DOMINATING SEQUENCES UNDER ATOMIC CHANGES WITH APPLICATIONS IN SIERPIŃSKI AND INTERVAL GRAPHS

Boštjan Brešar, Tanja Gologranc, Tim Kos

We continue the study of the Grundy domination number of a graph. A linear algorithm to determine the Grundy domination number of an interval graph is presented. The exact value of the Grundy domination number of an arbitrary Sierpiński graph is proven, and efficient algorithms to construct the corresponding sequences are presented. These results are obtained by using sharp bounds for the Grundy domination number of a vertex- and edge-removed graph, proven in this paper.

1. INTRODUCTION

Domination is one of the oldest and most studied topics in graph theory, and is known for many (practical) applications. Domination theory was comprehensively surveyed in two monographs from almost 20 years ago [**14, 13**], while a recent monograph [**15**] focuses only on the total domination, one of the most basic concepts in the theory. Many other variations of the classical domination number of a graph have been introduced in years. A very recent and natural one is the so-called Grundy domination number, which in some sense describes the worst scenario that can happen when a dominating set is built up in an online way. More precisely, the Grundy domination number of a graph is the maximum length of a sequence of its vertices, such that each time a vertex is added it dominates some vertex that was not dominated by previous vertices in the sequence [**5**]. (A total version of Grundy domination was introduced in [**6**].)

One of the focuses in this paper is on two rather famous classes of graphs, namely the interval graphs and the Sierpiński graphs. The interval graphs, i.e.,

the intersection graphs of intervals in the real line, have been introduced by Hajos about 60 years ago; they have many applications in large diversity of areas such as archeology, artificial intelligence, economics and planning, cf. [**3, 22**], and probably most intensively in mathematical biology, see e.g. [**1, 9, 23**]. On the other hand, Sierpiński graphs were introduced in relation to the Tower of Hanoi game, and were studied also in physics, interconnection networks, and topology; see more details in a very recent extensive survey [**19**]. The main feature of Sierpiński graphs is their fractal-like nature, and can be considered as a basic discrete version of fractals.

While domination and total domination number are computationally hard problems in general graphs [**11**], they can be efficiently determined in several classes of graphs, in particular, in the interval graphs and the Sierpiński graphs. An efficient algorithm for computing the domination number (and some related invariants) in strongly chordal graphs, which as a subclass contain the interval graphs, was first designed by FARBER in the 1980s [**10**]; KEIL soon followed with a linear time algorithm for determining the total domination number of interval graphs [**17**]. Even more can be said in the case of Sierpiński graphs, since exact formulas for the domination numbers of these graphs were established in [**21**], and more recently also the exact total domination numbers of arbitrary Sierpiński graphs were proven [**12**].

A motivation for studying dominating sequences was the domination game as introduced in [**7**]. Two players alternate turns in this game, one player wants to build a dominating set as quickly as possible, while the other (called Staller) wants to delay the process. By definition of legality of moves in the game it follows that the resulting number of moves when both players play optimally, called the game domination number of a graph, is bounded above by the Grundy domination number (in fact, a legal sequence whose length is the Grundy domination number is obtained when only Staller plays the game). The domination game has been intensively studied by several authors, and a lot of effort was given to resolve the (still open) 3/5-conjectures from [**18**]. In [**4**] the authors examined the possible changes of the game domination number under vertex- and edge-removal in a graph, and proposed a classification of the graphs with respect to the corresponding behaviour. (For a very recent paper on game domination see [**20**].)

In this paper we describe the behaviour of the Grundy domination number when an edge or a vertex is removed from a graph, see Section 2. We prove that in any graph G and $u \in V(G)$, the Grundy domination number of G drops by at most 2 when u is removed from G. Next, if e is an edge of an arbitrary graph G, then the Grundy domination number of $G - e$ is between one less than the Grundy domination number of G and one more than that number. Combining the edge-removal bound and the recursive fractal structure of Sierpiński graphs, we prove in Section 3 that the Grundy domination number of the Sierpiński graph S_p^n equals $p^{n-1} + \dfrac{p(p^{n-1} - 1)}{2}$. In addition, we present two efficient algorithms to construct a Grundy dominating sequence of a Sierpiński graph. The first algorithm is optimal, because it uses a recursive formula that builds only the Sierpiński labels of all vertices of the Grundy dominating sequence; the second algorithm is nice in the sense that the vertices are ordered lexicographically with respect to their Sierpiński

labels (hence, one could only follow this order and decide whether a given vertex can be put in the sequence or not). Finally, in Section 4 we make use of the vertex-removal formula (in fact, a version of this formula for the removal of a simplicial vertex) to construct an efficient algorithm for determining the Grundy domination number (resp. sequence) of an arbitrary interval graph. In the remainder of this section we present main formal definitions and notation, used throughout the paper.

