A NOTE ON THE NORDHAUS-GADDUM TYPE INEQUALITY TO THE SECOND LARGEST EIGENVALUE OF A GRAPH

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In honour of Dragoš Cvetković on the occasion of his 75th birthday.

Let G be a graph on n vertices and $\overline{G}$ its complement. In this paper, we prove a Nordhaus-Gaddum type inequality to the second largest eigenvalue of a graph G, $\lambda_2(G)$,

$$\lambda_2(G) + \lambda_2(\overline{G}) \leq -1 + \sqrt{\frac{n^2}{2} - n + 1},$$

when G is a graph with girth at least 5. Also, we show that the bound above is tight. Besides, we prove that this result holds for some classes of connected graphs such as trees, $k-$cyclic, regular bipartite and complete multipartite graphs. Based on these facts, we conjecture that our result holds to any graph.

1. INTRODUCTION

Let $G = (V, E)$ be a simple undirected graph with n vertices and m edges and let $\overline{G}$ be its complement. Given a vertex $v \in V$, the set of neighbors of v is $N(v) = \{w \in V | \{v, w\} \in E\}$ and $d(v) = |N(v)|$ is the degree of v. The girth of G is the length of the shortest cycle in G. Write $A(G) = [a_{ij}]$, where $a_{ij} = 1$ if $\{i, j\} \in E(G)$ and $a_{ij} = 0$, if $\{i, j\} \notin E(G)$, for the adjacency matrix of G. The

characteristic polynomial of G is $p_G(\lambda) = det(\lambda I - A)$ such that the roots λ_i for $i = 1, \cdots, n$ are the eigenvalues of $A(G)$ and can be arranged as $\lambda_1 \geq \cdots \geq \lambda_n$. The multiset $Spec(G) = \left\{ \lambda_1^{(s_1)}, \lambda_2^{(s_2)}, \ldots, \lambda_t^{(s_t)} \right\}$, where s_i is the multiplicity of λ_i, $1 \leq i \leq t$, is the spectrum of the graph G.

In 2007, Nikiforov [17] proposed the study of the Nordhaus-Gaddum type inequalities for all eigenvalues of a graph defining a function given by

$$(1) \qquad \max_{|G|=n} \left(|\lambda_k(G)| + |\lambda_k(\overline{G})| \right)$$

for all $k = 1, \ldots, n$. For the case $k = 1$, previous papers addressed the problem of finding lower and upper bounds to (1) as one can cite Nosal [19], Amin and Hakimi [1], Csikvári [4] and Terpai [27]. The latter obtained the following sharp upper bound:

$$\lambda_1(G) + \lambda_1(\overline{G}) \leq \frac{4}{3}n - 1.$$

The general case, for any k, was first introduced by Nikiforov in [17]. Particularly, in the case $k = 2$, Nikiforov and Yuan [18] obtained the best known upper bound to (1) as

$$(2) \qquad \lambda_2(G) + \lambda_2(\overline{G}) \leq -1 + \frac{n}{\sqrt{2}}.$$

In this paper, we propose a slightly improvement of inequality (2), that is,

$$(3) \qquad \lambda_2(G) + \lambda_2(\overline{G}) \leq -1 + \sqrt{\frac{n^2}{2} - n + 1},$$

what we will refer throughout the paper by NG_2-bound. Also, we show that NG_2-bound holds for some classes of graphs such as trees, $k-$cyclic, regular bipartite, complete multipartite graphs, generalized line graphs and exceptional graphs. For the first two classes, the proofs appear in Section 2 while the others are in Section 3. Also, in Section 3, we state our main result proving that inequality (3) is true for all graphs of girth at least 5. In addition, a family of graphs presented by Nikiforov in [17] is showed to be extremal to our NG_2-bound in Section 4.

2. PRELIMINARY RESULTS

In this section, we prove that inequality (3) holds for trees and $k-$cyclic graphs when $k \geq 0$. Some useful results are revisited to open this section. The first one is the well-known Weyl's inequality and the second one is an upper bound to the largest eigenvalue of trees.

For convenience, given a Hermitian matrix M of order n, we index its eigenvalues as $\alpha_1(M) \geq \cdots \geq \alpha_n(M)$.

