Now, let $\psi : K \to \mathsf{Aut}(\Gamma)$, where $(\sigma_s, \sigma_{\mathcal{B}}) \mapsto (\tau, \overline{\tau})$ is given in Claim 2. We next show that ψ is a homomorphism from K to $\mathsf{Aut}(\Gamma)$.

For any $\sigma, \sigma' \in \mathsf{Aut}(\mathcal{D})$, we have $(\sigma_s, \sigma_{\mathcal{B}}), (\sigma_s', \sigma_{\mathcal{B}}') \in K$. Hence, using Claim 1, we have

$$\begin{aligned} &\psi((\sigma_s, \sigma_{\mathcal{B}})(\sigma_s', \sigma_{\mathcal{B}}')) \\ &= \psi(\sigma_s\sigma_s', \sigma_{\mathcal{B}}\sigma_{\mathcal{B}}') = (\theta_1(\sigma_s\sigma_s')\cdots\theta_b(\sigma_s\sigma_s'), \eta_1(\sigma_{\mathcal{B}}\sigma_{\mathcal{B}}')\cdots\eta_{[v]_s}(\sigma_{\mathcal{B}}\sigma_{\mathcal{B}}')) \\ &= (\theta_1(\sigma_s)\theta_1(\sigma_s')\cdots\theta_b(\sigma_s)\theta_b(\sigma_s'), \eta_1(\sigma_{\mathcal{B}})\eta_1(\sigma_{\mathcal{B}}')\cdots\eta_{[v]_s}(\sigma_{\mathcal{B}})\eta_{[v]_s}(\sigma_{\mathcal{B}}')) \\ &= (\theta_1(\sigma_s)\cdots\theta_b(\sigma_s)\theta_1(\sigma_s')\cdots\theta_b(\sigma_s'), \eta_1(\sigma_{\mathcal{B}})\cdots\eta_{[v]_s}(\sigma_{\mathcal{B}})\eta_1(\sigma_{\mathcal{B}}')\cdots\eta_{[v]_s}(\sigma_{\mathcal{B}}')) \\ &= (\tau\tau', \overline{\tau}\,\overline{\tau'}) = (\tau, \overline{\tau})(\tau', \overline{\tau'}) = \psi(\sigma_s, \sigma_{\mathcal{B}})\psi(\sigma_s', \sigma_{\mathcal{B}}'). \end{aligned}$$

Thus, ψ is a homomorphism from K to $\mathsf{Aut}(\Gamma)$. This implies that $\mathsf{Aut}(\mathcal{D}) \lesssim \mathsf{Aut}(\Gamma)$ since $\mathsf{Aut}(\mathcal{D}) \cong K$ by Lemma 9. □

We are now ready to prove Theorem 8.

Proof of Theorem 8. First we show the necessity. Suppose that Γ is edge-transitive. Let $\Gamma_{i,j} = [\mathcal{N}^i, \mathcal{B}^j]$ for $i \in I_1$ and $j \in I_2$. Note that $\Gamma_{i,j}$ is an induced bipartite subgraph of Γ and is isomorphic to $IG_s(\mathcal{D})$. It follows that the induced subgroup of $\mathsf{Aut}(\Gamma)$ on $\Gamma_{i,j}$ is also edge-transitive on $\Gamma_{i,j}$, which implies that $IG_s(\mathcal{D})$ is edge-transitive, thus $\mathcal{D}$ is s-flag-transitive.

Next we show the sufficiency. Suppose that Γ is not edge-transitive. Then there must exist two edges of Γ, say $\{N_p^i, B_q^j\}$ and $\{N_{p'}^{i'}, B_{q'}^{j'}\}$, such that $\{N_p^i, B_q^j\}^\delta \neq \{N_{p'}^{i'}, B_{q'}^{j'}\}$ for any $\delta \in \mathsf{Aut}(\Gamma)$. Let

$$\begin{aligned} \Omega_1 &= \{N_1^1, \ldots, N_1^b\}, & \mho_1 &= \{B_1^1, \ldots, B_1^{[v]_s}\}, \\ &\vdots & &\vdots \\ \Omega_{[v]_s} &= \{N_{[v]_s}^1, \ldots, N_{[v]_s}^b\}, & \mho_b &= \{B_b^1, \ldots, B_b^{[v]_s}\}. \end{aligned}$$

Note that for any $\omega \in \mathrm{S}_{\Omega_p}$, $\omega' \in \mathrm{S}_{\mho_q}$, (ω, ω') is an automorphism of Γ, so there must exist some automorphism (ω, ω') such that $\{N_p^i, B_q^j\}^{(\omega,\omega')} = \{N_p^1, B_q^1\}$. Thus, it suffices to consider the non-edge-transitivity of $\Gamma_{1,1}$. In other words, there must exist two edges $\{N_p^1, B_q^1\}$, $\{N_{p'}^1, B_{q'}^1\} \in E(\Gamma_{1,1})$, but $\{N_p^1, B_q^1\}^\delta \neq \{N_{p'}^1, B_{q'}^1\}$ for any $\delta \in \mathsf{Aut}(\Gamma)$. This implies $IG_s(\mathcal{D})$ is not edge-transitive since $\Gamma_{1,1} \cong IG_s(\mathcal{D})$ and $\mathsf{Aut}(\mathcal{D}) \lesssim \mathsf{Aut}(\Gamma)$ by Theorem 10. Therefore $\mathcal{D}$ is not s-flag-transitive.

4. SEMISYMMETRIC GRAPHS

In this section, we shall consider biplanes and triplanes, i.e., symmetric 2-(v, k, λ_2) designs with $\lambda_2 = 2$ or 3. Our aim is to classify such designs $\mathcal{D}$ whose incidence graphs $\Gamma = \mathsf{IG}_{[v]_2,v}(\mathcal{D})$ are semisymmetric. For a vertex u in a graph G, let $N_i(u)$ denote the set of vertices of G at distance i from u. Our first aim is to show that every incidence graph Γ is not vertex-transitive.

Proposition 11. *If $\lambda_2 = 2$, then in $IG_2(\mathcal{D})$, we have $|N_2(\{\alpha, \beta\})| = 2v - 4$ for every $\{\alpha, \beta\} \in \mathcal{P}^{(2)}$, and $|N_2(B)| = v - 1$ for every $B \in \mathcal{B}$.*

Proof. Let α, β be two points of $\mathcal{D}$. If $\lambda_2 = 2$, then there are exactly two blocks, say B, B', such that $\{\alpha, \beta\} \subset B, B'$. Since $\mathcal{D}$ is symmetric, we have $B \cap B' = \{\alpha, \beta\}$ by Lemma 2.

Since $|B| = |B'| = k$, there are $[k]_2$ 2-subsets in each of B and B', and therefore there are $2[k]_2 - 2$ different 2-subsets in either B or B', different from $\{\alpha, \beta\}$. This implies that $|N_2(\{\alpha, \beta\})| = 2[k]_2 - 2$. Since $\lambda_2(v - 1) = k(k - 1)$ from relation (2), we have $|N_2(\{\alpha, \beta\})| = 2(v - 1) - 2 = 2v - 4$. On the other hand, for any block B, it is easy to see that $|N_2(B)| = v - 1$, since the intersection size of any two blocks is $\lambda_2 = 2$.

