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TOTAL ROMAN DOMINATION IN GRAPHS

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A Roman dominating function on a graph G is a function $f\colon V(G) \to \{0,1,2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Roman dominating function f is the sum, $\sum_{u\in V(G)} f(u)$, of the weights of the vertices. The Roman domination number is the minimum weight of a Roman dominating function in G. A total Roman domination function is a Roman dominating function with the additional property that the subgraph of G induced by the set of all vertices of positive weight has no isolated vertex. The total Roman domination number is the minimum weight of a total Roman domination function on G. We establish lower and upper bounds on the total Roman domination number. We relate the total Roman domination to domination parameters, including the domination number, the total domination number and Roman domination number.

1. INTRODUCTION

A *dominating set* in a graph G with vertex set $V(G)$ is a set S of vertices of G such that every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A dominating set of G of cardinality $\gamma(G)$ is called a $\gamma(G)$-set. The literature on the subject of domination parameters in graphs up to the year 1997 has been surveyed and detailed in the two books [**6, 7**].

A *total dominating set*, abbreviated TD-set, of a graph G with no isolated vertex is a set S of vertices of G such that every vertex in $V(G)$ is adjacent to at

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least one vertex in S. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set of G. A TD-set of G of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$-set. Total domination is now well studied in graph theory. The literature on the subject of total domination in graphs has been surveyed and detailed in the recent book [**10**]. A previous survey on total domination in graphs can also be found in [**9**].

In this paper we continue the study of Roman dominating functions in graphs. For a subset $S \subseteq V(G)$ of vertices of a graph G and a function $f\colon V(G) \longrightarrow \mathbb{R}$, we define $f(S) = \sum_{x \in S} f(x)$. A *Roman dominating function* on a graph G, abbreviated RD-*function*, is a function $f\colon V(G) \to \{0,1,2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The *weight*, $\omega(f)$, of f is defined as $f(V(G))$. The *Roman domination number*, denoted $\gamma_R(G)$, is the minimum weight of an RD-function in G; that is, $\gamma_R(G) = \min\{\omega(f) \mid f \text{ is an RD-function in } G\}$. An RD-function with minimum weight $\gamma_R(G)$ in G is called a $\gamma_R(G)$-*function*. For an RD-function f, let $V_i^f = \{v \in V(G)\colon f(v) = i\}$ for $i = 0,1,2$. Since these three sets determine f, we can equivalently write $f = (V_0^f, V_1^f, V_2^f)$. We observe that $\omega(f) = |V_1^f| + 2|V_2^f|$. The concept of a Roman dominating function was first defined by Cockayne, Dreyer, Hedetniemi, and Hedetniemi [**3**] and was motivated by an article in *Scientific American* by Ian Stewart entitled "Defend the Roman Empire" [**16**]. Roman domination in graphs is now very well studied, see, for example, [**1, 2, 4, 11, 12, 14, 15, 17**] and elsewhere.

Recently, Liu and Chang [**13**] introduced the concept of total Roman domination in graphs albeit in a more general setting. A *total Roman dominating function* of a graph G with no isolated vertex, abbreviated TRD-*function*, is a Roman dominating function $f = (V_0^f, V_1^f, V_2^f)$ on G with the additional property that the subgraph of G induced by the set of all vertices $V_1^f \cup V_2^f$ of positive weight under f has no isolated vertex. The *total Roman domination number* $\gamma_{tR}(G)$ is the minimum weight of an TRD-function on G. A TRD-function with minimum weight $\gamma_{tR}(G)$ in G is called a $\gamma_{tR}(G)$-*function*.

Applications of Roman domination were shown in Chambers et al. [**2**]. The concept of total Roman domination in graphs requires that every vertex in our graph (which represents a location in the Roman Empire) be secure in the sense that it is required to have a legion stationed in at least one of its neighboring locations. Thus, any location if attacked can be secured by sending a legion to it from an adjacent location. In the original paper by Liu and Chang [**13**] where the concept was introduced, algorithmic aspects of total Roman domination in graphs are studied.

Since every TRD-function in a graph G is a RD-function on G, we have the following straightforward observation.

Observation 1. *For every graph G with no isolated vertex, $\gamma_R(G) \le \gamma_{tR}(G)$.*

Moreover, we shall need the following properties of TRD-functions in a graph.

Observation 2. *Let G be a connected graph of order at least 3 and let $f = (V_0^f, V_1^f, V_2^f)$ be a $\gamma_{tR}(G)$-function. Then the following holds.*

(a) $|V_2^f| \le |V_0^f|$.

(b) *If x is a leaf and y a support vertex in G, then $x \notin V_2^f$ and $y \notin V_0^f$.*

(c) *If z has at least three leaf-neighbors, then $f(z) = 2$ and at most one leaf-neighbor of z belongs to V_1^f.*

In this paper, we relate the total Roman domination to domination parameters, including the domination number, the total domination number and Roman domination number. Further, we characterize the graphs with largest possible total Roman domination number, namely the order of the graph.

1.1. Notation

For basic notation and graph theory terminology not explicitly defined here, we in general follow Haynes, Hedetniemi and Slater [**6**]. Specifically, let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The integers $n(G) = |V(G)|$ and $m(G) = |E(G)|$ are the *order* and the *size* of G, respectively. The *open neighborhood* of a vertex v in G is the set $N_G(v) = \{u \in V(G) \,|\, uv \in E(G)\}$ and the *closed neighborhood of* v is $N_G[v] = \{v\} \cup N_G(v)$. For a set $S \subseteq V(G)$, its *open neighborhood* is the set $N_G(S) = \bigcup_{v \in S} N_G(v)$, and its *closed neighborhood* is the set $N_G[S] = N_G(S) \cup S$. The *boundary* of a set $S \subseteq V(G)$ is the set $\partial(S) = N_G(S) \setminus S$. The *$S$-external private neighborhood* of a vertex $v \in S$ is defined by $\mathrm{epn}(v, S) = \{w \in V(G) \setminus S \mid N_G[w] \cap S = \{v\}\}$. We define an *$S$-external private neighbor* of v to be a vertex in $\mathrm{epn}(v, S)$. If the graph G is clear from the context, we omit it in the above expressions. For example, we write $N[v]$ and $N(v)$ rather than $N_G[v]$ and $N_G(v)$, respectively.

If X and Y are sets of vertices in G, then X *dominates* Y if $Y \subseteq N[X]$ and X *totally dominates* Y if $Y \subseteq N(X)$. In particular, if X dominates $V(G)$, then X is a dominating set in G, while if X totally dominates $V(G)$, then X is a TD-set in G.

A *packing* in G is a set of vertices that are pairwise at distance at least 3 apart; that is, if u and v are distinct vertices that belong to a packing S, then $d_G(u, v) \ge 3$. Equivalently, a set S of vertices of G is a packing in G if the closed neighborhoods of vertices in S are pairwise disjoint.

The *degree* of a vertex v in G is $d_G(v) = |N_G(v)|$. The minimum and maximum degree of a graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For a set $S \subseteq V(G)$, the subgraph induced by S is denoted by $G[S]$. A *leaf* of G is a vertex of degree 1, while a *support vertex* of G is a vertex adjacent to a leaf. We denote by $L(G)$ and $S(G)$ the set of leaves and support vertices in G, respectively. An edge incident with a leaf is a *pendant edge*. A *strong support vertex* is a support vertex with at least two leaf-neighbors. If A and B are vertex disjoint sets in G, we denote by $[A, B]$ the set of edges of G with one end in A and the other end in B.