Theorem 1 ([**13**]). *Let M and N be Hermitian matrices of order n, and let $1 \leq i \leq n$ and $1 \leq j \leq n$. Then*

$$\alpha_i(M) + \alpha_j(N) \leq \alpha_{i+j-n}(M+N) \quad if \quad i+j \geq n+1,$$
$$\alpha_i(M) + \alpha_j(N) \geq \alpha_{i+j-1}(M+N) \quad if \quad i+j \leq n+1.$$

Theorem 2 ([**3**]). *If T is a tree of order n then $\lambda_1(T) \leq \sqrt{n-1}$. The equality holds if and only if $T \simeq S_n$, where S_n is the star with n vertices.*

Both results, Theorems 1 and 2, are used to prove in Proposition 3 that the proposed NG_2-bound holds for all trees.

Proposition 3. *If T is a tree of order n, then*

$$(4) \qquad \lambda_2(T) + \lambda_2(\overline{T}) \leq -1 + \sqrt{\frac{n^2}{2} - n + 1}.$$

Equality holds if and only if $T \simeq P_4$.

Proof. From computational experiments, it is easy to check that inequality (4) follows for all trees up to 6 vertices and among them the graph P_4 is the only one for which the equality holds since $\lambda_2(P_4) = \lambda_2(\overline{P_4}) = \frac{\sqrt{5}-1}{2}$.

Now, suppose that $n \geq 7$. From Theorem 1,

$$\lambda_2(\overline{T}) \leq -1 - \lambda_n(T).$$

Since $\lambda_n(T) = -\lambda_1(T)$, from Theorem 2, we get

$$(5) \qquad \lambda_2(\overline{T}) \leq -1 + \sqrt{n-1}.$$

From [**22**] [see Theorem 1], it is known that

$$(6) \qquad \lambda_2(T) \leq \sqrt{\frac{n}{2} - 1}.$$

Using inequalities (5) and (6), we obtain,

$$\lambda_2(T) + \lambda_2(\overline{T}) \leq -1 + \sqrt{\frac{n}{2} - 1} + \sqrt{n-1}$$

$$(7) \qquad\qquad\qquad \leq -1 + \sqrt{2\left(n - 1 + \frac{n}{2} - 1\right)} = -1 + \sqrt{3n-4}.$$

Note that, for $n \geq 7$,

$$3n - 4 = \left(\frac{n^2}{2} - n + 1\right) - \left(\frac{n^2}{2} - 4n + 5\right)$$

$$(8) \qquad\qquad < \frac{n^2}{2} - n + 1.$$

So, by (7) and (8), the result follows.

$\square$

Proposition 4 relates the spectral radius of a graph to NG_2-bound.

Proposition 4. *Let G be a graph of order n, let $x = \dfrac{n^2}{2} - n + 1$ and $y = -1 + \sqrt{x}$. If $\lambda_1(G) \leq \dfrac{\sqrt{x}}{2}$, then $\lambda_2(G) + \lambda_2(\overline{G}) \leq y$.*

Proof. Let G be a graph of order n. Suppose that $\lambda_1 \leq \sqrt{x}/2$. By applying Theorem 1 for the adjacency matrices of G and $\overline{G}$, we obtain $\lambda_2(\overline{G}) + \lambda_n(G) \leq -1$. Since $\lambda_n(G) < 0$ and $|\lambda_n(G)| \leq \lambda_1(G)$, we get

$$\lambda_2(\overline{G}) \quad \leq \quad -1 + \lambda_1(G),$$

and then

$$\lambda_2(G) + \lambda_2(\overline{G}) \leq -1 + 2\lambda_1(G) \leq -1 + \sqrt{\dfrac{n^2}{2} - n + 1} = y.$$

□

Now, let S_n^+ be the star graph plus one edge. For $n \geq 9$, the next result shows that S_n^+ has the largest spectral radius among all unicyclic graphs. By Proposition 5 with simple algebraic manipulations, we show that inequality (3) is true for all unicyclic graphs.

Proposition 5 ([**23**]). *If G is unicyclic of order n, then*

$$2 \leq \lambda_1(G) \leq \lambda_1(S_n^+).$$

Moreover, for every $n \geq 9$, $\lambda_1(S_n^+) \leq \sqrt{n}$.