Corollary 12. *If $\lambda_2 = 2$, then for any $N \in \mathcal{N}^i$ and $B \in \mathcal{B}^j$ in Γ, we have $|N_2(N)| \neq |N_2(B)|$. Hence Γ is not vertex-transitive.*

Proof. By Proposition 11, $|N_2(N)| = (2v - 4) + (2v - 3)(v - 1) = 2v^2 - 3v - 1$ and $|N_2(B)| = (v - 1) + v([v]_2 - 1) = \dfrac{v^2(v - 1)}{2} - 1$. If $2v^2 - 3v - 1 = \dfrac{v^2(v - 1)}{2} - 1$, then we have $v(v-2)(v-3)=0$, and thus $v=0,2$ or 3, a contradiction as $v>k\geq 3$. Thus $|N_2(N)|\neq|N_2(B)|$.

Proposition 13. *If $\lambda_2=3$, then in $IG_2(\mathcal{D})$, we have $|N_2(\{\alpha,\beta\})|=\frac{9v}{2}-\frac{23}{2}$ or $\frac{9v}{2}-\frac{27}{2}$ for every $\{\alpha,\beta\}\in\mathcal{P}^{(2)}$, and $|N_2(B)|=v-1$ for every $B\in\mathcal{B}$.*

Proof. Let α,β be two points of $\mathcal{D}$. If $\lambda_2=3$, then there are exactly three blocks, say B_1,B_2,B_3, containing them. By Lemma 2, we can assume $B_1\cap B_2=\{\alpha,\beta,\gamma\}$, $B_1\cap B_3=\{\alpha,\beta,\delta\}$, $B_2\cap B_3=\{\alpha,\beta,\varepsilon\}$.

First, we let $\gamma=\delta$, so that $B_1\cap B_2\cap B_3=\{\alpha,\beta,\gamma\}$. Since $B_1\cap B_2\cap B_3\subseteq B_2\cap B_3$, we have $\gamma=\varepsilon$. Since $|B_1|=|B_2|=|B_3|=k$, there are $[k]_2$ 2-subsets in each of B_1,B_2,B_3. Therefore, there are $3[k]_2-2[3]_2-1$ different 2-subsets contained in at least one of B_1,B_2 or B_3, different from $\{\alpha,\beta\}$. This implies that $|N_2(\{\alpha,\beta\})|=3[k]_2-2[3]_2-1$. Since $\lambda_2(v-1)=k(k-1)$ from (2), we have $|N_2(\{\alpha,\beta\})|=\frac{3}{2}\lambda_2(v-1)-7=\frac{9}{2}v-\frac{23}{2}$.

Next, if $\gamma\neq\delta$, then $\gamma,\delta,\varepsilon$ are mutually distinct. Similarly, one can see that there are $3[k]_2-3[3]_2$ different 2-subsets contained in at least one of B_1,B_2 or B_3, different from $\{\alpha,\beta\}$. This implies $|N_2(\{\alpha,\beta\})|=3[k]_2-9=\frac{3}{2}\lambda_2(v-1)-9=\frac{9}{2}v-\frac{27}{2}$.

Finally, for any block B, we have $|N_2(B)|=v-1$, as the intersection size of any two blocks is $\lambda_2=3$.

Corollary 14. *If $\lambda_2=3$ and $v\neq 5$, then for any $N\in\mathcal{N}^i$ and $B\in\mathcal{B}^j$ in Γ, we have $|N_2(N)|\neq|N_2(B)|$. Hence Γ is not vertex-transitive.*

Proof. By Proposition 13, we have $|N_2(N)|=\left(\frac{9v}{2}-\frac{23}{2}\right)+\left(\frac{9v}{2}-\frac{21}{2}\right)(v-1)=\frac{9}{2}v^2-\frac{21}{2}v-1$ or $|N_2(N)|=\left(\frac{9v}{2}-\frac{27}{2}\right)+\left(\frac{9v}{2}-\frac{25}{2}\right)(v-1)=\frac{9}{2}v^2-\frac{25}{2}v-1$, and $|N_2(B)|=(v-1)+v([v]_2-1)=\frac{v^2(v-1)}{2}-1$. If $\frac{9}{2}v^2-\frac{21}{2}v-1=\frac{v^2(v-1)}{2}-1$, we have $v(v-3)(v-7)=0$, which implies $v=0,3$ or 7, and so $v=7$ since $v>k\geq 3$. However, if $v=7$, then we have $k(k-1)=\lambda_2(v-1)=18$, which is impossible. Now, suppose that $\frac{9}{2}v^2-\frac{25}{2}v-1=\frac{v^2(v-1)}{2}-1$. This gives $v(v-5)^2=0$, and hence $v=5$, which contradicts our assumption. Consequently, $|N_2(N)|\neq|N_2(B)|$. □

We see that the proof of Corollary 14 would not hold if we have $v=5$. We would have $k=4$, and a 2-$(5,4,3)$ design has the property that $|N_2(N)|=|N_2(B)|$, for any $N\in\mathcal{N}^i$ and $B\in\mathcal{B}^j$ in Γ. We can address this problem in the following, more general result.

Theorem 15. *Let $\mathcal{D}$ be a symmetric 2-$(v,v-1,v-2)$ design, where $v\geq 4$. Then Γ is semisymmetric.*

Proof. We have $\mathcal{D}=(\mathcal{P},\mathcal{B})$, where $\mathcal{P}=\{\alpha_1,\alpha_2,\ldots,\alpha_v\}$ and $\mathcal{B}=\{\mathcal{P}\setminus\{\alpha_i\}:i\in I_1\}$. It is obvious that $\mathcal{D}$ is a symmetric 2-$(v,v-1,v-2)$ design. Note that the diameter of $IG_2(\mathcal{D})$ is 4 when $v = 4$, and is 3 when $v \geq 5$.

If $v = 4$, then $\mathcal{D}$ is a 2-$(4, 3, 2)$ design. Therefore Γ is not vertex-transitive by Corollary 12. Now we assume $v \geq 5$. Then $N_2(\{\alpha_p, \alpha_q\}^1) = (\bigcup_{i=1}^{v} \mathcal{N}^i) \setminus \{\{\alpha_p, \alpha_q\}^1\}$, and $N_3(\{\alpha_p, \alpha_q\}^1) = \{(\mathcal{P} \setminus \{\alpha_p\})^j, (\mathcal{P} \setminus \{\alpha_q\})^j : j \in I_2\}$. Let $B = \mathcal{P} \setminus \{\alpha_p\}$. Then $N_2(B^1) = (\bigcup_{j=1}^{[v]_2} \mathcal{B}^j) \setminus \{B^1\}$, and $N_3(B^1) = \{\{\alpha_p, \alpha_r\}^i : r \in I_1 \setminus \{p\}, i \in I_1\}$. It follows that there are $v - 1$ isolated vertices in the induced subgraph $[N_2(\{\alpha_p, \alpha_q\}^1), N_3(\{\alpha_p, \alpha_q\}^1)]$. There are $[v]_2 - 1$ isolated vertices in the induced subgraph $[N_2(B^1), N_3(B^1)]$. This implies that Γ is not vertex-transitive.