A tree obtained from a star on at least three vertices by subdividing every edge exactly once is called a *subdivided star*. A tree containing exactly two vertices that are not leaves (which are necessarily adjacent) is called a *double star*. The

corona of a graph H, denoted $\mathrm{cor}(H)$ or $H \circ K_1$ in the literature, is the graph obtained from H by adding a pendant edge to each vertex of H. We remark that the Roman domination number (among other domination parameters) of the generalized version of corona graphs has been studied in [**5**]. We use the standard notation $[k] = \{1, 2, \ldots, k\}$.

2. TOTAL ROMAN DOMINATION VERSUS (TOTAL, ROMAN) DOMINATION

In this section, we relate the total Roman domination with some standard domination parameters, including the ordinary domination number, the Roman domination number, and the total domination number.

2.1. Total Roman domination versus domination

We begin with the following useful property of total Roman dominating functions.

Lemma 3. *If G is a graph with no isolated vertex, then there exists a $\gamma_{tR}(G)$-function $f = (V_0^f, V_1^f, V_2^f)$ such that either V_2^f is a dominating set in G, or the set S of vertices not dominated by V_2^f satisfies $G[S] = kK_2$ for some $k \geq 1$, where $S \subseteq V_1^f$ and $\partial(S) \subseteq V_0^f$.*

Proof. Suppose that there is no $\gamma_{tR}(G)$-function $f' = (V_0^{f'}, V_1^{f'}, V_2^{f'})$ such that $V_2^{f'}$ is a dominating set in G. Among all $\gamma_{tR}(G)$-functions, let $f = (V_0^f, V_1^f, V_2^f)$ be chosen so that $|V_1^f|$ achieves a minimum value. Since f is a TRD-function, the set V_2^f dominates V_0^f. Further, the graph $G[V_1^f \cup V_2^f]$ contains no isolated vertex. Let V_{12}^f be the set of vertices in V_1^f that have a neighbor in V_2^f, and let $V_{11}^f = V_1^f \setminus V_{12}^f$. By supposition, V_2^f is not a dominating set of G, implying that the set of vertices not dominated by V_2^f, namely V_{11}^f, is non-empty.

Among all vertices in V_{11}^f, let v be one of minimum degree in $G[V_{11}^f]$. We show firstly that v has degree 1 in $G[V_{11}^f]$. If v is isolated in $G[V_{11}^f]$, let w be an arbitrary neighbor of v in $V_1^f \cup V_2^f$ and note that $w \in V_{12}^f$. If v has degree at least 2 in $G[V_{11}^f]$, let w be an arbitrary neighbor of v in V_{11}^f. In both cases, let $f' \colon V(G) \to \{0, 1, 2\}$ be defined as follows: $f'(v) = 0$, $f'(w) = 2$ and $f'(u) = f(u)$ for every vertex $u \notin \{v, w\}$. By our choice of the vertices v and w, the function f' is a TRD-function of G. Moreover, since $f(v) + f(w) = f'(v) + f'(w)$, we note that $f'(V(G)) = f(V(G))$, implying that $f' = (V_0^f \cup \{v\}, V_1^f \setminus \{v, w\}, V_2^f \cup \{w\})$ is a $\gamma_{tR}(G)$-function. However, the number of vertices having the value 1 under f' is less than $|V_1^f|$, contradicting our choice of f. Hence, the vertex v has degree 1 in $G[V_{11}^f]$.

Let w be the neighbor of v in V_{11}^f. If w has degree at least 2 in $G[V_1^f \cup V_2^f]$, then the function f' defined as before is once again a minimum TRD-function of G, contradicting our choice of f. Hence, w has degree 1 in $G[V_1^f \cup V_2^f]$. If v has

degree at least 2 in $G[V_1^f \cup V_2^f]$ (implying that v has a neighbor in V_{12}^f), then the function $g\colon V(G) \to \{0,1,2\}$ defined by $g(v) = 2$, $g(w) = 0$ and $g(u) = f(u)$ for every vertex $u \notin \{v,w\}$ is a minimum TRD-function of G that assigns the value 1 to fewer than $|V_1^f|$ vertices, contradicting our choice of f. Hence, v has degree 1 in $G[V_1^f \cup V_2^f]$. Therefore, v and w induce a K_2-component in $G[V_1^f \cup V_2^f]$; that is, $G[\{v,w\}] = K_2$. In particular, $N(v) \setminus \{w\} \subseteq V_0^f$ and $N(w) \setminus \{v\} \subseteq V_0^f$. Thus, $\partial(\{v,w\}) = N(\{v,w\}) \setminus \{v,w\} \subseteq V_0^f$. Since v is an arbitrary vertex of degree 1 in $G[V_{11}^f]$, we deduce that the vertices of degree 1 in $G[V_{11}^f]$ induce a subgraph in $G[V_1^f \cup V_2^f]$ that consists of the vertex disjoint union of copies of K_2.

We show next that every component of $G[V_{11}^f]$ is a K_2-component. Suppose, to the contrary, that $G[V_{11}^f]$ contains a component H that is not a K_2-component. As observed earlier, no vertex is isolated in $G[V_{11}^f]$ and every vertex of degree 1 in $G[V_{11}^f]$ belongs to a K_2-component of $G[V_1^f \cup V_2^f]$. Thus, every vertex of H has degree at least 2. We now let v' be a vertex of minimum degree in H. We note that $d_H(v') \geq 2$. Let w' be an arbitrary neighbor of v' in H. Let $h\colon V(G) \to \{0,1,2\}$ be defined as follows: $h(v') = 0$, $h(w') = 2$ and $h(u) = f(u)$ for every vertex $u \notin \{v',w'\}$. Then, $h = (V_0^f \cup \{v'\}, V_1^f \setminus \{v',w'\}, V_2^f \cup \{w'\})$ is a $\gamma_{tR}(G)$-function that assigns the value 1 to fewer than $|V_1^f|$ vertices, contradicting our choice of f. Therefore, every component of $G[V_{11}^f]$ is a K_2-component which is in fact a K_2-component of $G[V_1^f \cup V_2^f]$. From our earlier observations, this implies that letting $S = V_{11}^f$ we have $G[S] = kK_2$ for some $k \geq 1$, $S \subseteq V_1^f$, and $\partial(S) \subseteq V_0^f$. This completes the proof of Lemma 3. □

We are now in a position to relate the total Roman domination and the ordinary domination number.