Let $S = (v_1, \ldots, v_k)$ be a sequence of distinct vertices of a graph G. The corresponding set $\{v_1, \ldots, v_k\}$ of vertices from the sequence S will be denoted by $\widehat{S}$. A sequence $S = (v_1, \ldots, v_k)$, where $v_i \in V(G)$, is called a *legal (closed neigborhood) sequence* if, for each i

$$N[v_i] \setminus \cup_{j=1}^{i-1} N[v_j] \neq \emptyset.$$

(We also say that v_i is a *legal choice*, when the above inequality holds.) If for a legal sequence S, the set $\widehat{S}$ is a dominating set of G, then S is called a *dominating sequence* of G. Adopting the notation from domination theory, each vertex $u \in N[v_i] \setminus \cup_{j=1}^{i-1} N[v_j]$ is called a *private neighbor* of v_i with respect to $\{v_1, \ldots, v_i\}$. We will also use a more suggestive term by saying that v_i *footprints* the vertices from $N[v_i] \setminus \cup_{j=1}^{i-1} N[v_j]$, and that v_i is the *footprinter* of any $u \in N[v_i] \setminus \cup_{j=1}^{i-1} N[v_j]$. For a dominating sequence S any vertex in $V(G)$ has a unique footprinter in $\widehat{S}$. Thus the function $f_S : V(G) \to \widehat{S}$ that maps each vertex to its footprinter is well defined. Clearly the length k of a dominating sequence $S = (v_1, \ldots, v_k)$ is bounded from below by the domination number $\gamma(G)$ of a graph G. We call the maximum length of a legal dominating sequence in G the *Grundy domination number* of a graph G and denote it by $\gamma_{gr}(G)$. The corresponding sequence is called a *Grundy dominating sequence* of G or γ_{gr}-sequence of G. These concepts were introduced in [**5**].

Let $S_1 = (v_1, \ldots, v_n)$ and $S_2 = (u_1, \ldots, u_m), n, m \geq 0$, be two sequences. The *concatenation* of S_1 and S_2 is defined as the sequence $S_1 \oplus S_2 = (v_1, \ldots, v_n, u_1, \ldots, u_m)$.

2. GRUNDY DOMINATION NUMBER OF SUBGRAPHS, OBTAINED BY EDGE- OR VERTEX-DELETION

2.1. Edge-deletion

First consider the subgraphs obtained by the smallest possible atomic change, i.e. deletion of an edge. Unlike in the standard domination, where edge-deletion can only increase the domination number [**14**, Chapter 5], the following possibilities appear for the Grundy domination number.

Theorem 1. *If G is a graph and $e \in E(G)$, then*

$$\gamma_{gr}(G) - 1 \leq \gamma_{gr}(G - e) \leq \gamma_{gr}(G) + 1.$$

Moreover, there is an infinite family of graphs G such that all values of $\gamma_{gr}(G - e)$ between $\gamma_{gr}(G) - 1$ and $\gamma_{gr}(G) + 1$ are realized for different edges $e \in E(G)$.

Proof. Let S be a Grundy dominating sequence of G, and let $e = uv$ be an edge, deleted from G. Then, if $u, v \notin \widehat{S}$, S is also a legal sequence of $G-e$. In fact, the only case when S is not a legal sequence of $G - e$ is when $f_S(u) = v$ and $f_S^{-1}(v) = \{u\}$, or $f_S(v) = u$ and $f_S^{-1}(u) = \{v\}$. Without loss of generality, we may assume that $f_S(u) = v$ and $f_S^{-1}(v) = \{u\}$. Let S' be the sequence obtained by removing the vertex v from S. It is clear that S' is a legal sequence in $G - e$ of length $|\widehat{S}| - 1$. (If S' is not a dominating sequence, we can append u at the end of it, and obtain a legal dominating sequence of $G - e$.) We infer that $\gamma_{gr}(G) - 1 \le \gamma_{gr}(G - e)$.

For the other inequality consider a Grundy dominating sequence $S = (x_1, \ldots, x_k)$ of $G-e$, where $e = uv$ is deleted from G. If x_i is not in $N_{G-e}[u] \cup N_{G-e}[v]$, then it is clear that $N_G[x_i] \setminus \cup_{j=1}^{i-1} N_G[x_j] \ne \emptyset$, i.e., x_i is a legal choice also in G. Now, suppose that $x_i \in N_{G-e}[u]$. If $f_S^{-1}(x_i) \ne \{u\}$, then again x_i is a legal choice also in G. But even if $f_S^{-1}(x_i) = \{u\}$, and $v \notin \{x_1, \ldots, x_{i-1}\}$, x_i is a legal choice in G. Thus the only problem with legality of x_i is when $v = x_j$ for some $j < i$ (note that we are assuming $x_i \in N_{G-e}[u]$ and $f_S^{-1}(x_i) = \{u\}$, that is, the only vertex in $G - e$ footprinted by x_i is u). Now, let S' be the sequence obtained from S by removing x_i, i.e., $S' = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots x_k)$. From the above it is clear that S' is a legal sequence in G of length $\gamma_{gr}(G - e) - 1$, hence $\gamma_{gr}(G) \ge |\widehat{S'}| = \gamma_{gr}(G - e) - 1$.