On the other hand, it is obvious that $\mathsf{Aut}(\mathcal{D}) = \mathrm{S}_v$ (the symmetric group on v elements). Therefore the stabilizer of a block B is S_{v-1} which is 2-homogeneous on B when $v \geq 4$. This implies that $\mathcal{D}$ is 2-flag-transitive. Furthermore, Γ is edge-transitive by Theorem 8. Hence, Γ is semisymmetric. □

We recall, as remarked after Theorem 7, that in the next two theorems, the designs with the parameters in Theorems 6 and 7 all refer to the unique primitive flag-transitive designs.

Theorem 16. *If $\mathcal{D}$ is a non-trivial biplane with a 2-flag-transitive automorphism group G, then one of the following holds:*

(i) *$\mathcal{D}$ has parameters* $(16, 6, 2)$,

(ii) *$\mathcal{D}$ has parameters* $(7, 4, 2)$,

(iii) *$\mathcal{D}$ has parameters* $(11, 5, 2)$,

(iv) $G \leq A\Gamma L_1(q)$, *for some odd prime power q.*

Proof. We have $\mathcal{D}$ is flag-transitive, and thus $\mathcal{D}$ is block-transitive. By Theorem 6, it suffices to show that if (i), (ii) or (iii) in the theorem holds, then the stabilizer subgroup G_B, for any block B, is 2-homogeneous on B.

Let $\mathcal{D}$ be the 2-$(16, 6, 2)$ design. According to [**15**, Lemma 6.5], $\mathcal{D}$ has an automorphism group G which fixes B and is 4-transitive on B. This implies that G must be 2-homogeneous on B.

Let $\mathcal{D}$ be the 2-$(7, 4, 2)$ design. This is the complementary design of the Fano plane $PG(2, 2)$. According to [**28**, Lemma 11], $G = PSL_2(7)$ and the point stabilizer G_x of G on $\mathcal{D}$ is S_4. This implies that $|G_B| = |G_x| = 24$, and thus G_B must be 4-transitive on B as $|B| = 4$. Hence G_B must be 2-homogeneous on B.

Let $\mathcal{D}$ be the 2-$(11, 5, 2)$ design. This is the unique Hadamard 2-design of order 3. According to [**28**, Lemma 11], $G = PSL_2(11)$ and the point stabilizer of G on $\mathcal{D}$ is A_5. By [**15**, Lemma 6.2], G_B is 3-transitive on B, so that G_B must be 2-homogeneous on B.

In every case, $\mathcal{D}$ is 2-flag-transitive.

Theorem 17. *If $\mathcal{D}$ is a non-trivial triplane with a 2-flag-transitive automorphism group G, then one of the following holds:*

(i) $\mathcal{D}$ *has parameters* $(11, 6, 3)$,

(ii) $\mathcal{D}$ *has parameters* $(15, 7, 3)$,

(iii) $G \leq A\Gamma L_1(q)$, *where* q *is some power of a prime* p *and* $p \geq 5$.

Proof. We have $\mathcal{D}$ is flag-transitive, and thus $\mathcal{D}$ is block-transitive. By Theorem 7, it suffices to show that if (i) in the theorem holds, then $\mathcal{D}$ is not 2-flag-transitive, and if (ii) or (iii) holds, then the stabilizer subgroup G_B, for any block B, is 2-homogeneous on B.

Let $\mathcal{D}$ be the 2-$(45, 12, 3)$ design. According to [**35**, Theorem 1.2], $G = PSp_4(3) : 2$ is flag-transitive on $\mathcal{D}$. This implies $|G_B| = |G_x| = 1152$. Note that there are $[12]_2 = 66$ 2-subsets in the block B. Since $66 \nmid 1152$, G_B is not transitive on the set of 2-subsets of B. Thus $\mathcal{D}$ is not 2-flag-transitive.

Let $\mathcal{D}$ be the 2-$(11, 6, 3)$ design. This is the complementary design of the 2-$(11, 5, 2)$ design, the Hadamard 2-design of order 3. Thus the automorphism group of $\mathcal{D}$ is $G = PSL_2(11)$ and the point stabilizer G_x of G is A_5. By [**15**, Theorem 6.3], G_B is 2-homogeneous on B. Thus $\mathcal{D}$ is 2-flag-transitive.

Let $\mathcal{D}$ be the 2-$(15, 7, 3)$ design. By [**34**], $\mathcal{D} = PG_2(3, 2)$, a point-hyperplane design. According to [**34**, Theorem 1.1], the automorphism group of $\mathcal{D}$ is $G = A_7$ and G is 2-point-transitive [**34**, p.119]. Thus G_B is 2-transitive on B by [**15**, Theorem 5.6]. This implies G_B must be 2-homogeneous on B. Thus $\mathcal{D}$ is 2-flag-transitive. □

In light of Propositions 12 and 14, and Theorems 8, 15, 16 and 17, we obtain the biplanes and triplanes correspnding to the semisymmetric graphs as the follows.

Theorem 18. *Let* $\mathcal{D}$ *be any biplane or triplane. If* Γ *is a semisymmetric graph, then one of the following holds*:

(1) $\mathcal{D}$ *has parameters* $(16, 6, 2)$,

(2) $\mathcal{D}$ *has parameters* $(7, 4, 2)$,

(3) $\mathcal{D}$ *has parameters* $(11, 5, 2)$,

(4) $\mathcal{D}$ *has parameters* $(11, 6, 3)$,

(5) $\mathcal{D}$ *has parameters* $(15, 7, 3)$,

(6) $\mathsf{Aut}(\mathcal{D}) \leq A\Gamma L_1(q)$, *for some odd prime power* q. *In particular,* q *is some power of a prime* p *with* $p \geq 5$ *when* $\mathcal{D}$ *is a triplane.*

Finally, there are only finitely many known examples of symmetric designs for any fixed $\lambda_2 > 1$, and it has been conjectured that for any fixed $\lambda_2 > 1$, only finitely many exist. Thus we may propose the following question.

Question 19. *Find all the semisymmetric graphs constructed from any symmetric design with* $\lambda_2 > 1$.

To solve this question, we need consider the next question arisen from Theorem 8.

Question 20. *Classify all the* 2*-flag-transitive symmetric designs with some given* $\lambda_2 > 1$.

5. CONNECTEDNESS OF INCIDENCE GRAPHS

In this section, we shall investigate the connectedness of the incidence graphs that we have already seen. Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a t-(v, k, λ_t) design with $|\mathcal{B}| = b$, and $1 \leq s \leq t$. Recall from Lemma 3 that the s-incidence graph $IG_s(\mathcal{D})$ is biregular, and from Proposition 4 that the graph $\Gamma = \mathsf{IG}_{[v]_s,b}(\mathcal{D})$ is regular. Thus, we can possibly expect that these graphs are connected, and indeed, they may even have high vertex-connectivity. We begin with the following result, which states that $IG_s(\mathcal{D})$ is 2-connected for $1 \leq s < t$.

