

SOME GRAPH MAPPINGS THAT PRESERVE THE SIGN OF $\lambda_2 - r$

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In honour of Dragoš Cvetković on the occasion of his 75th birthday.

In this article we deal with the sign of $\lambda_2 - r$, $r > 0$, where λ_2 is the second largest eigenvalue of (adjacency matrix of) a simple graph and present some methods of determining it for some classes of graphs. The main result is a set of graph mappings that preserve the value of $\text{sgn}(\lambda_2 - r)$. These mappings induce equivalence relations among involved graphs, thus providing a way to indirectly apply the GRS-theorem (the generalization of so-called RS-theorem) to some GRS-undecidable (or RS-undecidable) graphs. To present possible applications, we revisit some of the previous results for reflexive graphs (graphs whose second largest eigenvalue does not exceed 2). We show how maximal reflexive graphs that belong to various families depending on their cyclic structure, can be reduced to RS-decidable graphs in terms of corresponding equivalence relations.

1. INTRODUCTION

The characteristic polynomial of a simple graph G on n vertices is defined by $P_G(\lambda) = \det(\lambda I - A)$, where A is $(0, 1)$ -adjacency matrix of G . Roots of the characteristic polynomial are the eigenvalues of G . Since A is a symmetric matrix, the eigenvalues are all real numbers and we assume their non-increasing order, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. The family of eigenvalues of G forms the spectrum of G [3]. The largest eigenvalue λ_1 is called the index of G and, if G is connected, then $\lambda_1 > \lambda_2$.

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The Interlacing Theorem. [3] Let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ be the eigenvalues of a simple graph G and $\mu_1 \geq \mu_2 \geq \dots \geq \mu_m$ the eigenvalues of its subgraph H . Then the inequalities $\lambda_{n-m+i} \leq \mu_i \leq \lambda_i$ ($i = 1, 2, \dots, m$) hold.

The second largest eigenvalue of a graph plays an important role both in theoretical investigations and in many applications. It is related to structural properties such as diameter and connectivity (cf. [5]). By the Interlacing theorem, the property $\lambda_2 \leq r$ is hereditary, i.e. if the second largest eigenvalue of a graph does not exceed r , then the same holds for any of its subgraphs.

So far graphs whose second largest eigenvalue does not exceed a constant $c > 0$ have been studied by various authors, and considered bounds include: $c = 1/3$ [1], $c = \sqrt{2} - 1$ [13], $c = (\sqrt{5} - 1)/2$ [4, 23], $c = 1$ [2, 11], $c = \sqrt{2}$ [7, 25], $c = (\sqrt{5} + 1)/2$ [9], $c = \sqrt{3}$ [7].

Graphs with the property $\lambda_2 \leq 2$ are called reflexive and they represent Lorentzian counterparts of the spherical and Euclidean graphs occurring in the theory of reflection groups [12]. Many classes of reflexive graphs have been a subject of investigations. Reflexive trees have been studied in [8, 11], bicyclic reflexive graphs with a bridge between the cycles in [19], see also [5, 14]. Various classes of multicyclic reflexive graphs have been determined in [10, 15, 16, 17, 18, 20, 21]. Reflexive bipartite regular graphs have been considered in [6].

Here, we present some new methods to establish whether $\lambda_2 < r$, $\lambda_2 > r$, or $\lambda_2 = r$, ($r > 0$) for some classes of graphs. First, in Section 2, we remind of crucial tools for further work, including the so-called GRS-theorem (the generalization of RS-theorem, which arose in the context of reflexive graphs) and the notions of GRS-decidable and GRS-undecidable graphs. In Sections 3 and 4 we present certain graph transformations that preserve or partially preserve the value of $\text{sgn}(\lambda_2 - r)$. Using these transformations we establish equivalence relations among the corresponding graphs. In Section 5 we present some applications. The aforementioned equivalence relations allow us to indirectly apply the GRS-theorem to some GRS-undecidable graphs by using a GRS-decidable representative of the corresponding equivalence class. To illustrate some of the possible applications, we show how maximal reflexive graphs (i.e. those that do not admit reflexive extensions) belonging to various families depending on their cyclic structure, determined in previous work [17, 18, 19], can be related to RS-decidable graphs with regard to the corresponding equivalence relations.