Theorem 4. *If G is a graph with no isolated vertex, then $2\gamma(G) \leq \gamma_{tR}(G)$. Further, if $2\gamma(G) = \gamma_{tR}(G)$, then $\gamma(G) = \gamma_t(G)$ or there exists a set S of vertices of G such that the following holds.*

(a) *$G[S] = kK_2$ for some $k \geq 1$.*
(b) *$\gamma(G - S) = \gamma_t(G - S)$.*
(c) *No neighbor of a vertex of S in G belongs to a $\gamma_t(G - S)$-set.*

Proof. By Lemma 3, there exists a $\gamma_{tR}(G)$-function $f = (V_0^f, V_1^f, V_2^f)$ such that either V_2^f is a dominating set of G or the set S of vertices not dominated by V_2^f satisfies $G[S] = kK_2$ for some $k \geq 1$, $S \subseteq V_1^f$, and $\partial(S) \subseteq V_0^f$. If V_2^f is a dominating set of G, then $\gamma(G) \leq |V_2^f| \leq \frac{1}{2}f(V(G)) = \frac{1}{2}\gamma_{tR}(G)$. If V_2^f is not a dominating set of G, then the set V_2^f is a dominating set of $G - S$ and can be extended to a dominating set of G by adding to it one vertex from each K_2-component of $G[S]$. In this case, $\gamma_{tR}(G) = f(V(G)) = 2|V_2^f| + |V_1^f| \geq 2|V_2^f| + |S|$, implying that $\gamma(G) \leq \gamma(G - S) + \frac{1}{2}|S| \leq |V_2^f| + \frac{1}{2}|S| \leq \frac{1}{2}\gamma_{tR}(G)$. This establishes the desired upper bound on $2\gamma(G)$.

Suppose that $\gamma_{tR}(G) = 2\gamma(G)$. Let $f = (V_0^f, V_1^f, V_2^f)$ be a $\gamma_{tR}(G)$-function chosen so that $|V_1^f|$ is a minimum. As shown in the proof of Lemma 3, the set V_2^f is a dominating set of G or the set S of vertices not dominated by V_2^f satisfies $G[S] = kK_2$ for some $k \geq 1$, where $S \subseteq V_1^f$ and $\partial(S) \subseteq V_0^f$. Suppose that V_2^f is a dominating set of G. Then, as observed earlier, $\frac{1}{2}\gamma_{tR}(G) = \gamma(G) \leq |V_2^f| \leq \frac{1}{2}f(V(G)) = \frac{1}{2}\gamma_{tR}(G)$. Hence we must have equality throughout this inequality chain. In particular, $\gamma(G) = |V_2^f|$ and $f(V(G)) = 2|V_2^f|$, implying that $V_1^f = \emptyset$. This in turn implies that the set V_2^f is a TD-set in G, and so $|V_2^f| = \gamma(G) \leq \gamma_t(G) \leq |V_2^f|$. Consequently, we must have equality throughout this inequality chain, implying that $\gamma(G) = \gamma_t(G)$. Hence we may assume that V_2^f is not a dominating set of G, for otherwise the desired result follows. In this case, the set S of vertices not dominated by V_2^f satisfies $G[S] = kK_2$ for some $k \geq 1$. Further, $S \subseteq V_1^f$ and $\partial(S) \subseteq V_0^f$. Thus, Part (a) follows readily.

As observed earlier, $\frac{1}{2}\gamma_{tR}(G) = \gamma(G) \leq \gamma(G-S) + \frac{1}{2}|S| \leq |V_2^f| + \frac{1}{2}|S| \leq \frac{1}{2}\gamma_{tR}(G)$. Hence we must have equality throughout this inequality chain. In particular, $\gamma(G-S) = |V_2^f|$ and $\gamma_{tR}(G) = 2|V_2^f|+|S|$. We show next that $|V_2^f| = \gamma_t(G-S)$. The set V_2^f is a TD-set of $G-S$, and so $\gamma_t(G-S) \leq |V_2^f|$. Conversely, if D is a $\gamma_t(G-S)$-set, then the function $g\colon V(G) \to \{0,1,2\}$ defined as follows: $g(v) = 2$ if $v \in D$, $g(v) = 1$ if $v \in S$, and $g(v) = 0$ for every vertex $u \notin D \cup S$ is a TRD-function of G, implying that $2|V_2^f|+|S| = \gamma_{tR}(G) \leq g(V(G)) = 2|D|+|S| = 2\gamma_t(G-S)+|S|$, and so $|V_2^f| \leq \gamma_t(G-S)$. Consequently, $|V_2^f| = \gamma_t(G-S)$. As observed earlier, $|V_2^f| = \gamma(G-S)$. Consequently, $\gamma(G-S) = \gamma_t(G-S)$. This establishes Part (b).

It remains to show that no neighbor of a vertex of S in G belongs to a $\gamma_t(G-S)$-set. Suppose, to the contrary, that there is a $\gamma_t(G-S)$-set, D^*, that contains a vertex that is adjacent in G to a vertex of S, say v_1. Recall that $G[S] = kK_2$ for some $k \geq 1$. Let v_2 be the neighbor of v_1 that belongs to S. We now consider the function $f^*\colon V(G) \to \{0,1,2\}$ defined as follows: $f^*(v) = 2$ if $v \in D^* \cup \{v_1\}$, $f^*(v) = 1$ if $v \in S \setminus \{v_1, v_2\}$, and $f^*(u) = f(u)$ for every vertex $u \notin D^* \cup \{v_1, v_2\}$. The function f^* is a TRD-function of G satisfying $f^*(V(G)) = f(V(G))$, implying that f^* is a $\gamma_{tR}(G)$-function. However, the vertices assigned the value 1 under f^* is equal to $|S| - 2 = |V_1^f| - 2$, contradicting our choice of f. Hence, there is no $\gamma_t(G-S)$-set that contains a vertex adjacent in G to a vertex of S. This establishes Part (c) and completes the proof of Theorem 4.

Theorem 5. *If G is a graph with no isolated vertex, then $\gamma_{tR}(G) \leq 3\gamma(G)$. Further, if $\gamma_{tR}(G) = 3\gamma(G)$, then every $\gamma(G)$-set is a packing in G.*

Proof. Let S be a minimum dominating set in G. Let S' denote the set of vertices in S that are isolated in $G[S]$ (possibly, $S' = \emptyset$). For each vertex $v \in S'$, we select one neighbor of v and denote it by v'. Let $S'' = \cup_{v \in S'}\{v'\}$. Let $f\colon V(G) \to \{0,1,2\}$ be defined as follows. For each vertex $v \in S$, let $f(v) = 2$. For each vertex $v \in S''$,

let $f(v) = 1$. For each vertex $v \in V(G) \setminus (S \cup S'')$, let $f(v) = 0$. Then, f is a TRD-function on G, implying that $\gamma_{tR}(G) \le f(V(G)) \le 2|S| + |S''| \le 2|S| + |S'| \le 3|S| = 3\gamma(G)$. This establishes the desired upper bound on $\gamma_{tR}(G)$.