Theorem 21. *Let* $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ *be a* t*-*(v, k, λ_t) *design. If* $1 \leq s < t$*, then* $IG_s(\mathcal{D})$ *is* 2*-connected.*

Proof. Let u be a vertex of $IG_s(\mathcal{D})$, and let U be the subset of $\mathcal{P}$ corresponding to u, so that U is an s-subset of $\mathcal{P}$ or a block of $\mathcal{B}$. It suffices to show that, for any two blocks $B, B' \in \mathbb{B} \setminus \{U\}$, there is a walk in $IG_s(\mathcal{D}) - u$ connecting B and B'. The theorem then follows since every vertex of $P^{(s)}$ in $IG_s(\mathcal{D})$ has $\lambda_s > 1$ neighbours in $\mathcal{B}$ (by Lemmas 1 and 3, and note that $\lambda_s > \lambda_t \geq 1$ by (1)), so that in $IG_s(\ D) - u$, every vertex of $P^{(s)} \setminus \{U\}$ has a neighbour in $B \setminus \{U\}$.

Let $B, B' \in\ B \setminus \{U\}$. First, suppose that $|B \cap B'| \geq s$. Then, there exists an s-subset $N \in\ P^{(s)} \setminus \{U\}$ contained in $B \cap B'$, unless if $|B \cap B'| = s$, $U \in \mathcal{P}^{(s)}$ and $U = B \cap B'$. If the latter situation does not occur, then '$B - N - B'$' is a suitable walk. Otherwise, we can choose $\alpha \in B \setminus B'$, $\alpha' \in B' \setminus B$ (since $|B \setminus B'| = |B' \setminus B| = k - s \geq 2$) and $\beta \in B \cap B'$ (since $|B \cap B'| = s \geq 1$), and set

$$N = ((B \cap B') \setminus \{\beta\}) \cup \{\alpha\} \subset B, \quad N' = ((B \cap B') \setminus \{\beta\}) \cup \{\alpha'\} \subset B'.$$

Then $N, N' \in \mathcal{P}^{(s)} \setminus \{U\}$ (since $N, N' \neq B \cap B'$). Now, we have $|N \cup N'| = s + 1 \leq t$, and hence there exists a block $B'' \in\ B$ with $B'' \supset N \cup N'$. Therefore, '$B - N - B'' - N' - B'$' is a suitable walk.

Now, suppose that $|B \cap B'| \leq s - 1$. Let $q = s - |B \cap B'| \geq 1$. If $U \not\supset B \cap B'$, then we choose $\alpha_1, \ldots, \alpha_q \in B \setminus B'$ and $\alpha'_1, \ldots, \alpha'_q \in B' \setminus B$. Otherwise, if $U \supset B \cap B'$, then since U is either an s-subset (if $U \in\ P^{(s)}$) or a k-subset (if $U \in\ B \setminus \{B, B'\}$), we can choose $\alpha_1 \in B \setminus (B' \cup U)$ and $\alpha'_1 \in B' \setminus (B \cup U)$, and then choose $\alpha_2, \ldots, \alpha_q \in B \setminus (B' \cup \{\alpha_1\})$ and $\alpha'_2, \ldots, \alpha'_q \in B' \setminus (B \cup \{\alpha'_1\})$. Note that $|B \setminus B'| = |B' \setminus B| = k - |B \cap B'| > q$, since $k > s$, so that we can indeed choose the α_i and α'_i. For $0 \leq i \leq q$, we set

$$N_i = (B \cap B') \cup \{\alpha'_1, \ldots, \alpha'_i, \alpha_{i+1}, \ldots, \alpha_q\},$$

which is an s-subset of $\mathcal{P}^{(s)}$. If $U \not\supset B \cap B'$, then since $N_i \supset B \cap B'$, we have $N_i \neq U$. If $U \supset B \cap B'$, then since $\{\alpha_1, \alpha_1'\} \cap N_i \neq \emptyset$ and $\alpha_1, \alpha_1' \notin U$, again we have $N_i \neq U$. Now, for every $1 \le i \le q$, we have $|N_{i-1} \cup N_i| = s + 1 \le t$, and there exists a block $B_i \in \ B$ with $B_i \supset N_{i-1} \cup N_i$. If $U \not\supset B \cap B'$, then since $B_i \supset B \cap B'$, we have $B_i \neq U$. If $U \supset B \cap B'$, then since $\{\alpha_1, \alpha_1'\} \cap (N_{i-1} \cup N_i) \neq \emptyset$ implies $\{\alpha_1, \alpha_1'\} \cap B_i \neq \emptyset$, and $\alpha_1, \alpha_1' \notin U$, again we have $B_i \neq U$. Therefore, '$B - N_0 - B_1 - N_1 - \cdots - B_q - N_q - B'$' is a suitable walk. □

Can $IG_s(\mathcal{D})$ possibly have higher vertex-connectivity? Can the vertex-connectivity be as large as the minimum degree of $IG_s(\mathcal{D})$? We leave these as an open problem.

Question 22. *Let $\mathcal{D}$ be a t-(v, k, λ_t) design. If $1 \le s < t$, then how large is the vertex- connectivity of $IG_s(\mathcal{D})$? Is $IG_s(\mathcal{D})$ m-connected, where $m = \min(\lambda_s, [k]_s)$ is the minimum degree of $IG_s(\mathcal{D})$, and $\lambda_s = \lambda_t \frac{[v-s]_{t-s}}{[k-s]_{t-s}}$?*

We note that the proof of Theorem 21 does not hold when $s = t$ and $\mathcal{D}$ is a sharp t-(v, k, λ_t) design, since then there may exist a $(t+1)$-subset of $\mathcal{P}$ with no block of $\mathcal{B}$ containing it. Now, we consider the connectedness of $IG_t(\mathcal{D})$. We first consider the special case $\lambda_t = 1$.

Theorem 23. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a t-(v, k, λ_t) design with $|\mathcal{B}| = b$. If $\lambda_t = 1$, then $IG_t(\mathcal{D})$ is not connected. Moreover, we have $IG_t(\mathcal{D}) \cong bK_{1,[k]_t}$, where $K_{1,[k]_t}$ is the star graph with $[k]_t$ edges.*

Proof. By Lemma 3, the degree of any vertex of $\mathcal{P}^{(t)}$ in $IG_t(\mathcal{D})$ is $\lambda_t = 1$. If $IG_t(\mathcal{D})$ is connected, then there exists some block $B \in \mathcal{B}$ such that $N \subset B$ for all $N \in \mathcal{P}^{(t)}$. It follows that $|B| = v$, which is a contradiction. By Lemma 3, we have $IG_t(\mathcal{D}) \cong bK_{1,[k]_t}$. □

Next, we consider the case when $\lambda_t > 1$.

Theorem 24. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a t-(v, k, λ_t) design with $\lambda_t > 1$. If the incidence graph of any $(t-1)$-derived design of $\mathcal{D}$ is connected, then $IG_t(\mathcal{D})$ is connected.*

Proof. It suffices to show that for any two t-subsets N, N' of $\mathcal{P}$ with $|N \cap N'| = t-1$, there is a path in $IG_t(\mathcal{D})$ connecting their corresponding vertices. The theorem follows since then, any two vertices of $\mathcal{P}^{(t)}$ are connected in $IG_t(\mathcal{D})$, and any vertex of $\mathcal{B}$ is adjacent to some vertex of $\mathcal{P}^{(t)}$.