To see the second part of the theorem, consider the family of graphs $H_{m,n}$, obtained by identifying a vertex of degree 1 of the path P_m, $m \ge 3$, with a vertex of the cycle C_n, $n \ge 3$. (The meaning of *edges of the cycle* and *edges of the path* in $H_{m,n}$ should be clear.) It is easy to see that $\gamma_{gr}(H_{m,n}) = m + n - 3$. Now, if e is an edge of the cycle in $H_{m,n}$, then $\gamma_{gr}(G_{m,n} - e) = m + n - 2$. If e is the pendant edge of the path in $H_{m,n}$, then $\gamma_{gr}(G_{m,n} - e) = m + n - 3 = \gamma_{gr}(G_{m,n})$. Finally, if e is any other edge (of the path), then $\gamma_{gr}(G_{m,n} - e) = m + n - 4$. $\square$

The following immediate consequence of Theorem 1 will be used later.

Corollary 2. *Let G be a graph, and let G' be obtained from G, by adding k edges to G. Then $\gamma_{gr}(G) - k \le \gamma_{gr}(G') \le \gamma_{gr}(G) + k$.*

2.2. Vertex-deletion

It is easy to see that $\gamma_{gr}(H) \le \gamma_{gr}(G)$, if H is an induced subgraph of G. Indeed, if S is a Grundy dominating sequence of H, then S is also a sequence in G, and it is clearly legal also with respect to G. If S is not a dominating sequence in G, one can add some vertices in a legal way at the end of S to make it a dominating sequence of G. Hence $\gamma_{gr}(G) \ge \gamma_{gr}(H)$, and so the graph property of having the Grundy domination number bounded from above by a constant belongs to hereditary properties.

Let us focus on the action of vertex deletion in a graph G. By the observation in the previous paragraph, the Grundy domination number cannot increase when a vertex is removed. The following result specifies how much it can decrease.

Theorem 3. *If G is a graph and $u \in V(G)$, then*

$$\gamma_{gr}(G) - 2 \le \gamma_{gr}(G - u) \le \gamma_{gr}(G).$$

Moreover, there is an infinite family of graphs G such that all values of $\gamma_{gr}(G-u)$ between $\gamma_{gr}(G) - 2$ and $\gamma_{gr}(G)$ are realized for different vertices $u \in V(G)$.

Proof. The bound $\gamma_{gr}(G - u) \leq \gamma_{gr}(G)$ immediately follows from the fact that the Grundy domination number of an induced subgraph H of G is not greater than that of G.

For bounding $\gamma_{gr}(G-u)$ from below, let S be a Grundy dominating sequence in G, and let v be the vertex in S that footprints u, i.e., $v = f_S(u)$. Consider the sequence S' obtained from S by removing v, and, if $u \in \widehat{S}$, also removing u. We claim that S' is a legal sequence in $G - u$. Indeed, since v and u are not in S', we derive that each vertex x from S' in $G - u$ footprints all the vertices that are footprinted by x with respect to S in G (while x with respect to S' in $G - u$ could also footprint some additional vertices in $N_G(u) \cup N_G[v]$). Since a legal sequence S' can be completed to a dominating sequence of $G - u$, we get $\gamma_{gr}(G) - 2 \leq |\widehat{S'}| \leq \gamma_{gr}(G - u)$.

To see the second part of the theorem, consider the family of graphs $G_{m,n}$, obtained by identifying a vertex of degree 1 of the path P_m, $m \geq 4$, with a vertex of the complete graph K_n, $n \geq 3$. It is clear that $\gamma_{gr}(G_{m,n}) = m$. Now, if u is the vertex of degree 1 in $G_{m,n}$ or its neighbor or the identified vertex, then $\gamma_{gr}(G_{m,n} - u) = m - 1$. If u is a vertex of the complete graph (and not the identified vertex), then $\gamma_{gr}(G_{m,n} - u) = m$. Finally, if u is any other vertex (in the path), then $\gamma_{gr}(G_{m,n} - u) = m - 2$.   □

Better bounds can be obtained for special types of vertices, namely for twin and simplicial vertices, whose definitions we recall now. A vertex $v \in V(G)$ is a *simplicial* vertex of a graph G, if $N(v)$ induces a complete graph. Two vertices u and v in G are called *twins* if $N[u] = N[v]$. A vertex $v \in V(G)$ is called a *twin vertex* if there exists $u \in V(G)$, such that u and v are twins. The following result will be applied later (the second statement was known already in [**5**]).

Proposition 4. *Let G be a graph and $u \in V(G)$.*

(i) *If u is a simplicial vertex, then $\gamma_{gr}(G - u) \geq \gamma_{gr}(G) - 1$.*

(ii) *If u is a twin vertex, then $\gamma_{gr}(G - u) = \gamma_{gr}(G)$.*

Proof. (i) Let S be a Grundy dominating sequence in G, u a simplicial vertex, and let $f_S(u) = v$. If $u \notin \widehat{S}$, then the sequence obtained from S by removing v is a legal sequence in $G - u$, implying $\gamma_{gr}(G - u) \geq \gamma_{gr}(G) - 1$. Suppose now that $u \in \widehat{S}$. Then u footprints one vertex from the clique $N[u]$. Thus u is in S before any $x \in N(u) \cap \widehat{S}$ which means that $f_S(u) = u$. Thus the sequence obtained from S by removing u is a legal sequence in $G - u$, implying $\gamma_{gr}(G - u) \geq \gamma_{gr}(G) - 1$.