Proposition 6. *If G is unicyclic of order n, then*

$$\lambda_2(G) + \lambda_2(\overline{G}) \leq -1 + \sqrt{\dfrac{n^2}{2} - n + 1}.$$

Proof. From computational experiments, it is easy to check that the result follows for all unicyclic graphs up to 9 vertices. Now, suppose that G is an unicyclic graph with $n \geq 10$. Note that,

$$(9) \qquad\qquad 4n \quad < \quad \dfrac{n^2}{2} - n + 1.$$

From inequality (9) and Proposition 5, we have

$$\lambda_1(G) \quad \leq \quad \sqrt{n}$$
$$< \quad \dfrac{1}{2}\sqrt{\dfrac{n^2}{2} - n + 1}.$$

By Proposition 4, we obtain the desired inequality.

□

For a non-negative integer k, a connected graph of order n and size m is $k-$cyclic if $m = n + k$. For $k = 0$, the result folllows from Proposition 6. For each $k \geq 1$, we give the proof in Proposition 7 under some condition on n.

Proposition 7. *Let k be a positive integer. If G is a $k-$cyclic graph of order $n \geq 5 + \sqrt{16k + 31}$, then*

$$\lambda_2(G) + \lambda_2(\overline{G}) \leq -1 + \sqrt{\frac{n^2}{2} - n + 1}.$$

Proof. From Hong [**12**],

$$(10) \qquad\qquad \lambda_1(G) \leq \sqrt{n + 2k + 1}.$$

For each $n \geq 5 + \sqrt{16k + 31}$, we get

$$4(n + 2k + 1) = \frac{n^2}{2} - n + 1 + (-\frac{n^2}{2} + 5n + 8k + 3)$$

$$(11) \qquad\qquad \leq \frac{n^2}{2} - n + 1.$$

Using equations (10) and (11),

$$\lambda_1(G) \leq \frac{1}{2}\sqrt{\frac{n^2}{2} - n + 1},$$

for all $n \geq 5 + \sqrt{16k + 31}$.

From Proposition 4, the result follows.

$\square$

This section is ended by two results that will be useful to prove our main results.

Proposition 8 ([**16**]). *For a graph G of order n and girth at least 5,*

$$\lambda_1(G) \leq \min\{\Delta, \sqrt{n - 1}\}.$$

Proposition 9 ([**24**]). *For $n \geq 2$, if $G \not\cong K_n$ is a graph of order n then $\lambda_2(G) \geq 0$. The equality holds if and only if G is a complete multipartite graph.*

3. MAIN RESULTS

For a natural number $p, 1 \leq p \leq n$, a graph G on n vertices is a *complete split graph*, denoted by $CS(n, p)$, if it can be partitioned into an independent set of p vertices and a clique of $n - p$ vertices such that every vertex in the independent set is connected to every vertex of the clique. Notice that $CS(n, p)$ is isomorphic to

the complete multipartite graph $K_{s_1,s_2,\cdots,s_{n-p+1}}$ such that $s_i = 1$ for $1 \le i \le n - p$ and $s_{n-p+1} = p$.

Next, we prove that the complete graph on n vertices is the only one graph which attains $\lambda_2(K_n) + \lambda_2(\overline{K_n}) = -1$. So, assuming that G is not isomorphic to K_n, it turns out that the lower bound to $\lambda_2(G) + \lambda_2(\overline{G})$ is obtained for a complete split graph $CS(n,p)$.

Proposition 10. *Let G be a graph with order $n \ge 3$ with at least one edge. Then, one of the following statements holds,*

(i) *If $G \not\simeq K_n$ then $\lambda_2(G) + \lambda_2(\overline{G}) \ge 0$. Moreover, if G has no isolated vertices, $\lambda_2(G) + \lambda_2(\overline{G}) = 0$ if and only if G is a complete split graph;*

(ii) *$\lambda_2(G) + \lambda_2(\overline{G}) = -1$ if and only if $G \simeq K_n$;*

(iii) *There is no graph G such that $\lambda_2(G) + \lambda_2(\overline{G}) \in (-1, 0)$.*

Proof. Let G be a graph of order $n \ge 3$ with at least one edge.