Let $N, N' \in \mathcal{P}^{(t)}$ with $|N \cap N'| = t - 1$. Let $N \cap N' = N_0$ and $\mathcal{P} \setminus N_0 = \{\alpha_1, \ldots, \alpha_{v-t+1}\}$. For $1 \le i \le v - t + 1$, let $N_i = \{\alpha_i\} \cup N_0$, so that the N_i are all the t-subsets of $\mathcal{P}$ containing N_0. By Lemma 1, $\mathcal{D}$ is also a $(t-1)$-design, and there are λ_{t-1} blocks of $\mathcal{B}$ containing N_0, say $B_1, \ldots, B_{\lambda_{t-1}}$, where $\lambda_{t-1}(k - t + 1) = \lambda_t(v - t + 1)$. Let $[N_0]$ denote the subgraph of $IG_t(\mathcal{D})$ induced by the vertices $N_1, \ldots, N_{v-t+1}, B_1, \ldots, B_{\lambda_{t-1}}$, and note that N, N' are two vertices of $[N_0]$. Now, if we obtain a $(t-1)$-derived design $\mathcal{D}_{N_0}$ by deleting the points of N_0 from $\mathcal{P}$, one at a time, then $\mathcal{D}_{N_0}$ is a 1-$(v-t+1, k-t+1, \lambda_t)$ design, and its incidence graph has bipartition classes $\mathcal{P} \setminus N_0$ and $\{B_1 \setminus N_0, \ldots, B_{\lambda_{t-1}} \setminus N_0\}$. It is then clear that $[N_0]$ is isomorphic to the incidence graph of $\mathcal{D}_{N_0}$. Therefore by assumption, $[N_0]$ is connected, and there exists a path connecting N and N' in $[N_0]$. □

Theorem 24 makes a strong assumption about the connectedness of $(t-1)$-derived designs of a t-design. For the assumption of Theorem 24, we conjecture that:

Conjecture 25. *Let $\mathcal{D}$ be a t-(v, k, λ_t) design with $\lambda_t > 1$. Then the incidence graph of any $(t-1)$-derived design of $\mathcal{D}$ is connected.*

Note that any t-(v, k, λ_t) design can be derived into a 2-$(v-t+2, k-t+2, \lambda_t)$ design. It follows that the conjecture is equivalent to the conjecture below.

Conjecture 26. *The incidence graph of a derived design of a 2-(v, k, λ_2) design $\mathcal{D}$, where $\lambda_2 > 1$, is connected.*

Remark. Conjecture 26 is true when $\mathcal{D}$ is symmetric. However, it is complicated if $\mathcal{D}$ is not symmetric, since the derived designs of $\mathcal{D}$ need not be isomorphic. Indeed, if $\mathcal{D}$ is a symmetric 2-(v, k, λ_2) design, then by Lemma 2, the cardinality of the intersection of any two blocks is λ_2. Let $\mathcal{D}_p$ be the derived design of $\mathcal{D}$ for any point p of $\mathcal{P}$. Let B, B' be any two blocks of $\mathcal{D}$ containing p. Then $B \setminus \{p\}$ and $B' \setminus \{p\}$ are two blocks of $\mathcal{D}_p$. Note that if $\lambda_2 > 1$, then $|(B \setminus \{p\}) \cap (B' \setminus \{p\})| \geq 1$ and there exists some point q such that $q \in (B \setminus \{p\}) \cap (B' \setminus \{p\})$. Therefore, there is a path '$B \setminus \{p\} - q - B' \setminus \{p\}$' connecting $B \setminus \{p\}$ and $B' \setminus \{p\}$ in the incidence graph $IG(\mathcal{D}_p)$. This implies that $IG(\mathcal{D}_p)$ is connected.

By the remark and Theorem 24, we have the following corollaries.

Corollary 27. *Let $\mathcal{D}$ be a symmetric t-(v, k, λ_t) design with $\lambda_t > 1$. Then $IG_t(\mathcal{D})$ is connected.*

Corollary 28. *Let $\mathcal{D}$ be a symmetric 2-(v, k, λ_2) design with $\lambda_2 > 1$. Then the diameter of $IG_2(\mathcal{D})$ is at most 4.*

Example 29. Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a Hadamard 3-design 3-$(8, 4, 1)$. By Lemma 1, it is also a 2-$(8, 4, 3)$ design, where $\mathcal{P} = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and

$$\begin{aligned}\mathcal{B} = \Big\{ & \{1,3,7,8\}, \{1,2,4,8\}, \{2,3,5,8\}, \{3,4,6,8\}, \{4,5,7,8\}, \\ & \{1,5,6,8\}, \{2,6,7,8\}, \{1,2,3,6\}, \{1,2,5,7\}, \{1,3,4,5\}, \\ & \{1,4,6,7\}, \{2,3,4,7\}, \{2,4,5,6\}, \{3,5,6,7\} \Big\}.\end{aligned}$$

Let $B_1 = \{1, 3, 7, 8\}$ and $B_2 = \{2, 4, 5, 6\}$. According to Theorem 23, there is no path connecting B_1 and B_2 in $IG_3(\mathcal{D})$. But there is a path connecting B_1 and B_2 in $IG_2(\mathcal{D})$. The path is '$B_1 - N_1 - B_3 - N_2 - B_4 - N_3 - B_2$' where $B_3 = \{1, 2, 3, 6\}$, $B_4 = \{1, 2, 4, 8\}$, $N_1 = \{1, 3\}$, $N_2 = \{1, 2\}$ and $N_3 = \{2, 4\}$.

By the construction of the graphs Γ and Theorems 21, 23, we have the following two theorems.

Theorem 30. *Let $\mathcal{D}$ be a t-(v,k,λ_t) design with b blocks, and $1 \le s < t$. Let $d = \min([v]_s, b)$. Then $\Gamma = \mathsf{IG}_{[v]_s,b}(\mathcal{D})$ is $2d$-connected.*

Proof. Let Γ' be a graph obtained by deleting at most $2d-1$ vertices of Γ. Without loss of generality, at most one vertex of $\mathcal{N}^1 \cup \mathcal{B}^1$ is deleted. Let H be the subgraph of Γ' induced by $(\mathcal{N}^1 \cup \mathcal{B}^1) \cap V(\Gamma')$. Thus, H is isomorphic to either $IG_s(\mathcal{D})$ or $IG_s(\mathcal{D})-u$, where u is some vertex in $IG_s(\mathcal{D})$. By Theorem 21, H is connected. By Lemmas 1 and 3, the graph $IG_s(\mathcal{D})$ is biregular, with degrees $\lambda_s > 1$ and $[k]_s > 1$, if $1 \le s < t$. This means that in Γ', if $N \in \mathcal{N}^i \cap V(\Gamma')$ for some $i \in I_1 \setminus \{1\}$, then N is adjacent to some $B \in \mathcal{B}^1 \cap V(H)$. Similarly, for any $B \in \mathcal{B}^j \cap V(\Gamma')$, where $j \in I_2 \setminus \{1\}$, we have B is adjacent to some $N \in \mathcal{N}^1 \cap V(H)$. Thus Γ' is connected, and therefore Γ is $2d$-connected.