Suppose next that $\gamma_{tR}(G) = 3\gamma(G)$. Let S be an arbitrary $\gamma(G)$-set and let S' and S'' be as defined earlier in the proof. As observed earlier, $3\gamma(G) = \gamma_{tR}(G) \le 2|S| + |S''| \le 2|S| + |S'| \le 3|S| = 3\gamma(G)$. Consequently, we must have equality throughout this inequality chain. In particular, $|S'| = |S|$, implying that $S' = S$ and therefore that S is an independent set. Further, $|S''| = |S'|$. Let u and v be distinct vertices in S. Since S is an independent set, $d_G(u, v) \ge 2$. We show that $d_G(u, v) \ge 3$. Suppose, to the contrary, that $d_G(u, v) = 2$. Let w be a common neighbor of u and v, and choose $u' = v' = w$ where, as before, u' and v' are the vertices chosen to be adjacent to u and v, respectively. With this choice of u' and v', we note that $|S''| < |S'|$, a contradiction. Therefore, $d_G(u, v) \ge 3$. This is true for every pair of distinct vertices in S, implying that S is a packing in G. Thus, every $\gamma(G)$-set is a packing in G. □

We remark that the upper bound of Theorem 5 is sharp. For example, for $k \ge 2$, let G be obtained from the disjoint union of k stars $K_{1,t}$, where $t \ge 3$, by selecting one leaf from each star and adding any number of edges joining these t selected leaves so that the resulting graph is connected. Then, $\gamma(G) = k$ and the set of k support vertices of G form the unique $\gamma(G)$-set (which we observe is a packing in G). Let $f\colon V(G) \to \{0, 1, 2\}$ be a TRD-function of G and consider a central vertex v of one of the original stars used to construct G. If $f(v) \le 1$, then every leaf-neighbor of v in G has value at least 1 under f, implying that $f([N[v]) \ge t \ge 3$. If $f(v) = 2$, then at least one neighbor of v is assigned a positive value under f, once again implying that $f([N[v]) \ge 3$. This is true for each of the k central vertices of the original stars. Therefore, $f(V(G)) \ge 3k$. This is true for every TRD-function f of G, implying that $\gamma_{tR}(G) \ge 3k = 3\gamma(G)$. As shown earlier, $\gamma_{tR}(G) \le 3\gamma(G)$ holds for every graph. Consequently, $\gamma_{tR}(G) = 3\gamma(G)$.

It remains an open problem to find a necessary and sufficient condition for equality to hold in Theorem 5. We remark that if G is a graph with no isolated vertex such that every $\gamma(G)$-set is a packing in G, then it is not necessarily true that $\gamma_{tR}(G) = 3\gamma(G)$. For example, for $k \ge 3$, suppose that G_k is obtained from a star $K_{1,k}$ with central vertex v by subdividing $k - 1$ edges twice and subdividing the remaining edge exactly once. Then, $\gamma(G_k) = k$ and the set of k support vertices of G_k form the unique $\gamma(G_k)$-set which is a packing in G_k. However, the function $f\colon V(G_k) \to \{0, 1, 2\}$ that assigns the value 1 to every leaf and to every support vertex, the value 2 to v, and the value 0 to the remaining vertices of G_k is a TRD-function of G_k of weight $2k + 2$, and so $\gamma_{tR}(G_k) \le 2(k + 1) < 3k = 3\gamma(G_k)$. The graph G_4, for example is illustrated in Figure 1, where the darkened vertices form a $\gamma(G_4)$-set and the given function f is a TRD-function of G_4 of weight $2(k + 1) = 10$.

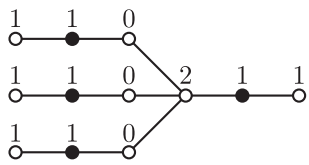

Figure 1. The graph G_4.

2.2. Total Roman domination versus total domination

In this section, we relate the total Roman domination and the total domination number.

Theorem 6. *If G is a graph with no isolated vertex, then*

$$\gamma_t(G) \le \gamma_{tR}(G) \le 2\gamma_t(G). \tag{1}$$

Further, the following holds.

(a) $\gamma_t(G) = \gamma_{tR}(G)$ *if and only if G is the disjoint union of copies of K_2.*

(b) *If $\gamma_{tR}(G) = 2\gamma_t(G)$ and S is an arbitrary $\gamma_t(G)$-set, then* $\mathrm{epn}(v, S) \neq \emptyset$ *for all $v \in S$.*

Proof. Let $f = (V_0^f, V_1^f, V_2^f)$ be an arbitrary $\gamma_{tR}(G)$-function. Then, $V_1^f \cup V_2^f$ is a TD-set of G, implying that $\gamma_t(G) \le |V_1^f| + |V_2^f| \le |V_1^f| + 2|V_2^f| = \gamma_{tR}(G)$. This establishes the lower bound in the statement of the theorem. Suppose that $\gamma_t(G) = \gamma_{tR}(G)$. Then we must have equality throughout this inequality chain. In particular, $V_2^f = \emptyset$, implying that $V(G) = V_1^f$. Since f is an arbitrary $\gamma_{tR}(G)$-function, this implies that $(\emptyset; V(G); \emptyset)$ is the only $\gamma_{tR}(G)$-function. Let $P\colon v_1v_2\ldots v_k$ be a longest path in G. We note that all neighbors of v_1 belong to P. If $k \ge 3$, then the function $f' = (\{v_1\}, V(G) \setminus \{v_1, v_2\}, \{v_2\})$ is a $\gamma_{tR}(G)$-function, contradicting our earlier observation that $(\emptyset; V(G); \emptyset)$ is the only $\gamma_{tR}(G)$-function. Therefore, $k = 2$ and G is the disjoint union of copies of K_2. Conversely, if G is the disjoint union of copies of K_2, then it is immediate that $\gamma_t(G) = \gamma_{tR}(G) = |V(G)|$. This completes the proof of Part (a).

The upper bound follows immediately from the observation that assigning a weight of 2 to each vertex of a given $\gamma_t(G)$-set and assigning a weight of 0 to all remaining vertices of G yields a TRD-function on G, implying that $\gamma_{tR}(G) \le 2\gamma_t(G)$. Suppose that $\gamma_{tR}(G) = 2\gamma_t(G)$ and let S be an arbitrary $\gamma_t(G)$-set. Suppose, to the contrary, that $\mathrm{epn}(v, S) = \emptyset$ for some vertex $v \in S$. The function $f = (V(G) \setminus S, \{v\}, S \setminus \{v\})$ is a TRD-function of G, implying that $\gamma_{tR}(G) \le f(V(G)) = 1 + 2(|S| - 1) < 2|S| = 2\gamma_t(G)$, a contradiction. Therefore, $\mathrm{epn}(v, S) \neq \emptyset$ for all $v \in S$. This proves Part (b). □

By Theorem 6, if G is a graph with no isolated vertex, then $\gamma_{tR}(G) \ge \gamma_t(G)$. Further, $\gamma_{tR}(G) = \gamma_t(G)$ if and only if G is the disjoint union of copies of K_2. We next characterize the connected graphs G satisfying $\gamma_{tR}(G) = \gamma_t(G) + 1$.

Proposition 7. *Let G be a connected graph of order $n \ge 3$. Then, $\gamma_{tR}(G) = \gamma_t(G) + 1$ if and only if $\Delta(G) = n - 1$.*

Proof. Suppose that $\gamma_{tR}(G) = \gamma_t(G) + 1$. By Lemma 3, there exists a $\gamma_{tR}(G)$-function $f = (V_0^f, V_1^f, V_2^f)$ such that either V_2^f is a dominating set of G or the set S of vertices not dominated by V_2^f satisfies $G[S] = kK_2$ for some $k \ge 1$, $S \subseteq V_1^f$, and $\partial(S) \subseteq V_0^f$. If $V_2^f = \emptyset$, then $V(G) = S$ and $G = G[S]$ is a disjoint union of copies of K_2. But then $\gamma_{tR}(G) = n = \gamma_t(G)$, a contradiction. Hence, $V_2^f \neq \emptyset$.