(i) Let $G \not\simeq K_n$. By Proposition 9, $\lambda_2(G) \ge 0$. Besides, since $G \not\simeq nK_1$ and $\overline{G} \not\simeq K_n$ then
$$\lambda_2(G) + \lambda_2(\overline{G}) \ge 0.$$

For an integer $p \ge 2$, let $G \simeq CS(n,p)$. Since $CS(n,p) \simeq K_{1,\ldots,1,p}$, from Proposition 9, we get $\lambda_2(G) = 0$. It is known that $P_{\overline{G}}(x) = x^{n-p}(x+1)^{p-1}(x - p + 1)$. Then $\lambda_2(\overline{G}) = 0$ and so, $\lambda_2(G) + \lambda_2(\overline{G}) = 0$.

Now suppose that $\lambda_2(G) + \lambda_2(\overline{G}) = 0$. Hence, $G \not\simeq K_n$ and, from Proposition 9, $\lambda_2(G) = \lambda_2(\overline{G}) = 0$. Also, $G \simeq K_{p_1,\ldots,p_k}$, for positive integers $1 \le p_1 \le \cdots \le p_k$ such that $1 < p_k$.

Consequently, $\overline{G} \simeq K_{p_1} \cup \cdots \cup K_{p_k}$ and $\lambda_2(\overline{G}) = p_{k-1} - 1$. However, $\lambda_2(\overline{G}) = 0$. Then, $p_{k-1} = 1$ and so, $p_1 = p_2 = \cdots = p_{k-1} = 1$.

(ii) If $G \simeq K_n$, $\lambda_2(G) = -1$ and $\lambda_2(\overline{G}) = 0$. Reciprocally, if $\lambda_2(G) + \lambda_2(\overline{G}) = -1$ then $\lambda_2(G) < 0$ or $\lambda_2(\overline{G}) < 0$. From Proposition 9, we have $G \simeq K_n$.

(iii) The proof of this case follows straightforward from (i) and (ii).

$\square$

Proposition 11. *Let G be a graph of order n and girth at least 5. Then,*

$$\lambda_2(G) + \lambda_2(\overline{G}) \le -1 + \sqrt{\frac{n^2}{2} - n + 1}.$$

Proof. The proof will be done in two cases.

Case 1: Let G be a graph of order n and girth g such that $n, g \in [5, 8]$.

Under such conditions, there exist 26 graphs and all of them are unicyclic (see in [**21**]). From Proposition 6, we get

$$\lambda_2(G) + \lambda_2(\overline{G}) \leq -1 + \sqrt{\frac{n^2}{2} - n + 1}.$$

Case 2: Let G be a graph of order $n \geq 9$ and girth $g \geq 5$. Note that

$$
\begin{aligned}
4(n-1) &= \left(\frac{n^2}{2} - n + 1\right) - \left(\frac{n^2}{2} - 5n + 5\right) \\
&= \frac{n^2}{2} - n + 1 - \frac{1}{2}(n - 5 + \sqrt{15})(n - 5 - \sqrt{15}) \\
&< \frac{n^2}{2} - n + 1.
\end{aligned}
$$

(12)

Theorem 8 and inequality (12) give us

$$
\begin{aligned}
\lambda_1(G) &\leq \sqrt{n-1} \\
&< \frac{1}{2}\sqrt{\frac{n^2}{2} - n + 1}.
\end{aligned}
$$

From Proposition 4,

$$\lambda_2(G) + \lambda_2(\overline{G}) \leq -1 + \sqrt{\frac{n^2}{2} - n + 1}.$$

$\square$

For regular graphs, it seems that NG_2-bound is true and some insights from [**7**] can be useful in order to prove it. Here, we prove that inequality (3) holds for every connected regular bipartite graph.

Proposition 12. *If G is a connected r-regular bipartite graph of order n, then*

$$\lambda_2(G) + \lambda_2(\overline{G}) \leq -1 + \sqrt{\frac{n^2}{2} - n + 1}.$$

Proof. For a r-regular graph G, it is known [see Theorem 3.1 of [**26**]] that

(13) $$\lambda_2(G) \leq \frac{n}{2} - r.$$

From Theorem 1 and since G is a r-regular bipartite graph,

$$\lambda_2(\overline{G}) = -1 - \lambda_n(G) = r - 1.$$

However, from (13),

(14) $$\lambda_2(G) + \lambda_2(\overline{G}) \leq -1 + \frac{n}{2}.$$

Notice that for $n \in \mathbb{R}$,

$$(15) \qquad \frac{n}{2} \leq \sqrt{\frac{n^2}{2} - n + 1}.$$

Hence, from inequalities (14) and (15), the proof is completed. $\square$

Graphs with the property $\lambda_2 \leq 1$ are investigated in several papers and they are completely described within certain families (see for exemple [5], [8], [9], [14], [15], [20] and [25]). The next result shows that all graphs with the mentioned property satisfy NG_2-bound.