(ii) Let v be a twin of u and let S be a Grundy dominating sequence in G. If $u \notin \widehat{S}$, then S is a legal sequence of $G - u$, implying $\gamma_{gr}(G - u) \geq \gamma_{gr}(G)$. Suppose now that $u \in \widehat{S}$. Then $v \notin \widehat{S}$ and the sequence S' obtained from S by replacing u with v is a Grundy dominating sequence in G not containing u. Hence also in

this case we have $\gamma_{gr}(G - u) \geq \gamma_{gr}(G)$. Combining this with Theorem 3 we obtain $\gamma_{gr}(G - u) = \gamma_{gr}(G)$.

3. DOMINATING SEQUENCES OF SIERPIŃSKI GRAPHS

Set $[n] = \{1, 2, \ldots, n\}$ and $[n]_0 = \{0, 1, \ldots, n - 1\}$. The Sierpiński graph S_p^n $(n, p \geq 1)$ is defined on the vertex set $[p]_0^n$, two different vertices $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ being adjacent if and only if there exists an $h \in [n]$ such that

1. $u_t = v_t$, for $t = 1, 2, \ldots, h - 1$;

2. $u_h \neq v_h$; and

3. $u_t = v_h$ and $u_h = v_t$ for $t = h + 1, h + 2, \ldots, n$.

In the rest, we will shortly write $\langle u_1 u_2 \ldots u_n \rangle$ for $(u_1, u_2, \ldots, u_n)$ and u_j will be called j-*th bit* of a vertex u. A vertex of the form $\langle ii \ldots i \rangle = \langle i^n \rangle$ of S_p^n is called an *extreme vertex*. The extreme vertices of S_p^n are of degree $p - 1$ while the degree of any other vertex is p.

In other words, S_p^n can be constructed from p copies of S_p^{n-1} as follows. For each $i \in [p]_0$ concatenate i to the left of the vertices in a copy of S_p^{n-1} and denote the obtained graph with iS_p^{n-1}. Then for each $i \neq j$ join copies iS_p^{n-1} and jS_p^{n-1} by the single edge $e_{ij}^{(n)} = \{ij^{n-1}, ji^{n-1}\}$. In Fig. 1 the construction of S_3^3 is illustrated.
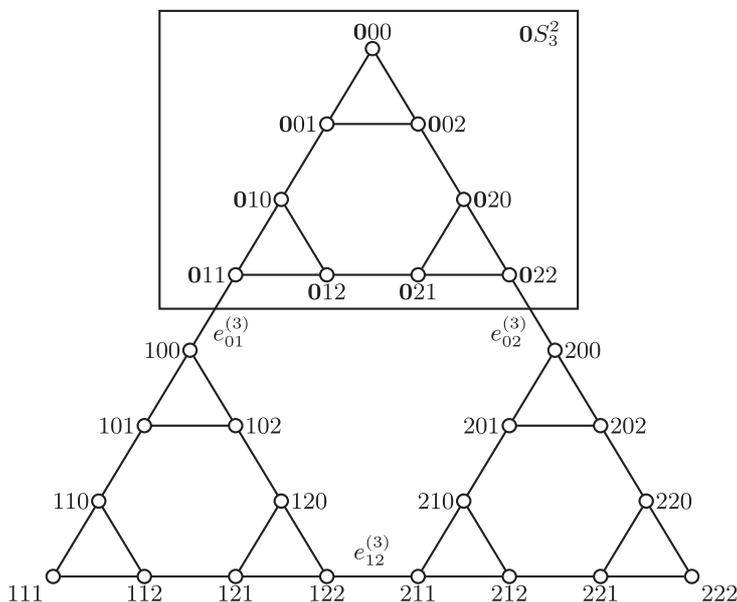

Figure 1. The Sierpiński graph S_3^3.

A dominating sequence of S_p^n built in the proof of the following result will be denoted by A_p^n. Note that we can construct a dominating sequence of iS_p^n by concatenating i to the left of every vertex in the sequence A_p^n, and we denote this sequence by iA_p^n.

Theorem 5. *If $n, p \geq 1$ and S_p^n is a Sierpiński graph, then*

$$\gamma_{gr}(S_p^n) = p^{n-1} + \frac{p(p^{n-1} - 1)}{2}.$$

Proof. Fix $p \geq 1$. It is clear that $\gamma_{gr}(S_p^1) = \gamma_{gr}(K_p) = 1$. Let G be a graph of p disjoint copies of S_p^{n-1}, where $n \geq 2$. It is easy to see that $\gamma_{gr}(G) = p \cdot \gamma_{gr}(S_p^{n-1})$. To construct S_p^n from G, $\binom{p}{2}$ edges have to be added. Hence by Corollary 2, if $\binom{p}{2}$ edges are added, the Grundy domination number increases by at most $\binom{p}{2}$. It follows that $\gamma_{gr}(S_p^n) \leq \gamma_{gr}(G) + \binom{p}{2} = p \cdot \gamma_{gr}(S_p^{n-1}) + \binom{p}{2}$. Note that the recursion $a_n = pa_{n-1} + \binom{p}{2}$ with base $a_1 = 1$ can be converted to the explicit form $a_n = p^{n-1} + \frac{p(p^{n-1} - 1)}{2}$. Thus we derive the upper bound