Since $V_1^f \cup V_2^f$ is a TD-set of G, we note that $\gamma_{tR}(G) - 1 = \gamma_t(G) \le |V_1^f| + |V_2^f| \le |V_1^f| + 2|V_2^f| - 1 = \gamma_{tR}(G) - 1$. Hence, we must have equality throughout this inequality chain. In particular, $|V_2^f| = 2|V_2^f| - 1$, and so $|V_2^f| = 1$. Let $V_2^f = \{v\}$.

We show that V_2^f is a dominating set of G. Suppose, to the contrary, that V_2^f is not a dominating set of G and consider the set S of vertices not dominated by V_2^f. Since $\partial(S) \subseteq V_0^f$, there must be a vertex in $V_1^f \setminus S$ that is adjacent to v. Using the notation employed in the proof of Lemma 3, let V_{12}^f be the set of vertices in V_1^f that have a neighbor in V_2^f. In our case, every vertex in V_{12}^f is a neighbor of v. If $|V_{12}^f| \ge 2$, then removing all but one vertex from V_{12}^f and adding it to the set V_0^f produces a new TRD-function of G of weight less that $f(V(G))$, a contradiction. Hence, $|V_{12}^f| = 1$. Thus, $\gamma_{tR}(G) = |S| + |V_{12}^f| + 2|V_2^f| = |S| + 3$. Recall that $G[S] = kK_2$ for some $k \ge 1$. Let $\mathcal{C}$ denote the set of k components of $G[S]$ and let $F \in \mathcal{C}$ be an arbitrary K_2-component in $G[S]$. Since G is connected, there is a vertex v_F in $V(G) \setminus V(F)$ that is adjacent to a vertex of $V(F)$ in G. Since $\partial(S) \subseteq V_0^f$, we note that $v_F \in V_0^f$ and therefore v_F is adjacent to v in G. Let u_F be a vertex in $V(F)$ that is adjacent to v_F in G and let

$$S' = \bigcup_{F \in \mathcal{C}} \{u_F, v_F\}.$$

The set $S' \cup \{v\}$ is a TD-set of G, implying that $\gamma_t(G) \le |S'| + 1 \le |S| + 1 = \gamma_{tR}(G) - 2$, a contradiction. Therefore, V_2^f is a dominating set of G. As observed earlier, $V_2^f = \{v\}$. The vertex v is therefore a dominating vertex of G (in the sense that it dominates $V(G)$), implying that $\Delta(G) = n-1$. Conversely, if $\Delta(G) = n-1$, then $\gamma_t(G) = 2$ and $\gamma_{tR}(G) = 3$. Thus, $\gamma_{tR}(G) = \gamma_t(G) + 1$. □

Cockayne et al. [**3**] established the following relationship between the domination number and the Roman domination number of a graph: For every graph G,

$$\gamma(G) \le \gamma_R(G) \le 2\gamma(G). \tag{2}$$

The graphs G satisfying $\gamma(G) = \gamma_R(G)$ were characterized in [**3**]. A graph G for which $\gamma_R(G) = 2\gamma(G)$ is defined in [**3**] to be a *Roman graph*. While the class of Roman trees has been characterized in [**8**], it remains an open problem to characterize Roman graphs in general. We now define an analogous concept for the total Roman domination number and define a graph G to be a *total Roman graph* if $\gamma_{tR}(G) = 2\gamma_t(G)$. Examples of total Roman graphs include the corona, $\mathrm{cor}(H)$, of a graph H with no isolated vertex and cycles or paths on n vertices where $n \equiv 0 \pmod 4$.

We present next a trivial necessary and sufficient condition for a graph to be a total Roman graph.

Proposition 8. *Let G be a graph without isolated vertices. Then, G is a total Roman graph if and only if there exists a $\gamma_{tR}(G)$-function $f = (V_0^f, V_1^f, V_2^f)$ such that $V_1^f = \emptyset$.*

Proof. Suppose that G is a total Roman graph. Let S be an arbitrary $\gamma_t(G)$-set. As observed in the proof of Theorem 6(b), the function $f = (V_0^f, V_1^f, V_2^f)$ that assigns a weight of 2 to each vertex of S and a weight of 0 to all remaining vertices of G is a TRD-function on G, and so $2\gamma_t(G) = \gamma_{tR}(G) \le f(V(G)) = 2|V_2^f| = 2|S| = 2\gamma_t(G)$. Hence, we must have equality throughout this inequality chain. In particular, $\gamma_{tR}(G) = f(V(G))$, implying that f is a $\gamma_{tR}(G)$-function satisfying $V_1^f = \emptyset$. Conversely, suppose there exists a $\gamma_{tR}(G)$-function $g = (V_0^g, V_1^g, V_2^g)$ satisfying $V_1^g = \emptyset$. Since the set $V_1^g \cup V_2^g = V_2^g$ is a TD-set of G, we note that $\gamma_t(G) \le |V_2^g| = \frac{1}{2}\gamma_{tR}(G)$, or, equivalently, $2\gamma_t(G) \le \gamma_{tR}(G)$. By Theorem 6, $\gamma_{tR}(G) \le 2\gamma_t(G)$ for all graphs G. Consequently, $\gamma_{tR}(G) = 2\gamma_t(G)$, and so G is a total Roman graph. $\square$

It remains an open problem to find a nontrivial necessary and sufficient condition for a graph to be a total Roman graph, or to characterize the total Roman graphs. Recall that by Theorem 6(b), if G is a total Roman graph and S is an arbitrary $\gamma_t(G)$-set, then $\mathrm{epn}(v, S) \ne \emptyset$ for all $v \in S$. However this condition is not sufficient for a graph to be a total Roman graph. For example, for $k \ge 3$, suppose that H_k is obtained from a star $K_{1,k}$ with central vertex v by subdividing $k-1$ edges three times and subdividing the remaining edge exactly twice. Let A be the set k support vertices in H_k and let B be the set of vertices of degree 2 in H_k that are adjacent to a support vertex. Then, $\gamma_t(H_k) = 2k$ and the set $S = A \cup B$ is the unique $\gamma_t(H_k)$-set. We note that $\mathrm{epn}(v, S) \ne \emptyset$ for all $v \in S$. However, the function $f\colon V(H_k) \to \{0, 1, 2\}$ that assigns the value 0 to every leaf, the value 2 to every vertex in A, the value 1 to every vertex in B, the value 2 to v, and the value 0 to the remaining vertices of H_k is a TRD-function of H_k of weight $3k+2$, and so $\gamma_{tR}(H_k) \le 3k+2 < 4k = 2\gamma_t(H_k)$. The graph H_4, for example is illustrated in Figure 2, where the darkened vertices form a $\gamma_t(H_4)$-set and the given function f is a TRD-function of H_4 of weight $3k + 2 = 14$.

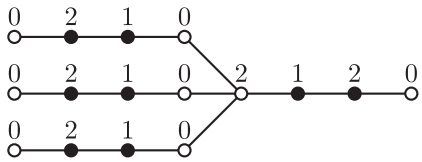

Figure 2. The graph H_4.