2. SOME FORMER AND AUXILIARY RESULTS

Throughout the paper we use the value $P_G(r)$, where P_G is the characteristic polynomial of G , to determine whether the second largest eigenvalue $\lambda_2(G)$ exceeds r . Therefore, using Schwenks Lemma with its corollaries, along with the Interlacing theorem, we locate $\lambda_2(G)$ with respect to r .

Schwenk's Lemma. ([22]) Given a graph G , let $C(v)$ and $C(uv)$ denote the set

of all cycles containing a vertex v and an edge uv of G , respectively. Then,

$$(1) \quad P_G(\lambda) = \lambda P_{G-v}(\lambda) - \sum_{u \in \text{Adj}(v)} P_{G-v-u}(\lambda) - 2 \sum_{C \in C(v)} P_{G-V(C)}(\lambda),$$

$$(2) \quad P_G(\lambda) = P_{G-uv}(\lambda) - P_{G-v-u}(\lambda) - 2 \sum_{C \in C(uv)} P_{G-V(C)}(\lambda),$$

where $\text{Adj}(v)$ denotes the set of neighbors of v , while $G - V(C)$ is the graph obtained from G by removing the vertices belonging to the cycle C .

Corollary 1 (Heilbronner [3, Theorem 2.12, p. 59]). Let G be a graph obtained by joining a vertex v_1 of a graph G_1 to a vertex v_2 of a graph G_2 by an edge. Let G'_1 and G'_2 be the subgraphs of G_1 and G_2 , respectively, obtained by deleting the vertices v_1 and v_2 from G_1 and G_2 . Then,

$$P_G(\lambda) = P_{G_1}(\lambda)P_{G_2}(\lambda) - P_{G'_1}(\lambda)P_{G'_2}(\lambda).$$

Corollary 2 (Heilbronner [3, Theorem 2.11, p. 59]). Let G be a graph with a pendent edge v_1v_2 , v_1 being of degree 1. Then $P_G(\lambda) = \lambda P_{G_1}(\lambda) - P_{G_2}(\lambda)$, where G_1 (G_2) are the graphs obtained from G (resp. G_1), by deleting the vertex v_1 (v_2).

Now, let us mention an important theorem. Its initial form is called the RS-theorem (by Radosavljević and Simić [19]), and it refers to reflexive graphs, while its generalization is called the GRS-theorem by Mihailović [9]. This theorem gives answers about the second largest eigenvalue of a graph with a cut-vertex. More precisely, it tells whether $\lambda_2(G) < r$, $\lambda_2(G) = r$ or $\lambda_2(G) > r$ depending on the indices of the (connected) components arising after the removal of the cut-vertex. The RS-theorem is the special case of the GRS-theorem for $r = 2$. There are cases in which this theorem does not give the answer about the sign of $\lambda_2(G) - r$. We call corresponding graphs GRS-undecidable (or RS-undecidable), otherwise graphs are GRS-decidable (or RS-decidable). This theorem is important for graphs with a cut-vertex because in that case it is stronger than the Interlacing theorem.

GRS-Theorem. [9] Let G be a graph with a cut-vertex u . Let G_1, G_2, \dots, G_k be the (connected) components of the graph $G - u$. Then:

- (1) If $\lambda_2(G_i) > r$ for at least one of the components G_1, G_2, \dots, G_k ; or if at least two of the components have indices greater than r ; or if only one of the components has index greater than r and at least one of the remaining graphs has index r , then $\lambda_2(G) > r$.
- (2) If at least two of the graphs G_1, G_2, \dots, G_k have indices r , and for the rest of them the indices do not exceed r , then $\lambda_2(G) = r$.
- (3) If at most one of the graphs G_1, G_2, \dots, G_k has index r , and for the rest of them the indices are less than r , then $\lambda_2(G) < r$.