$$(1) \qquad \gamma_{gr}(S_p^n) \leq p^{n-1} + \frac{p(p^{n-1} - 1)}{2}.$$

For the reversed inequality, we will construct a legal dominating sequence in S_p^n of length $p \cdot \gamma_{gr}(S_p^{n-1}) + \binom{p}{2}$, which will be denoted by A_p^n. First, we define another sequence $iB_p^n = (\langle i(i+1)^{n-1} \rangle, \langle i(i+2)^{n-1} \rangle, \dots, \langle i(p-2)^{n-1} \rangle, \langle i(p-1)^{n-1} \rangle)$, where $i \in [p]_0$. So in iB_p^n are just the extreme vertices of iS_p^{n-1}, where i is smaller then all other bits. The sequence A_p^n will be constructed recursively. For the recursion base the dominating sequence of S_p^1 is $A_p^1 = (\langle 0 \rangle)$. The sequence A_p^n, for $n > 1$ is constructed as follows:

$$A_p^n = 0A_p^{n-1} \oplus 0B_p^n \oplus 1A_p^{n-1} \oplus 1B_p^n \oplus (p-2)A_p^{n-1} \oplus (p-1)B_p^n \oplus (p-1)A_p^{n-1}.$$

We can also write

$$A_p^n = \bigoplus_{i=0}^{p-1} \left(iA_p^{n-1} \oplus iB_p^n \right),$$

where $(p-1)B_p^n$ is an empty sequence. In Fig. 2 the γ_{gr}-sequences of S_3^1, S_3^2 and S_3^3 are illustrated.

Clearly $\langle 0^n \rangle$ is the first vertex in A_p^n. Now, we show that $\langle 0^n \rangle$ is the only extreme vertex in the sequence. To see that, we have to expand the equation to the bottom of recursion. Then each vertex in the sequence comes from some iA_p^1 or some iB_p^ℓ, where $i \in [p]_0$ and $1 < \ell \leq n$. If a vertex comes from iA_p^1, then its

last bit is 0. The only extreme vertex with the last bit 0 is $\langle 0^n \rangle$. If a vertex is in some iB_p^ℓ, then it is not an extreme vertex in A_p^n, since its ℓ-th bit is i and its last $(\ell - 1)$ bits are greater then i. So the only extreme vertex in A_p^n is $\langle 0^n \rangle$.

It is easy to see, that vertices are pairwise different. If $i, j \in [p]_0$ and $i \neq j$, then vertices in iA_p^{n-1} and vertices in jA_p^{n-1} differ already in the first bit. The same holds for vertices in iB_p^n and jB_p^n and vertices in iA_p^{n-1} and jB_p^n. In some iB_p^n are just the vertices that are extreme vertices in S_p^{n-1}, and in iA_p^{n-1} the only extreme vertex is $\langle i0^{n-1} \rangle$. But vertex $\langle i0^{n-1} \rangle$ is not in iB_p^n since $0 \leq i$. So, vertices that are in iA_p^{n-1} are not in iB_p^n.

To show that the sequence A_p^n is legal, we check that every vertex in the sequence is footprinting at least one vertex. We mentioned already that every vertex in A_p^n comes either from iA_p^1 or some iB_p^ℓ, where $i \in [p]_0$ and $1 < \ell \leq n$. If the vertex v comes from a sequence iB_p^ℓ, its form is $v = \langle x_1 x_2 \ldots x_{n-\ell} a b^{\ell-1} \rangle$, where $a < b$ and $x_1, \ldots, x_{n-\ell}, a, b \in [p]_0$. The vertex v footprints $u = \langle x_1 x_2 \ldots x_{n-\ell} b a^{\ell-1} \rangle$, because $a < b$, and u and all its other neighbors cannot be in A_p^n before v.

If v comes from some iA_p^1, its form is $v = \langle x_1 x_2 \ldots x_{n-2} i 0 \rangle$, and we claim that v footprints at least the vertex $u = \langle x_1 x_2 \ldots x_{n-2} i (p-1) \rangle$. Clearly, u is not footprinted before by some other vertex from jA_p^n. Vertices that are in some jB_p^ℓ and are also in $N[u]$ are just some vertices of iB_p^2. Since iB_p^2 is in A_p^n after iA_p^1, it follows that v footprints u.

It is obvious that the vertices of A_p^n dominate the whole graph, since already the vertices of $\bigoplus_{i=0}^{p-1} iA_p^{n-1}$ dominates it. So A_p^n is a (legal) dominating sequence and its length is

$$|A_p^n| = \sum_{i=0}^{p-1} \left(|iA_p^{n-1}| + |iB_p^n| \right) = \sum_{i=0}^{p-1} \left(|A_p^{n-1}| + (p-1-i) \right) = p \cdot |A_p^{n-1}| + \binom{p}{2}.$$

Since $|A_p^1| = 1$, we can transform the recursion to the explicit form

$$|A_p^n| = p^{n-1} + \frac{p(p^{n-1} - 1)}{2}.$$

It follows that

$$(2) \qquad\qquad \gamma_{gr}(S_p^n) \geq |A_p^n| = p^{n-1} + \frac{p(p^{n-1} - 1)}{2}.$$