Recall that for every graph G with no isolated vertex, $\gamma(G) \le \gamma_t(G) \le 2\gamma(G)$. Let G be a total Roman graph. If $\gamma_t(G) > \frac{3}{2}\gamma(G)$, then by Theorem 5, we note that $\frac{3}{2}\gamma(G) < \gamma_t(G) = \frac{1}{2}\gamma_{tR}(G) \le \frac{3}{2}\gamma(G)$, a contradiction. Hence, $\gamma_t(G) \le \frac{3}{2}\gamma(G)$. Further, suppose that $\gamma_{tR}(G) = \gamma_R(G)$. By Inequality (2), we note then that $2\gamma(G) \ge \gamma_R(G) = \gamma_{tR}(G) = 2\gamma_t(G) \ge 2\gamma(G)$. Hence, we must have equality throughout this inequality chain. In particular, $2\gamma(G) = \gamma_R(G)$, implying that G is a Roman graph. We state these observations formally as follows.

Observation 9. *If G is a total Roman graph, then $\gamma_t(G) \le \frac{3}{2}\gamma(G)$. Further, if $\gamma_{tR}(G) = \gamma_R(G)$, then G is a Roman graph.*

We remark that the upper bound in Observation 9 is tight, as may be seen

by taking G to be a path P_{12k} where $k \geq 1$. In this case, G is a total Roman graph, $\gamma_t(G) = 6k$, and $\gamma(G) = 4k$, implying that $\gamma_t(G) = \frac{3}{2}\gamma(G)$.

Suppose that G is a Roman graph and $\gamma(G) = \gamma_t(G)$. Then, by Inequality (1), we note that $2\gamma_t(G) = 2\gamma(G) = \gamma_R(G) \leq \gamma_{tR}(G) \leq 2\gamma_t(G)$. Hence, we must have equality throughout this inequality chain. In particular, $\gamma_R(G) = 2\gamma_t(G)$, implying that G is a total Roman graph. We state this observation formally as follows.

Observation 10. *If G is a Roman graph and $\gamma(G) = \gamma_t(G)$, then G is a total Roman graph.*

2.3. Total Roman domination versus Roman domination

In this section we relate the total Roman domination number and the Roman domination number. By Observation 1, for every graph G with no isolated vertex, $\gamma_R(G) \leq \gamma_{tR}(G)$. We establish next an upper bound on the total Roman domination number in terms of the Roman domination number.

Theorem 11. *If G is a graph of order n with no isolated vertex, then $\gamma_{tR}(G) \leq 2\gamma_R(G) - 1$. Further, $\gamma_{tR}(G) = 2\gamma_R(G) - 1$ if and only if $\Delta(G) = n - 1$.*

Proof. Among all $\gamma_R(G)$-functions, let $f = (V_0^f, V_1^f, V_2^f)$ be chosen so that $|V_2^f|$ is a maximum. Suppose firstly that $V_1^f \neq \emptyset$. If some vertex in V_1^f is adjacent to a vertex in V_2^f, then we can simply re-assign it a weight of 0 to produce a new RD-function of weight less than $f(V(G))$, a contradiction. Hence, no vertex in V_1^f is dominated by V_2^f. We show that V_1^f is an independent set in G. Suppose, to the contrary, that u and v are adjacent vertices in V_1^f. Then, $f' = (V_0^f, V_1^f \setminus \{u, v\}, V_2^f \cup \{v\})$ is a RD-function of G. Moreover, since $f(u)+f(v) = f'(u)+f'(v)$, we note that $f'(V(G)) = f(V(G))$, implying that f' is a $\gamma_R(G)$-function. However, the vertices assigned the value 2 under f' exceeds $|V_2^f|$, contradicting our choice of f. Hence, V_1^f is an independent set in G. We show next that V_1^f is a packing in G. Suppose, to the contrary, that u and v are two distinct vertices in V_1^f at distance 2 apart. Let w be a common neighbor of u and v. Since V_1^f is an independent set and since neither u nor v are dominated by V_2^f, we note that $w \in V_0^f$. In this case, we let $f' = ((V_0^f \setminus \{w\}) \cup \{u, v\}, V_1^f \setminus \{u, v\}, V_2^f \cup \{w\})$ and note that f' is a RD-function of G. Moreover, since $f(u) + f(v) + f(w) = f'(u) + f'(v) + f'(w)$, we note that $f'(V(G)) = f(V(G))$, implying that f' is a $\gamma_R(G)$-function. However, the vertices assigned the value 2 under f' exceeds $|V_2^f|$, once again a contradiction. Therefore, V_1^f is a packing in G.

As observed earlier, no vertex in V_1^f is dominated by V_2^f. Further, V_1^f is a packing in G. Thus, all neighbors of a vertex in V_1^f belong to V_0^f. For each vertex $w \in V_1^f$, we select an arbitrary neighbor, w', of w and we let $W = \cup_{w \in V_1^f}\{w'\}$. We note that $|W| = |V_1^f|$ and $W \subseteq V_0^f$. Further, each vertex in W is adjacent to at least one vertex in V_2^f, implying that $|V_2^f| \geq 1$. We also observe that $\gamma_R(G) = |V_1^f|+2|V_2^f|$. We now consider the function $g = (V_0^g, V_1^g, V_2^g) = (V_0^f \setminus W, V_1^f \cup W, V_2^f)$.

Suppose that there are no isolated vertices in $G[V_1^g \cup V_2^g]$. In this case, g is a TRD-function of G, implying that

$$\gamma_{tR}(G) \leq g(V(G)) = 2|V_1^f| + 2|V_2^f| = 2\gamma_R(G) - 2|V_2^f| \leq 2(\gamma_R(G) - 1).$$

Suppose that there are isolated vertices in $G[V_1^g \cup V_2^g]$. Let U be the set of isolated vertices in $G[V_1^g \cup V_2^g]$. Since $G[V_1^g]$ contains no isolated vertex, we note that $U \subseteq V_2^f$. As observed earlier, each vertex in W is adjacent to at least one vertex in V_2^f, implying that $U \subset V_2^f$. For each vertex $u \in U$, we select an arbitrary neighbor, u', of u and we let $Z = \cup_{u \in U}\{u'\}$. We note that $|Z| \leq |U| < |V_2^f|$ and $Z \subseteq V_0^g$. We now consider the function $h = (V_0^g \setminus Z, V_1^g \cup Z, V_2^g)$. In this case, h is a TRD-function of G, implying that $\gamma_{tR}(G) \leq h(V(G)) = g(V(G)) + |Z| \leq g(V(G)) + |V_2^f| - 1$. Thus, since $g(V(G)) = 2|V_1^f| + 2|V_2^f|$, we note that

$$\gamma_{tR}(G) \leq 2|V_1^f| + 3|V_2^f| - 1 < 2|V_1^f| + 4|V_2^f| - 1 = 2\gamma_R(G) - 1.$$

Hence we may assume that $V_1^f = \emptyset$, for otherwise $\gamma_{tR}(G) < 2\gamma_R(G) - 1$ and the desired bound follows. Let U be the set of isolated vertices in $G[V_2^f]$. If $U = \emptyset$, then f is a TRD-function of G, implying that $\gamma_{tR}(G) \leq f(V(G)) = \gamma_R(G)$. Consequently, by Observation 1, $\gamma_{tR}(G) = \gamma_R(G)$. Hence we may assume that $U \neq \emptyset$. For each vertex $u \in U$, we select an arbitrary neighbor, u', of u and we let $U' = \cup_{u \in U}\{u'\}$. We note that $|U'| \leq |U| \leq |V_2^f|$. We now consider the function $f' = (V_0^f \setminus U', V_1^f \cup U', V_2^f)$. The function f' is a TRD-function of G, implying that $\gamma_{tR}(G) \leq f'(V(G)) = f(V(G)) + |U'| \leq f(V(G)) + |V_2^f|$. Thus, since in this case $f(V(G)) = 2|V_2^f|$, we note that

$$\gamma_{tR}(G) \leq 2|V_2^f| + |V_2^f| = \frac{3}{2}\gamma_R(G) \leq 2\gamma_R(G) - 1.$$

Further, if $\gamma_{tR}(G) = 2\gamma_R(G) - 1$, then $\gamma_R(G) = 2$, implying that $|V_2^f| = 1$ and $\Delta(G) = n - 1$. Conversely, if $\Delta(G) = n - 1$, then $\gamma_{tR}(G) = 3 = 2\gamma_R(G) - 1$, which completes the proof.