3. MAPPINGS THAT PRESERVE THE SIGN OF $\lambda_2 - r$

Let A, B, X and Y be rooted graphs with roots a, b, x and y , respectively, including cases of graphs with only one vertex (a, b, x and y , respectively). Let us introduce the following notation: $\bar{A} = A - a$, $\bar{B} = B - b$, $\bar{X} = X - x$ and $\bar{Y} = Y - y$, and let us denote by $A \cdot B$ the coalescence of graphs A and B formed by identifying the vertices a and b . Now, let us form some new graphs by interconnecting graphs A, B, X and Y . Let us denote by $V(A, B, X)$ the graph constructed from graphs A, B and X by adding bridges ax and bx (as in Fig.1(a)).

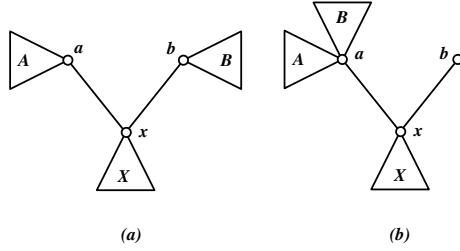


Figure 1:

If x_1 and x_2 are two vertices of X , then by $U(A, B, X)$ we denote the graph constructed from graphs A, B and X by adding bridges ax_1 and bx_2 (Fig.2(a)).

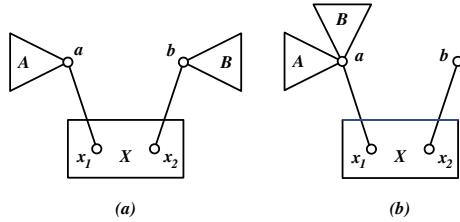


Figure 2:

By adding three new edges ax , bx and ab to A, B and X we get $T(A, B, X)$ (Fig.3(a)). By adding four edges ax , bx , ay and by to A, B, X and Y we get $Q(A, B, X, Y)$ (Fig.4(a)).

Corresponding families of graphs of types $V(A, B, X)$, $U(A, B, X)$, $T(A, B, X)$ and $Q(A, B, X, Y)$ we label **V**, **U**, **T** and **Q**, respectively. Now, let us define four mappings:

- (1) $\alpha : \mathbf{V} \rightarrow \mathbf{V}$, $\alpha(V(A, B, X)) = V(A \cdot B, b, X)$, Fig.1,
- (2) $\beta : \mathbf{U} \rightarrow \mathbf{U}$, $\beta(U(A, B, X)) = U(A \cdot B, b, X)$, Fig.2,
- (3) $\gamma : \mathbf{T} \rightarrow \mathbf{T}$, $\gamma(T(A, B, X)) = T(A \cdot B, b, X)$, Fig.3,
- (4) $\delta : \mathbf{Q} \rightarrow \mathbf{Q}$, $\delta(Q(A, B, X, Y)) = Q(A \cdot B, b, X, Y)$, Fig.4.

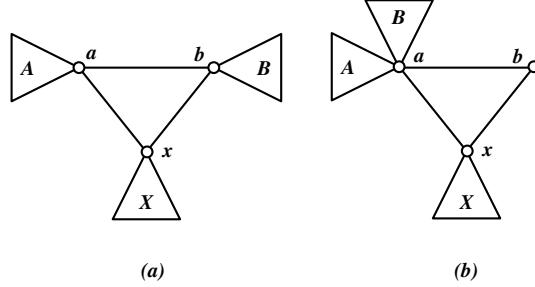


Figure 3:

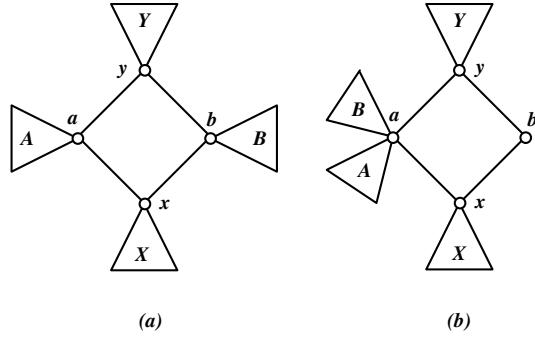


Figure 4:

Now, we denote by r the index of X , i.e. $\lambda_1(X) = r$, and, additionally, we assume the condition $P_{X-x_1}(r) = P_{X-x_2}(r)$, for graphs from \mathbf{U} . In order to prove that these mappings preserve the sign of $\lambda_2 - r$, first we prove the following two lemmas.