Combining inequalities (1) and (2), we get

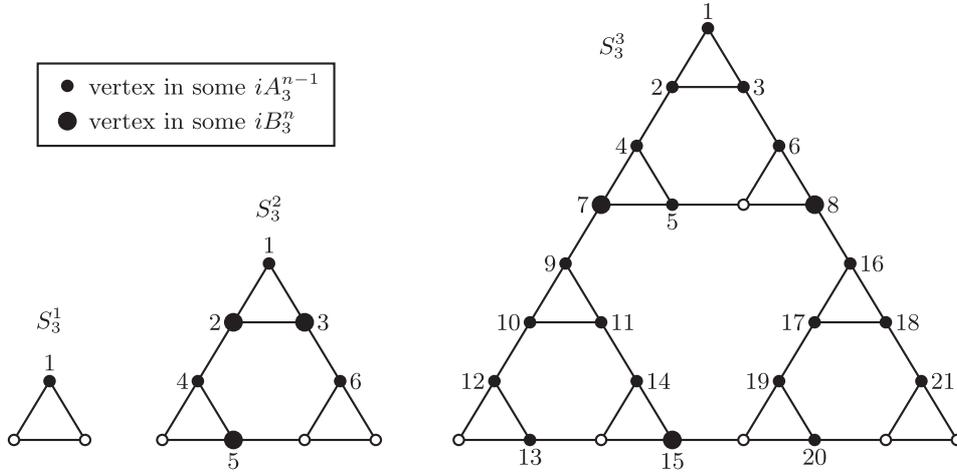$$\gamma_{gr}(S_p^n) = p^{n-1} + \frac{p(p^{n-1} - 1)}{2}. \qquad\qquad \square$$

Figure 2. The Sierpiński graphs S_3^1, S_3^2 and S_3^3 with γ_{gr}-sequences.

The proof of Theorem 5 also yields the construction of a Grundy dominating sequence of S_p^n. In the construction we are building the Grundy dominating sequence by concatenating smaller sequences and by concatenating one bit to the left of every vertex in the sequence. Thus the time complexity of the construction is the same as the length of the Grundy dominating sequence multiplied by n, since each vertex is labelled by n bits. We derive that the time complexity of the construction is $O(np^n)$. In addition, the algorithm constructs along the way all Grundy dominating sequences of S_p^ℓ, for $\ell \leq n$. Since the complexity of the algorithm (which simply builds the sequence A_p^n) is the same as generating the labels of vertices in S_p^n that form a Grundy dominating sequence, we infer that this complexity is best possible.

Corollary 6. *The time complexity of constructing a Grundy dominating sequence* (A_p^n) *of the Sierpiński graph* S_p^n *is* $O(np^n)$, *and this is best possible.*

There exists an alternative (and easier) construction of a γ_{gr}-sequence of S_p^n, of which construction time complexity is slightly worse than above, since we are going through all vertices in S_p^n. Nevertheless, let us present also this construction, because the vertices in the sequence are nicely listed in the lexicographical order. This means that we order the vertices lexicographically by their labels and in this order we add to the sequence each legal vertex (a vertex whose neighborhood is not contained in the union of neighborhoods of previously chosen vertices). We denote the sequence by L_p^n. A vertex v is in L_p^n if and only if

- the last bit of v is 0 or

- $v = \langle x_1 x_2 \ldots x_{n-\ell} a b^{\ell-1} \rangle$, where $2 \leq \ell \leq n$, $b > a$ and $x_1, \ldots, x_{n-\ell}, a, b \in [p]_0$.

Let us show that L_p^n is legal. If the last bit of v is 0 ($v = \langle x_1 x_2 \ldots x_{n-1} 0 \rangle$) then v footprints at least the vertex $u = \langle x_1 x_2 \ldots x_{n-1} (p-1) \rangle$. Note that u is

not footprinted by any other vertex of L_p^n since all its other neighbors are lexicographically greater than v. So if they are in L_p^n then they are in L_p^n after v. If $v = \langle x_1 x_2 \ldots x_{n-\ell} a b^{\ell-1} \rangle$, where $2 \leq \ell \leq n$, $b > a$ and $x_1, \ldots, x_{n-\ell}, a, b \in [p]_0$ then v footprints $u = \langle x_1 x_2 \ldots x_{n-\ell} b a^{\ell-1} \rangle$. Since $b > a$, all other neighbors of u are also lexicographically greater then v (note that they are of the form $\langle x_1 x_2 \ldots x_{n-\ell} b a^{\ell-2} y \rangle$, where $y \in [p]_0 \setminus \{a\}$). It is clear that the vertices of L_p^n dominate the whole graph, since already the vertices with the last bit 0 dominate it. So L_p^n is a (legal) dominating sequence.
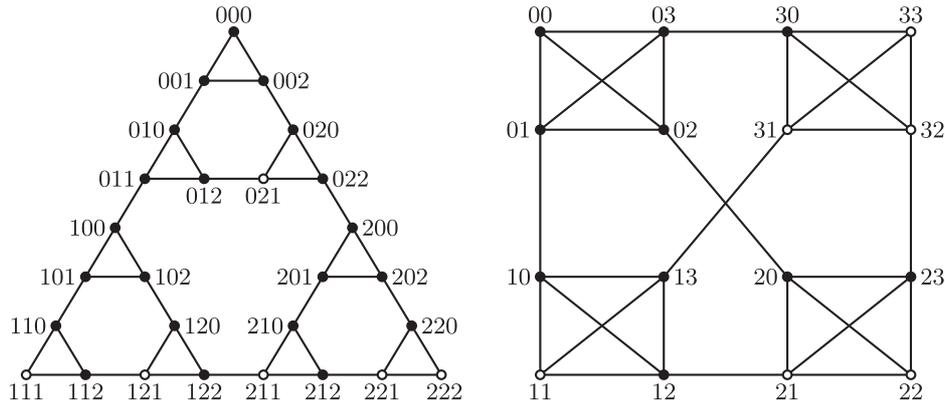