3. GRAPHS WITH LARGE TOTAL ROMAN DOMINATION NUMBER

In this section, we characterize the graphs with largest possible total Roman domination number, namely the order of the graph. Let $\mathcal{G}$ be the family of graphs that can be obtained from a 4-cycle $v_1v_2v_3v_4v_1$ by adding $k_1+k_2 \geq 1$ vertex-disjoint paths P_2 and joining v_1 to the end of k_1 such paths and joining v_2 to the end of k_2 such paths (possibly, $k_1 = 0$ or $k_2 = 0$). Let $\mathcal{H}$ be the family of graphs that can be obtained from a double star by subdividing each pendant edge once and subdividing the non-pendant edge $r \geq 0$ times. A graph G in the family $\mathcal{G}$ and a graph H in the family $\mathcal{H}$ are illustrated in Figure 3(a) and 3(b), respectively.

(a) $G \in \mathcal{G}$ (b) $H \in \mathcal{H}$

Figure 3. Graphs in the families $\mathcal{G}$ and $\mathcal{H}$.

We show firstly that the total Roman domination number of a nontrivial path or a cycle is the order of the path or cycle.

Proposition 12. *If G is a nontrivial path or a cycle on n vertices, then $\gamma_{tR}(G) = n$.*

Proof. If $G = P_2$, then the result is immediate. Hence we may assume that $G = P_n$ or $G = C_n$, where $n \geq 3$. Let $f = (V_0^f, V_1^f, V_2^f)$ be a $\gamma_{tR}(G)$-function. Suppose that there is a vertex v in V_0^f that is adjacent to two vertices u and w that both belong to V_2^f. Thus, uvw is a path in G where $f(v) = 0$ and $f(u) = f(w) = 2$. Since $G[V_1^f \cup V_2^f]$ contains no isolated vertices, $d_G(u) = d_G(w) = 2$ and the neighbor of u (respectively, w) different from v in G has value at least 1 under f. Let $f' \colon V(G) \to \{0, 1, 2\}$ be defined as follows: $f'(u) = f'(v) = f'(w) = 1$ and $f'(x) = f(x)$ for every vertex $x \notin \{u, v, w\}$. Since f is a TRD-function of G, so too is the function f'. Moreover, since $f'(u)+f'(v)+f'(w) < f(u)+f(v)+f(w)$, we note that $f'(V(G)) < f(V(G)) = \gamma_{tR}(G)$, a contradiction. Hence, every vertex in V_0^f is adjacent to exactly one vertex in V_2^f, and so $|V_0^f| \leq |V_2^f|$. By Observation 2(a), $|V_2^f| \leq |V_0^f|$. Consequently, $|V_0^f| = |V_2^f|$. Therefore, $\gamma_{tR}(G) = |V_1^f| + 2|V_2^f| = |V_0^f| + |V_1^f| + |V_2^f| = n$. □

We are now in a position to state the following result.

Theorem 13. *Let G be a connected graph of order $n \geq 2$. Then, $\gamma_{tR}(G) = n$ if and only if one of the following holds.*

(a) *G is a path or a cycle.*
(b) *G is a corona,* $\mathrm{cor}(F)$, *of some graph F.*
(c) *G is a subdivided star.*
(d) *$G \in \mathcal{G} \cup \mathcal{H}$.*

Proof. Suppose that $\gamma_{tR}(G) = n$. If $\Delta(G) \leq 2$, then G is a path or a cycle and the result follows from Proposition 12. Hence we may assume that $\Delta(G) \geq 3$. In particular, this implies that $n \geq 4$. Let $V(G) = V$. We proceed further with a series of claims that give us structural properties of the graph G.

Claim A. *Every support vertex is adjacent to exactly one leaf.*

Proof of Claim A. Suppose, to the contrary, that G contains a strong support vertex v. Let u and w be two leaf-neighbors of v. Let $f \colon V \to \{0, 1, 2\}$ be defined

as follows: $f(u) = f(w) = 0$, $f(v) = 2$, and $f(x) = 1$ for every vertex $x \notin \{u, v, w\}$. Then, f is a TRD-function of G, and so $\gamma_{tR}(G) \leq f(V) = n-1$, a contradiction. (□)

Claim B. *If $d_G(v) \geq 3$ for some vertex v of G, then at most two neighbors of v are not support vertices.*

Proof of Claim B. Let v be a vertex of degree at least 3 in G, and let $N_v = \{v_1, v_2, v_3\}$ be an arbitrary subset consisting of three neighbors of v. By Claim A, at most one vertex in N_v is a leaf. Suppose, to the contrary, that no vertex of N_v is a support vertex. Let $f\colon V \to \{0, 1, 2\}$ be defined as follows: $f(v_1) = f(v_2) = 0$, $f(v) = 2$, and $f(x) = 1$ for every vertex $x \notin \{v, v_1, v_2\}$. If f is a TRD-function of G, then $\gamma_{tR}(G) \leq f(V) = n - 1$, a contradiction. Hence, f is not a TRD-function of G, implying that there exists a vertex v_{12} such that $N_G(v_{12}) \subseteq \{v_1, v_2\}$. By supposition, the vertex v_{12} is not a leaf. Thus, $N_G(v_{12}) = \{v_1, v_2\}$. Analogously, there exists a vertex, v_{23} say, in G adjacent to both v_2 and v_3 but to no other vertex in G. Let $g\colon V \to \{0, 1, 2\}$ be defined as follows: $g(v_{12}) = g(v_{23}) = 0$, $f(v_2) = 2$, and $f(x) = 1$ for every vertex $x \notin \{v_2, v_{12}, v_{23}\}$. Then, g is a TRD-function of G, and so $\gamma_{tR}(G) \leq g(V) = n - 1$, a contradiction. (□)

As an immediate consequence of Claim B, $\delta(G) = 1$. If every vertex of G is a leaf or a support vertex, then G is a corona, $\mathrm{cor}(H)$, of some graph H. Hence we may assume that at least one vertex of G is neither a leaf nor a support vertex. Let $L = L(G)$ and $S = S(G)$ denote the set of leaves and support vertices, respectively, in G. Let $W = V \setminus (L \cup S)$. By assumption, $W \neq \emptyset$. Let $H = G[W]$. As an immediate consequence of Claim B, every vertex in W has at most two neighbors in W. We state this formally as follows.