Theorem 31. *Let $\mathcal{D}$ be a t-$(v,k,1)$ design (not necessarily sharp) with b blocks. Then $\Gamma \cong bK_{[v]_t,[v]_t}$, where $\Gamma = \mathsf{IG}_{[v]_t,b}(\mathcal{D})$ and $K_{[v]_t,[v]_t}$ is the $[v]_t$ by $[v]_t$ balanced complete bipartite graph. Conversely, if $\Gamma \cong bK_{[v]_t,[v]_t}$ where $\Gamma = \mathsf{IG}_{[v']_{t'},b'}(\mathcal{D})$, then $\mathcal{D}$ is a t'-$(v',k',1)$ Steiner design with b blocks.*

Proof. If $\mathcal{D}$ is a t-$(v,k,1)$ design, then $IG_t(\mathcal{D})$ is isomorphic to $bK_{1,[k]_t}$, the union of b star graphs $K_{1,[k]_t}$. By the construction of Γ, Γ is the union of b isomorphic bipartite graphs where one bipartition has $b[k]_t$ vertices which are b copies of some t-subsets, the other bipartition has $[v]_t$ vertices which are $[v]_t$ copies of some block. Therefore $\Gamma \cong bK_{[v]_t,[v]_t}$. Conversely, suppose $\Gamma \cong bK_{[v]_t,[v]_t}$. Let $\Gamma = \mathsf{IG}_{[v']_{t'},b'}(\mathcal{D})$. Since $\mathcal{D}$ is simple, it follows that the vertices of one bipartition of any of these bipartite graphs are copies of some block of $\mathcal{D}$. It implies that $b' = b$ and $\mathcal{D}$ is a t'-$(v',k',1)$ design. Thus $\mathcal{D}$ is a Steiner design with b blocks. $\square$

We end with the following open question, which is the analogue of Question 22 for Γ.

Question 32. *Let $\mathcal{D}$ be a t-(v,k,λ_t) design with b blocks. If $1 \le s < t$, then how large is the vertex-connectivity of $\Gamma = \mathsf{IG}_{[v]_s,b}(\mathcal{D})$? Is Γ d-connected, where $d = [v]_s\lambda_s$ is the degree of every vertex of Γ, and $\lambda_s = \lambda_t \dfrac{[v-s]_{t-s}}{[k-s]_{t-s}}$?*

6. ENERGY OF INCIDENCE GRAPHS

We study the energy of the incidence graphs Γ. Recall that if a design is not a symmetric 2-design, then the size of $B \cap B'$ for any two different blocks B, B' is difficult to determine, and the computation of the eigenvalues of the adjacency matrix of Γ is not simple. Thus we mainly consider symmetric 2-(v,k,λ_2) designs in this section. In this case, note that from (2), we have $k = \dfrac{v-1}{k-1}\lambda_2 > \lambda_2$, and hence $v > k > \lambda_2$.

Recall that for a graph G, say with vertex set $V(G) = \{u_1, \dots, u_n\}$, the *adjacency matrix* of G is the matrix $\mathbf{A}(G) = (a_{ij})_{n\times n}$ where $a_{ij} = 1$ if u_i and u_j are adjacent, and $a_{ij} = 0$ otherwise. It is well known that $\mathbf{A}(G)$ is a real symmetric matrix with real eigenvalues. The *spectrum* of G is $\mathfrak{S}(\mathbf{A}(G))$, i.e., the spectrum of $\mathbf{A}(G)$, which is the collection of the eigenvalues of $\mathbf{A}(G)$. Now, suppose that G is a bipartite graph with vertex bipartition V_1 and V_2. Let $V_1 = \{u_1, \dots, u_m\}$ and $V_2 = \{u_{m+1}, \dots, u_n\}$. The *biadjacency matrix* of G is $\mathbf{B}(G) = (b_{ij})_{m\times(n-m)}$, where $b_{ij} = 1$ if u_i is adjacent to u_{j+m}, and $b_{ij} = 0$ otherwise.

Let $\mathcal{D}$ be a t-(v, k, λ_t) design with b blocks, and $1 \le s \le t$. The *s-incidence matrix* of $\mathcal{D}$, which concerns the relationship between s-subsets and blocks of $\mathcal{D}$, is exactly the biadjacency matrix of $IG_s(\mathcal{D})$. That is, the $[v]_s \times b$ matrix $A_s = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1, & \text{if the } i\text{-th } s\text{-subset is contained in the } j\text{-th block;} \\ 0, & \text{otherwise.} \end{cases}$$

For symmetric 2-designs, we have the following lemma.

Lemma 33. *Let $\mathcal{D}$ be a symmetric 2-(v, k, λ_2) design, and A_2 be the 2-incidence matrix of $\mathcal{D}$. Then*

$$A_2^T A_2 = ([k]_2 - [\lambda_2]_2)I + [\lambda_2]_2 J,$$

where I is the $v \times v$ identity matrix and J is the $v \times v$ matrix with all entries equal to 1. By convention, $[\lambda_2]_2 = 0$ for $\lambda_2 = 1$.

Proof. Obviously $A_2^T A_2$ is a $v \times v$ matrix whose (i, j)-th entry is the real inner product of the i-th and the j-th columns of A_2. If $i = j$, then this is just the number of 2-subsets in the i-th block which is $[k]_2$. When $\lambda_2 > 1$, the (i, j)-th entry is $[\lambda_2]_2$ if $i \neq j$. This is obvious since the size of the intersection of any two blocks is λ_2. When $\lambda_2 = 1$, it follows that the (i, j)-th entry is zero. □

Now, we can compute the energy of the incidence graph Γ of a symmetric 2-design.

Theorem 34. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a symmetric 2-(v, k, λ_2) design with $|\mathcal{B}| = b = v$, and $\Gamma = \mathsf{IG}_{[v]_2,b}(\mathcal{D}) = \mathsf{IG}_{[v]_2,v}(\mathcal{D})$. Then, the energy of Γ is given by*

$$\mathcal{E}(\Gamma) = 2[v]_2\big(\lambda_2 + \sqrt{\lambda_2(v - \lambda_2)(v - 1)}\big). \tag{3}$$

In particular, if $\lambda_2 = 1$, then $\mathcal{E}(\Gamma) = 2v[v]_2$.