Lemma 1. Let G be a graph from \mathbf{V} , \mathbf{U} or \mathbf{T} . If $\lambda_1(\bar{A}) < r$ and $\lambda_1(\bar{B}) < r$ hold, then $\lambda_3(G) < r < \lambda_1(G)$.

Proof: It is obvious that $\lambda_1(G) > r$ holds. By GRS-theorem applied to the vertex b of the graph $G - A$ it follows $\lambda_2(G - A) < r$, hence $\lambda_2(G - a) < r$, and therefore $\lambda_3(G) < r$, by the Interlacing theorem. \square

Lemma 2. Let G be a graph from \mathbf{Q} . If $\lambda_1(\bar{A}) < r$, $\lambda_1(\bar{B}) < r$ and $\lambda_1(Y) \leq r$ holds, then $\lambda_4(G) < r < \lambda_1(G)$.

Proof: Again, $\lambda_1(G) > r$ holds. By Lemma 1 $\lambda_3(G - Y) < r$ holds, hence $\lambda_3(G - y) < r$, and therefore $\lambda_4(G) < r$, by the Interlacing theorem. \square

Note that if G is a graph such that $\lambda_3(G) < r < \lambda_1(G)$ holds, then $\text{sgn}(\lambda_2(G) - r) = \text{sgn}(P_G(r))$. Therefore, in this case, by calculating $P_G(r)$ we can determine whether $\lambda_2(G)$ is greater than, equal to, or less than r .

Theorem 1. Let G be a graph from **V**, **U**, **T** or **Q**, and let $\lambda_1(X) = r$. Then mappings $\alpha, \beta, \gamma, \delta$ preserve the sign of $\lambda_2(G) - r$.

Proof: In all four cases, if $\lambda_1(\bar{A}) \geq r$ or $\lambda_1(\bar{B}) \geq r$, it follows $\text{sgn}(\lambda_2(G) - r) = 1$, and analogously $\text{sgn}(\lambda_2(\alpha(G)) - r) = \text{sgn}(\lambda_2(\beta(G)) - r) = \text{sgn}(\lambda_2(\gamma(G)) - r) = \text{sgn}(\lambda_2(\delta(G)) - r) = 1$ so we further assume that $\lambda_2(\bar{A}), \lambda_2(\bar{B}) < r$.

(1) For $G \in \mathbf{V}$, applying Heilbronner's lemma, we get

$$P_G(\lambda) = P_X(\lambda)P_A(\lambda)P_B(\lambda) - P_{\bar{X}}(\lambda)(P_{\bar{A}}(\lambda)P_B(\lambda) + P_A(\lambda)P_{\bar{B}}(\lambda))$$

and for $\lambda = r$, it follows $P_G(r) = -P_{\bar{X}}(r)(P_{\bar{A}}(r)P_B(r) + P_A(r)P_{\bar{B}}(r))$.