Figure 3. The Sierpiński graph S_3^3 and S_4^2 with vertices of L_3^3 and L_4^2 bolded.

To show that L_p^n is a γ_{gr}-sequence, we have to check its length. There are p^{n-1} vertices with the last bit 0. Now we have to compute how many vertices have the form $\langle x_1 x_2 \ldots x_{n-\ell} a b^{\ell-1} \rangle$, where $2 \leq \ell \leq n$, $b > a$ and $x_1, \ldots, x_{n-\ell}, a, b \in [p]_0$. If ℓ and all $x_1, \ldots, x_{n-\ell}$ are fixed, then there are $\dfrac{p(p-1)}{2}$ such vertices. So altogether we have

$$\sum_{\ell=2}^{n} \left(p^{n-\ell} \cdot \frac{p(p-1)}{2} \right) = \frac{p(p-1)}{2} \sum_{\ell=2}^{n} p^{n-\ell} = \frac{p(p-1)}{2} \cdot \frac{p^{n-1}-1}{p-1} = \frac{p(p^{n-1}-1)}{2}$$

vertices that satisfy those conditions. This implies that the length of L_p^n is $p^{n-1} + \dfrac{p(p^{n-1}-1)}{2}$, so L_p^n is indeed a γ_{gr}-sequence of S_p^n. In Fig. 3, S_3^3 and S_4^2 with L_3^3 and L_4^2 are illustrated.

4. DOMINATING SEQUENCES OF INTERVAL GRAPHS

In this section we present an algorithm that generates a Grundy dominating sequence of an arbitrary interval graph. We use the results of Section 2 concerning the deletion of simplicial vertices and twins.

An *interval representation* of a graph is a family of intervals of the real line assigned to vertices so that vertices are adjacent if and only if the corresponding

intervals intersect. A graph is an *interval graph* if it has an interval representation. For more details on interval graphs see [**3, 22**].

Let us present the ordering and establish the notation of vertices in an interval graph. Let $G = (V, E)$ be an interval graph with an interval representation $I_G : V(G) \to \{[a, b]; \ a, b \in \mathbb{R}, \ a \le b\}$, and vertices $V = \{v_1, \dots, v_n\}$ sorted in the non-decreasing order according to the right endpoints of corresponding intervals. In other words, $I_G(v_i) = [a_i, b_i]$, and $b_1 \le b_2 \le \dots \le b_n$. It is clear that v_1 is a simplicial vertex of G. Let $\widehat{A} = \{a_1, b_1, \dots a_n, b_n\}$ be the (multi)set of interval endpoints. We will also make use of the non-decreasing sequence A_{I_G} of the real numbers from $\widehat{A}$ of length $2n$, such that all elements of $\widehat{A}$ are used; in the case $a_i = b_j$, for some $i, j \in \{1, 2, \dots, n\}$, a_i is in the sequence before b_j. We call the sequence A_{I_G} the *interval endpoints sequence*. In Fig. 4 an example of interval representation and interval endpoints sequence is presented.
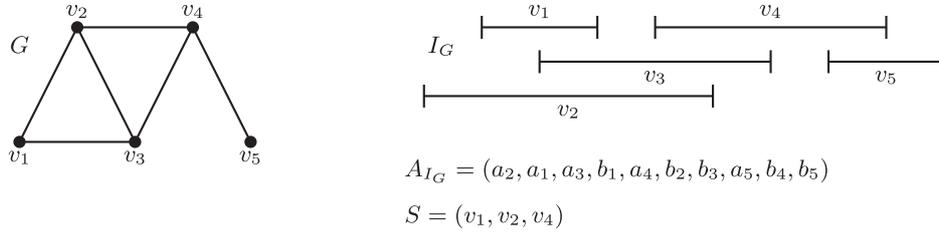

$$A_{I_G} = (a_2, a_1, a_3, b_1, a_4, b_2, b_3, a_5, b_4, b_5)$$

$$S = (v_1, v_2, v_4)$$

Figure 4. An interval graph G with interval representation I_G, interval endpoints sequence A_{I_G} and Grundy dominating sequence S returned by Algorithm 1.

Algorithm 1: Grundy dominating sequence of an interval graph G.