Proposition 13. *Let G be a graph of order $n \geq 2$. If $\min\left\{\lambda_2(G), \lambda_2(\overline{G})\right\} \leq 1$ then*

$$\lambda_2(G) + \lambda_2(\overline{G}) \leq -1 + \sqrt{\frac{n^2}{2} - n + 1}.$$

Proof. According to [21], the inequality is satisfied for every graph of order $2 \leq n \leq 7$. Suppose that G is a graph with $n \geq 8$ vertices. In this case,

$$(16) \qquad \left(\frac{n}{2} + 1\right)^2 \leq \frac{n^2}{2} - n + 1.$$

Let $\lambda_2(G) \leq 1$. From (16) and $\lambda_2(G) \leq n/2 - 1$ [see Theorem 2 of [11]],

$$\begin{aligned} \lambda_2(G) + \lambda_2(\overline{G}) \quad &\leq \quad -1 + \left(\frac{n}{2} + 1\right) \\ &\leq \quad -1 + \sqrt{\frac{n^2}{2} - n + 1} \end{aligned}$$

and, the proof is completed.

$\square$

Generalized line graphs and exceptional graphs were well studied in [6] and satisfies $\lambda_n \geq -2$. The next result shows that such graphs satisfy NG_2-bound.

Proposition 14. *Let G be a graph of order n such that $\lambda_n(G) \geq -2$. Then,*

$$\lambda_2(G) + \lambda_2(\overline{G}) \leq -1 + \sqrt{\frac{n^2}{2} - n + 1}.$$

Proof. Let G a graph under the conditions above. We have, $\lambda_2(\overline{G}) \leq 1$ [see Theorem 3 in [5]] and, the result follows directly from Proposition 13.

$\square$

4. A CLASS OF EXTREMAL GRAPHS TO THE NG_2-BOUND

In this section, we show an infinite family of graphs of girth 3 that satisfies inequality (3). Such graphs were studied by Nikiforov in [17], but here we will define it using the H-join operation. Let H be a graph with vertex set $V(H) = \{v_i; 1 \le i \le k\}$. Let $\mathcal{F} = \{G_i; 1 \le i \le k\}$ be a family of graphs G_i of order n_i. For each $v_i \in V(H), G_i \in \mathcal{F}$ is assigned to v_i. The H-join of a graph G, denoted by $G = H[G_1, G_2, \ldots, G_k]$, in $\mathcal{F}$ has vertex set

$$V(G) = \left(\bigcup_{i=1}^{k} V(G_i) \right)$$

and edge set

$$E(G) = \left(\bigcup_{i=1}^{k} E(G_i) \right) \bigcup \left(\bigcup_{v_i v_j \in E(H)} \{uw : u \in V(G_i), w \in V(G_j)\} \right).$$

If $H \simeq P_4$, $p, q \ge 1$ and $n = 2(p+q)$, then

$$H_{p,q,q,p} = P_4[K_p, \overline{K_q}, \overline{K_q}, K_p].$$

For $n = 2(p+q) + 1$,

$$H_{p,q,q,p+1} = P_4[K_p, \overline{K_q}, \overline{K_q}, K_{p+1}].$$

Let us define the following family of graphs:

$$\mathcal{H}(P_4) = \{H_{p,q,q,p} , H_{p,q,q,p+1} \; ; \; p, q \ge 1\}.$$

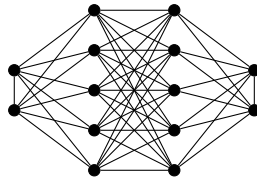Figure 1 displays the $H-$join graph given by $H_{2,5,5,2} \in \mathcal{H}(P_4)$.

Figure 1: $H_{2,5,5,2} \in \mathcal{H}(P_4)$.