Proof. Note that Γ is a balanced bipartite graph with vertex bipartition $V_1 = \bigcup_{i=1}^{b} \mathcal{N}^i$ and $V_2 = \bigcup_{j=1}^{[v]_2} \mathcal{B}^j$, where each $\mathcal{N}^i$ is a copy of $\mathcal{P}^{(2)}$ and each $\mathcal{B}^j$ is a copy of $\mathcal{B}$, so that Γ has $2v[v]_2$ vertices. Let A_2 be the 2-incidence matrix of $\mathcal{D}$, and $\mathbf{B}$ and $\mathbf{A}$ be the biadjacency and adjacency matrices of Γ. After a proper ordering of the vertices of Γ, we see that

$$\mathbf{B} = \begin{pmatrix} A_2 & \cdots & A_2 \\ \vdots & & \vdots \\ A_2 & \cdots & A_2 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} & \mathbf{B} \\ \mathbf{B}^T & \end{pmatrix},$$

where each empty space of $\mathbf{A}$ contains all zeros. Note that there are $[v]_2$ matrices A_2 in each row of $\mathbf{B}$, and b matrices A_2 in each column of $\mathbf{B}$, so $\mathbf{B}$ is a $b[v]_2 \times b[v]_2$ square matrix. Since

$$\mathbf{A}^2 = \begin{pmatrix} \mathbf{B}\mathbf{B}^T & \\ & \mathbf{B}^T\mathbf{B} \end{pmatrix},$$

the eigenvalues of $\mathbf{A}$ are the singular values of $\mathbf{B}^T$ together with their negatives. Note that the singular values of a matrix M are the positive square roots of the eigenvalues of MM^T. For $\mathbf{B}^T\mathbf{B}$, we have

$$\begin{aligned}\mathbf{B}^T\mathbf{B} &= \begin{pmatrix} A_2^T & \cdots & A_2^T \\ \vdots & & \vdots \\ A_2^T & \cdots & A_2^T \end{pmatrix} \begin{pmatrix} A_2 & \cdots & A_2 \\ \vdots & & \vdots \\ A_2 & \cdots & A_2 \end{pmatrix} = \begin{pmatrix} bA_2^TA_2 & \cdots & bA_2^TA_2 \\ \vdots & & \vdots \\ bA_2^TA_2 & \cdots & bA_2^TA_2 \end{pmatrix}_{[v]_2\times[v]_2} \\ &= bJ_{[v]_2\times[v]_2} \otimes (A_2^TA_2).\end{aligned}$$

It is easy to compute the spectrum of $bJ_{[v]_2\times[v]_2}$, and by Lemma 33, we can find the spectrum of $A_2^TA_2$. These are

$$\begin{pmatrix} b[v]_2 & 0 \\ 1 & [v]_2-1 \end{pmatrix}, \qquad \begin{pmatrix} [k]_2+(v-1)[\lambda_2]_2 & [k]_2-[\lambda_2]_2 \\ 1 & v-1 \end{pmatrix}.$$

By Lemma 5, the spectrum of $\mathbf{B}^T\mathbf{B}$ is

$$\begin{pmatrix} ([k]_2+(v-1)[\lambda_2]_2)b[v]_2 & ([k]_2-[\lambda_2]_2)b[v]_2 & 0 \\ 1 & v-1 & v([v]_2-1) \end{pmatrix}.$$

We have

$$\begin{aligned} & ([k]_2+(v-1)[\lambda_2]_2)b[v]_2 \\ =\ & b[v]_2[k]_2 + b(v-1)[v]_2[\lambda_2]_2 \\ =\ & b[v]_2[k]_2 + b^2(v-1)[k]_2\frac{\lambda_2-1}{2} && \text{(since } [v]_2\lambda_2 = b[k]_2) \\ =\ & b[k]_2\left([v]_2 + v(v-1)\frac{\lambda_2-1}{2}\right) && \text{(since } b=v) \\ =\ & b[k]_2[v]_2\lambda_2 = ([v]_2\lambda_2)^2, \end{aligned}$$

and

$$\begin{aligned} & ([k]_2-[\lambda_2]_2)b[v]_2 \\ =\ & [v]_2^2\lambda_2 - [v]_2[\lambda_2]_2 b && \text{(since } [v]_2\lambda_2 = b[k]_2) \\ =\ & [v]_2\lambda_2\left([v]_2 - v\frac{\lambda_2-1}{2}\right) && \text{(since } b=v) \\ =\ & \frac{v-\lambda_2}{2}v[v]_2\lambda_2 = \frac{v^2\lambda_2}{4}(v-\lambda_2)(v-1). \end{aligned}$$

It follows that the spectrum of $\mathbf{B}^T\mathbf{B}$ is

$$\begin{pmatrix} ([v]_2\lambda_2)^2 & \frac{v^2\lambda_2}{4}(v-\lambda_2)(v-1) & 0 \\ 1 & v-1 & v([v]_2-1) \end{pmatrix}.$$

This implies that the singular values of $\mathbf{B}^T$ are

$$\begin{pmatrix} [v]_2\lambda_2 & \frac{v}{2}\sqrt{\lambda_2(v-\lambda_2)(v-1)} & 0 \\ 1 & v-1 & v([v]_2-1) \end{pmatrix}.$$

Thus the spectrum of $\mathbf{A}$ is

$$\begin{pmatrix} [v]_2\lambda_2 & \frac{v}{2}\sqrt{\lambda_2(v-\lambda_2)(v-1)} & 0 & -\frac{v}{2}\sqrt{\lambda_2(v-\lambda_2)(v-1)} & -[v]_2\lambda_2 \\ 1 & v-1 & 2v([v]_2-1) & v-1 & 1 \end{pmatrix}.$$

Therefore, the energy of Γ is

$$\mathcal{E}(\Gamma) = 2[v]_2\big(\lambda_2 + \sqrt{\lambda_2(v-\lambda_2)(v-1)}\big),$$

and (3) holds. □

We remark that for the case $\lambda_2 = 1$, Theorem 31 gives $\Gamma \cong bK_{[v]_2,[v]_2}$. It is easy to compute the spectrum of Γ, and obtain $\mathcal{E}(\Gamma) = 2v[v]_2$.

It was conjectured in [**8**] that the complete graph K_n, with energy $2(n-1)$, has the largest energy among all n-vertex graphs. However, this conjecture has been disproved in [**30**]. The n-vertex graphs G for which $\mathcal{E}(G) > 2(n-1)$ are defined as *hyperenergetic graphs*. Completely characterizing the hyperenergetic graphs is a challenging problem, and one may refer to [**9**] and their references. In the next result, we will completely characterize the symmetric 2-(v,k,λ_2) designs whose incidence graphs Γ are hyperenergetic. Recall that for any symmetric 2-(v,k,λ_2) design, we have $v > k > \lambda_2$.

Theorem 35. *Let $\mathcal{D}$ be a symmetric 2-(v,k,λ_2) design. Then $\Gamma = \mathsf{IG}_{[v]_2,v}(\mathcal{D})$ is not hyperenergetic if $\lambda_2 = 1,2,3,4$ or $\mathcal{D}$ has parameters $(v,k,\lambda_2) = (7,6,5),(8,7,6),(9,8,7)$. Otherwise, Γ is hyperenergetic, i.e., if $v \ge \lambda_2+3$ for $\lambda_2 = 5,6,7$, or $\lambda_2 \ge 8$.*

Proof. Recall that Γ has $2v[v]_2$ vertices. For $\lambda_2 = 1,2,3,4$, note that $(v-\lambda_2)(v-1) \le \Big(v - \frac{\lambda_2+1}{2}\Big)^2$, and hence

$$\begin{aligned} 2[v]_2\big(\lambda_2 + \sqrt{\lambda_2(v-\lambda_2)(v-1)}\big) &\le 2[v]_2\Big(\lambda_2 + 2\Big(v - \frac{\lambda_2+1}{2}\Big)\Big) \\ &= 2[v]_2(2v-1) \\ &< 4v[v]_2 - 2. \end{aligned}$$

When $(v,\lambda_2) = (7,5),(8,6),(9,7)$, substituting these values into (3) gives $\mathcal{E}(\Gamma) \approx 535.33$, 849.25 and 1265.98, while $4v[v]_2 - 2 = 586$, 894 and 1294. Hence, Γ is not hyperenergetic in all of these cases.