Input: An interval graph G with vertices $(v_1, v_2, \dots, v_n)$ (where v_i corresponds to $[a_i, b_i]$), ordered according to their right end-points, and the interval endpoints sequence A_{I_G}.

Output: Grundy dominating sequence S of a graph G.

1   $S = ()$;
2   $newInterval = $ false;
3   $A = A_{I_G}$;
4   **while** $A \ne ()$ **do**
5   Choose $e \in A$ such that $A = (e) \oplus A'$;
6   **if** e *is some* a_i **then**
7         $newInterval = $ true;
8   **else if** e *is some* b_i *and* $newInterval$ *is true* **then**
9         $S = S \oplus (v_i)$;
10        $newInterval = $ false;
11      $A = A'$

Theorem 7. *Algorithm* 1 *returns a Grundy dominating sequence of an interval graph G. In addition, $\gamma_{gr}(G)$ equals the number of consecutive subsequences of the form (a_i, b_j) in the interval endpoints sequence A_{I_G} for any interval representation I_G of G.*

Proof. The proof goes by induction on the number of vertices of an interval graph G). For $|V| = 1$ the length of the sequence of our algorithm is 1 which is clearly optimal.

Suppose now that the algorithm returns a Grundy dominating sequence for any interval graph with at most $n - 1$ vertices and let G be an arbitrary interval graph with n vertices. Let $V(G) = \{v_1, \ldots, v_n\}$ and let $I_G : V(G) \rightarrow \{[a, b] ; a, b \in \mathbb{R}, a \leq b\}$ be an interval representation of G with $I_G(v_i) = [a_i, b_i]$ ordered according to their right end-points, i.e., $b_1 \leq b_2 \leq \cdots \leq b_n$. Let A_{I_G} be the corresponding interval endpoints sequence. Since $G - v_1$ is an interval graph with $n - 1$ vertices, the algorithm returns the Grundy dominating sequence S' in $G - v_1$ (using the induction hypothesis). Since the vertex with the smallest right end-point is the first vertex in the sequence produced by the algorithm, v_2 is the first vertex of S'.

There are two options for v_1. First, suppose that v_1 is a twin vertex in G. Then v_1 and v_2 are twins, and, using Proposition 4(ii), $\gamma_{gr}(G) = \gamma_{gr}(G - v_1)$. Hence the sequence S' is also a Grundy dominating sequence in G. As v_1 and v_2 are twins, the sequence S obtained from S' by replacing v_2 with v_1 is also a Grundy dominating sequence in G. Note that S is exactly the sequence returned by Algorithm 1. Indeed, since v_1 and v_2 are two consecutive vertices with respect to the right endpoints ordering and are twins, no interval endpoint lies between b_1 and b_2; i.e., b_1 and b_2 are consecutive endpoints in the sequence A_{I_G}. Hence, after v_1 is put to S (and v_2 is not), the algorithm follows the same steps as the algorithm in $G - v_1$. The proof of this case is complete.

Finally suppose that v_1 is not a twin in G. Since v_1 is simplicial in G, by Proposition 4(i) we infer $\gamma_{gr}(G) \leq \gamma_{gr}(G - v_1) + 1$. As v_2 is the first vertex of S' and $N[v_1] \subsetneq N[v_2]$, $S = (v_1) \oplus S'$ is a legal dominating sequence in G.

Proposition 4(i) again implies that $\gamma_{gr}(G) = \gamma_{gr}(G - v_1) + 1$, which means that S is a Grundy dominating sequence in G. Since S is the sequence returned by Algorithm 1, the proof of the correctness of the algorithm is complete.

For the second statement in the theorem, one just needs to note that the algorithm counts the number of consecutive subsequences of A_{I_G}, in which the first vertex is a left endpoint a_i and the second vertex is a right endpoint b_j. Since the Grundy domination number of G is independent from the choice of its interval representation I_G we infer that the number of such subsequnces is also invariant under the interval representation.   □

Let G be an arbitrary interval graph on n vertices and m edges. It is easy to see that the time complexity of Algorithm 1 is $O(n)$, since the length of the interval endpoints sequence is $2n$. It is known that the time complexity of constructing interval representation of an interval graph is $O(n + m)$ [**2, 16, 8**]. To sort vertices according to their right endpoints and to construct interval endpoints sequence

$O(n \log n)$ time is needed, since we just need to sort endpoints of intervals. Thus the time complexity of preparing input data for Algorithm 1 is $O(n \log n + m)$. We derive the following result.

Corollary 8. *Let G be an interval graph on n vertices and m edges. The time complexity of preprocessing input data for Algorithm 1 is $O(n \log(n) + m)$, and Algorithm 1 is linear with complexity $O(n)$.*

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Faculty of Natural Sciences and Mathematics,   (Received March 6, 2016)
University of Maribor,   (Revised August 31, 2016)
Koroška cesta 160, Maribor
Slovenia

and

Institute of Mathematics, Physics and Mechanics,
Jadranska ulica 19, Ljubljana
Slovenia
E-mails: bostjan.bresar@um.si
         tanja.gologranc1@um.si

Institute of Mathematics, Physics and Mechanics,
Jadranska ulica 19, Ljubljana
Slovenia
E-mail: tim.kos@imfm.si