It is easy to see that the complement operation is closed in the family $\mathcal{H}(P_4)$ if and only if n is even, that is,

$$(17) \qquad \overline{H_{p,q,q,p}} = \overline{P_4(K_p, \overline{K_q}, \overline{K_q}, K_p)} = P_4(K_q, \overline{K_p}, \overline{K_p}, K_q) = H_{q,p,p,q} \in \mathcal{H}(P_4)$$

and,

$$\overline{H_{p,q,q,p+1}} = \overline{P_4(K_p, \overline{K_q}, \overline{K_q}, K_{p+1})} = P_4(K_q, \overline{K_p}, \overline{K_{p+1}}, K_q) \notin \mathcal{H}(P_4).$$

Let $1 \le p \le q$ be integer numbers such that $n = 2(p+q)$ and let $r = p-q-1$, $s = q(q + 2p + 2) + (p - 1)^2$ and $t = q(q + 6p - 2) + (p - 1)^2$. From Cardoso et al. [2], the spectrum of $H_{p,q,q,p}$ is as follows

(18)
$$Spec(H_{p,q,q,p}) = \left\{ \frac{r - \sqrt{t}}{2}, \frac{r + 2q - \sqrt{s}}{2}, -1^{(2p-2)}, 0^{(2q-2)}, \frac{r + \sqrt{t}}{2}, \frac{r + 2q + \sqrt{s}}{2} \right\}.$$

Immediately from (17) and (18), we obtain Proposition 15.

Proposition 15. *Let p and q natural numbers such that $1 \le p \le q$. Then,*

$$\lambda_2(\overline{H_{p,q,q,p}}) = \lambda_2(H_{p,q,q,p}) + q - p.$$

In the next proposition, we prove that for each $n \equiv 0 \pmod 4$, $H_{\frac{n}{4},\frac{n}{4},\frac{n}{4},\frac{n}{4}}$ is a P_4-join graph which is extremal to NG_2-bound.

Proposition 16. *If $G \simeq H_{p,q,q,p}$, then*

$$\lambda_2(H_{p,q,q,p}) + \lambda_2(\overline{H_{p,q,q,p}}) = -1 + \sqrt{(q + 6p - 2)q + (p - 1)^2}.$$

Besides, the maximum value for the sum is attained if and only if $p = \left\lfloor \dfrac{n}{4} \right\rfloor$ and $q = \left\lceil \dfrac{n}{4} \right\rceil$.

Proof. Consider the spectrum of $H_{p,q,q,p}$ in (18) and Proposition 15. If $n = 2(p+q)$,

$$\lambda_2(G) + \lambda_2(\overline{G}) \;=\; -1 + \frac{\sqrt{-16p^2 + 8np + (n - 2)^2}}{2}.$$

For $\alpha = \sqrt{2}\sqrt{(n - 1)^2 + 1} > 0$, define $h : \left[\dfrac{n - \alpha}{4}, \dfrac{n + \alpha}{4} \right] \to \mathbb{R}$ such that $h(x) = -1 + \dfrac{\sqrt{-16x^2 + 8nx + (n - 2)^2}}{2}.$

As the function h is continuous in the interval $\left[\dfrac{n - \alpha}{4}, \dfrac{n + \alpha}{4} \right]$ and differentiable in $\left(\dfrac{n - \alpha}{4}, \dfrac{n + \alpha}{4} \right)$, h admits a maximum value at $x = \dfrac{n}{4}$. Besides as p is a positive number, $\lambda_2(G) + \lambda_2(\overline{G})$ reaches the maximum value for $p = \left\lfloor \dfrac{n}{4} \right\rfloor$. Consequentely, $q = \left\lceil \dfrac{n}{4} \right\rceil$.

□

5. CONCLUSION

By Harary [**10**], if G is a graph of order $n \geq 6$ then G or $\overline{G}$ has girth 3. In the light of this result, the Propositions 3 and 11 enable us to conclude that there exist many graphs of girth 3 for which the NG_2-bound is verified. These facts, in addition to the computational experiments performed for all graphs up to 8 vertices, lead us to conjecture the following result:

Conjecture 17. *Let G be a graph on n vertices. Then,*

$$\lambda_2(G) + \lambda_2(\overline{G}) \leq -1 + \sqrt{\frac{n^2}{2} - n + 1},$$

with equality if and only if $G \simeq H_{\frac{n}{4}, \frac{n}{4}, \frac{n}{4}, \frac{n}{4}}$, for $n \equiv 0 \pmod 4$.

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