Now assume that $v \geq \lambda_2 + 3$ for $\lambda_2 = 5, 6, 7$, or $\lambda_2 \geq 8$. We need to prove that

$$2[v]_2\big(\lambda_2 + \sqrt{\lambda_2(v-\lambda_2)(v-1)}\,\big) > 4v[v]_2 - 2.$$

This follows if $\lambda_2 + \sqrt{\lambda_2(v-\lambda_2)(v-1)} \geq 2v$, which is easily seen to be equivalent to $v \geq \dfrac{\lambda_2(\lambda_2-3)}{\lambda_2-4}$. Thus if $\lambda_2 \geq 8$, we have $v \geq \lambda_2 + 2 \geq \dfrac{\lambda_2(\lambda_2-3)}{\lambda_2-4}$. If $\lambda_2 = 6, 7$, we have $v \geq \lambda_2 + 3 \geq \dfrac{\lambda_2(\lambda_2-3)}{\lambda_2-4}$. If $\lambda_2 = 5$, then we are done if $v \geq 10$. Now, equation (2) gives $k(k-1) = (v-1)\lambda_2$, and this is impossible for the remaining parameters $(v, \lambda_2) = (8, 5), (9, 5)$. We conclude that Γ is hyperenergetic. □

Next, we recall that if $\mathcal{D}$ is a symmetric 2-(v, k, λ_2) design, then its complementary design $\overline{\mathcal{D}}$ is a symmetric 2-$(v, v-k, v-2k+\lambda_2)$ design. Here, when we consider complementary designs, we will assume that $v \geq 2k$. Let $\overline{\Gamma} = \mathsf{IG}_{[v]_2,v}(\overline{\mathcal{D}})$. Then by (3), the energy of $\overline{\Gamma}$ is

$$\mathcal{E}(\overline{\Gamma}) = 2[v]_2\big(v - 2k + \lambda_2 + \sqrt{(v-2k+\lambda_2)(2k-\lambda_2)(v-1)}\,\big). \tag{4}$$

Now, we compare the energies of Γ and $\overline{\Gamma}$.

Theorem 36. *Let $\mathcal{D}$ be a symmetric 2-(v, k, λ_2) design with $v \geq 2k$, and $\overline{\mathcal{D}}$ be its complementary design. Then $\mathcal{E}(\Gamma) \leq \mathcal{E}(\overline{\Gamma})$.*

Proof. Recall that $v > k > \lambda_2$. Now,

$$(v-2k+\lambda_2)(2k-\lambda_2) - \lambda_2(v-\lambda_2) = 2(k-\lambda_2)(v-2k) \geq 0,$$

since $v \geq 2k$. Hence $(v-2k+\lambda_2)(2k-\lambda_2) \geq \lambda_2(v-\lambda_2)$, and by (3) and (4), we can easily obtain $\mathcal{E}(\Gamma) \leq \mathcal{E}(\overline{\Gamma})$.

Corollary 37. *Let $\mathcal{D}$ be a symmetric 2-(v, k, λ_2) design with $v \geq 2k$. Then Γ and $\overline{\Gamma}$ are hyperenergetic graphs if $\lambda_2 \geq 5$.*

Example 38. Let $\mathcal{D}$ be the projective plane of order $n \geq 2$. Then $\mathcal{D}$ is a symmetric 2-$(n^2+n+1, n+1, 1)$ design, and $\overline{\mathcal{D}}$ is a symmetric 2-(n^2+n+1, n^2, n^2-n) design. By Theorem 35, we have Γ is not hyperenergetic. Also $\overline{\Gamma}$ is not hyperenergetic if $n = 2$, and hyperenergetic if $n \geq 3$.

Example 39. Let $\mathcal{D}$ be the Hadamard 2-$(4n-1, 2n-1, n-1)$ design of order $n \geq 2$, which is a symmetric design. Then $\overline{\mathcal{D}}$ is a 2-$(4n-1, 2n, n)$ design. By Theorem 35, Γ is not hyperenergetic if $n = 2, 3, 4, 5$, and hyperenergetic if $n \geq 6$. Also, $\overline{\Gamma}$ is not hyperenergetic if $n = 2, 3, 4$, and hyperenergetic if $n \geq 5$.

Remark. Observe that there are infinitely many symmetric 2-$(v, k, 1)$ designs, since there are infinitely many projective planes of finite order. It has been conjectured that for any fixed $\lambda_2 > 1$, there are only finitely many symmetric 2-(v, k, λ_2) designs. By Theorem 35, this conjecture, when restricted to $\lambda_2 = 2, 3, 4$, is equivalent to the following conjecture: *Among all symmetric 2-(v, k, λ_2) designs $\mathcal{D}$ with $\lambda_2 > 1$, all but finitely many of the* *graphs* $\Gamma = \mathsf{IG}_{[v]_2,v}(\mathcal{D})$ *are hyperenergetic.* Note that there are infinitely many symmetric 2-(v,k,λ_2) designs with $\lambda_2 > 1$, by considering the Hadamard 2-designs, and hence there are infinitely many hyperenergetic graphs Γ.

Using Theorem 35, we can also deduce the following result, which may be considered as a partial result to the above conjecture.

Theorem 40. *Consider all symmetric* 2-(v,k,λ_2) *designs with* $v \geq 2k$, *and for such a design* $\mathcal{D}$, *let* $\overline{\mathcal{D}}$ *be its complementary design. Then, all but finitely many of the graphs* $\overline{\Gamma} = \mathsf{IG}_{[v]_2,v}(\overline{\mathcal{D}})$ *are hyperenergetic.*

Proof. Recall that if $\mathcal{D}$ is a symmetric 2-(v,k,λ_2) design, then $\overline{\mathcal{D}}$ is a symmetric 2-$(v, v-k, v-2k+\lambda_2)$ design. By Theorem 35, it suffices to check that there are finitely many sets of the parameters $(v, v-k, v-2k+\lambda_2)$ with $\overline{\lambda}_2 := v-2k+\lambda_2 \in \{1,2,3,4\}$. Since $v \geq 2k$, we have $1 \leq \lambda_2 \leq \overline{\lambda}_2 \leq 4$, and $v = 2k + \overline{\lambda}_2 - \lambda_2 \leq 2k+3$. By (2), we have $k(k-1) = (v-1)\lambda_2$, and using this equation for every such pair $(\lambda_2, \overline{\lambda}_2)$, we can easily see that the only possible sets of parameters for $(v, v-k, v-2k+\lambda_2)$ are $(7,4,2)$, $(11,6,3)$, and $(15,8,4)$.

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