Claim C. $\Delta(H) \leq 2$.

Claim D. *Every support vertex in G has degree* 2.

Proof of Claim D. By the connectivity of G, there is at least one edge in $[S, W]$ that joins a vertex of S and a vertex of W. Let $vw \in E(G)$, where $v \in S$ and $w \in W$. By Claim A, every support vertex is adjacent to exactly one leaf. Let v' be the leaf-neighbor of v. We show that the support vertex v has degree 2. Suppose, to the contrary, that v has degree at least 3 in G. Let u be a neighbor of v different from v' and w. By Claim B, the vertex u is a support vertex. Let u' be the leaf-neighbor of u. Let $f\colon V \to \{0, 1, 2\}$ be defined as follows: $f(u') = f(v') = f(w) = 0$, $f(v) = f(u) = 2$, and $f(x) = 1$ for every vertex $x \notin \{u, u', v, v', w\}$. Then, f is a TRD-function of G, and so $\gamma_{tR}(G) \leq f(V) = n - 1$, a contradiction. Therefore, $N_G(v) = \{v', w\}$ and $d_G(v) = 2$. Hence, every support vertex of G that is adjacent to a vertex of W has degree 2 in G. By the connectivity of G, this implies that every support vertex of G has degree 2. (□)

By Claim D, every support vertex in G has degree 2. Thus, by the connectivity of G, the graph H is necessarily connected. We state this formally as follows.

Claim E. *H is connected.*

Let W_S be the set of vertices in W that are adjacent to a (support) vertex in S. By Claim C, $\Delta(H) \le 2$.

Claim F. *If $w \in W_S$ has degree 2 in H, then the two neighbors of w in H are not adjacent and $G - N_H(w)$ has exactly one isolated vertex.*

Proof of Claim F. Let $w \in W_S$ and suppose that $d_H(w) = 2$. Let $N_H(w) = \{w_1, w_2\}$. Let $f\colon V \to \{0,1,2\}$ be defined as follows: $f(w_1) = f(w_2) = 0$, $f(w) = 2$, and $f(x) = 1$ for every vertex $x \notin \{w, w_1, w_2\}$. If f is a TRD-function of G, then $\gamma_{tR}(G) \le f(V) = n - 1$, a contradiction. Hence, f is not a TRD-function of G, implying that there exists a vertex x such that $N_G(x) = \{w_1, w_2\}$. Let X be the set of all such vertices x that are adjacent to both w_1 and w_2 but to no other vertex in G. Let $g\colon V \to \{0,1,2\}$ be defined as follows: $g(x) = 0$ if $x \in X$, $g(w_1) = 2$, and $g(x) = 1$ for every vertex $x \notin X \cup \{w_1\}$. Then, g is a TRD-function of G, and so $n = \gamma_{tR}(G) \le g(V) = n - |X| + 1$. Thus, $|X| \le 1$, implying that $|X| = 1$. Thus, there is exactly one common neighbor, say w_{12}, of w_1 and w_2 that is adjacent to no other vertex of G; that is, $N_G(w_{12}) = \{w_1, w_2\}$. Notice that w_{12} is neither a leaf nor a support vertex, and so w_{12} belongs to H. If w_1 and w_2 are adjacent, then they have degree 3 in H, which is a contradiction with Claim C. (□)

Claim G. *The graph H is either a path or a 4-cycle, C_4. Further, the following holds.*

(a) *If $H = C_4$, then at least two adjacent vertices of H have degree 2 in G.*
(b) *If $H = P_k$, then every internal vertex of H has degree 2 in G.*

Proof of Claim G. By Claim E, H is connected. By Claim C, H is either a path or a cycle. Suppose that H is a cycle C_k for some $k \ge 3$. Let H be given by $w_1 w_2 \ldots w_k w_1$. Renaming the vertices of H if necessary, we may assume that $w_2 \in W_S$. By Claim F, the vertices w_1 and w_3 are not adjacent and $G - \{w_1, w_3\}$ has exactly one isolated vertex. This is only possible if $k = 4$ (and the isolated vertex in $G - \{w_1, w_3\}$ is the vertex w_4). In this case, we note that $N(w_4) = \{w_1, w_3\}$, and so $d_G(w_4) = 2$. Suppose that both w_1 and w_3 have degree at least 3 in G. Thus, $W_S = \{w_1, w_2, w_3\}$. By considering the vertex w_1 instead of the vertex w_2, an analogous argument shows that $d_G(w_3) = 2$, a contradiction. Hence, at least one of w_1 and w_3 has degree 2 in G. This proves Part (a). Part (b) follows readily from Claim F. (□)

We now return to the proof of Theorem 13. By Claim G, the graph H is either a path or a 4-cycle, C_4. If $H = C_4$, then by Claim G we deduce that $G \in \mathcal{G}$. If $H = P_1$, then G is obtained from a star by subdividing each of its edge once; that is, H is a subdivided star. If $H = P_k$ where $k \ge 2$, then G is obtained from a double star by subdividing each pendant edge once and subdividing the non-pendant edge $k - 2$ times; that is, $G \in \mathcal{H}$. Therefore, if $\gamma_{tR}(G) = n$, then the four conditions in the statement of the theorem hold.

Conversely, suppose that the graph G satisfies one of the four conditions in the statement of the theorem. Suppose, to the contrary, that $\gamma_{tR}(G) < n$. By Proposition 12, G is neither a path nor a cycle. Hence, G satisfies condition (b), (c) or (d) in the statement of the theorem. Let $f = (V_0^f, V_1^f, V_2^f)$ be a $\gamma_{tR}(G)$-function. Then, $|V_2^f| < |V_0^f|$, which implies that there is a vertex $v \in V_2^f$ with at least two neighbors belonging to V_0^f. If v has degree precisely 2 in G, then v is isolated in the graph $G[V_1^f \cup V_2^f]$, a contradiction. Hence, v has degree at least 3 in G. If $G \notin \mathcal{G}$, then the structure of the graph G implies that every such vertex v of degree at least 3 has at most one neighbor that is not a support vertex. This in turn implies that at least one support vertex of v belongs to V_0^f, contradicting Observation 2(b). If $G \in \mathcal{G}$, then the vertex v of degree at least 3 belongs to the 4-cycle in G. Further, the two neighbors of v that belong to V_0^f belong to the 4-cycle of G. The remaining vertex of the 4-cycle is therefore isolated in the graph $G[V_1^f \cup V_2^f]$, a contradiction. Hence, $\gamma_{tR}(G) = n$. This completes the proof of Theorem 13.

4. CLOSING REMARKS

We close with the following three open problems that we have yet to settle.

Problem 1. Characterize the graphs G achieving the upper bound in Theorem 4; that is, characterize the graphs G satisfying $2\gamma(G) = \gamma_{tR}(G)$.

Problem 2. Characterize the graphs G achieving the upper bound in Theorem 5; that is, characterize the graphs G satisfying $\gamma_{tR}(G) = 3\gamma(G)$.

Problem 3. Characterize the graphs G achieving the upper bound in Theorem 6; that is, characterize the total Roman graphs G (satisfying $\gamma_{tR}(G) = 2\gamma_t(G)$).

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