Using Schwenk's and Heilbronner's lemmas we get

$$\begin{aligned} P_{\alpha(G)}(\lambda) &= (P_{\bar{A}}(\lambda)P_B(\lambda) + P_A(\lambda)P_{\bar{B}}(\lambda) - \lambda P_{\bar{A}}(\lambda)P_{\bar{B}}(\lambda))(\lambda P_X(\lambda) - P_{\bar{X}}(\lambda)) - \\ &\quad - \lambda P_{\bar{X}}(\lambda)P_{\bar{A}}(\lambda)P_{\bar{B}}(\lambda) \\ &= \lambda P_X(\lambda)(P_{\bar{A}}(\lambda)P_B(\lambda) + P_A(\lambda)P_{\bar{B}}(\lambda) - \lambda P_{\bar{A}}(\lambda)P_{\bar{B}}(\lambda)) - \\ &\quad - P_{\bar{X}}(\lambda)(P_{\bar{A}}(\lambda)P_B(\lambda) + P_A(\lambda)P_{\bar{B}}(\lambda)), \end{aligned}$$

hence, $P_{\alpha(G)}(r) = -P_{\bar{X}}(r)(P_{\bar{A}}(r)P_B(r) + P_A(r)P_{\bar{B}}(r))$. Therefore, $P_G(r) = P_{\alpha(G)}(r)$, which means that $\text{sgn}(\lambda_2(G) - r) = \text{sgn}(\lambda_2(\alpha(G)) - r)$.

(2) For $G \in \mathbf{U}$, by Heilbronner's lemma we get

$$P_G(\lambda) = (P_X(\lambda)P_A(\lambda) - P_{X-x_1}(\lambda)P_{\bar{A}}(\lambda))P_B(\lambda) - \\ - P_{\bar{B}}(\lambda)(P_A(\lambda)P_{X-x_2}(\lambda) - P_{\bar{A}}(\lambda)P_{X-x_1-x_2}(\lambda))$$

and for $\lambda = r$, it follows

$$P_G(r) = -P_{X-x_1}(r)(P_{\bar{A}}(r)P_B(r) + P_A(r)P_{\bar{B}}(r)) + P_{X-x_1-x_2}(r)P_{\bar{A}}(r)P_{\bar{B}}(r),$$

since $P_{X-x_1}(r) = P_{X-x_2}(r)$.

By Schwenk's and Heilbronner's lemma we also get

$$P_{\beta(G)}(\lambda) = (P_{\bar{A}}(\lambda)P_B(\lambda) + P_A(\lambda)P_{\bar{B}}(\lambda) - \lambda P_{\bar{A}}(\lambda)P_{\bar{B}}(\lambda))(\lambda P_X(\lambda) - P_{X-x_2}(\lambda)) - \\ - P_{\bar{A}}(\lambda)P_{\bar{B}}(\lambda)(\lambda P_{X-x_1}(\lambda) - P_{X-x_1-x_2}(\lambda)),$$

and for $\lambda = r$ it follows

$$P_{\beta(G)}(r) = -P_{X-x_1}(r)(P_{\bar{A}}(r)P_B(r) + P_A(r)P_{\bar{B}}(r)) + P_{X-x_1-x_2}(r)P_{\bar{A}}(r)P_{\bar{B}}(r),$$

since $P_{X-x_1}(r) = P_{X-x_2}(r)$. Therefore, $P_G(r) = P_{\beta(G)}(r)$, which means that $\text{sgn}(\lambda_2(G) - r) = \text{sgn}(\lambda_2(\beta(G)) - r)$.

(3) For $G \in \mathbf{T}$, by Heilbronner's lemma we get

$$\begin{aligned} P_G(\lambda) &= P_X(\lambda)P_{G-X}(\lambda) + P_{\bar{X}}(\lambda)(\lambda P_{G-X}(\lambda) - P_{\bar{A}}(\lambda)P_B(\lambda) - \\ &\quad - P_A(\lambda)P_{\bar{B}}(\lambda) - 2P_{\bar{A}}(\lambda)P_{\bar{B}}(\lambda)) - \lambda P_{\bar{X}}(\lambda)P_{G-X}(\lambda) = \\ &= P_X(\lambda)P_{G-X}(\lambda) - P_{\bar{X}}(\lambda)(P_{\bar{A}}(\lambda)P_B(\lambda) + P_A(\lambda)P_{\bar{B}}(\lambda) + 2P_{\bar{A}}(\lambda)P_{\bar{B}}(\lambda)), \end{aligned}$$

and for $\lambda = r$, it follows $P_G(r) = -P_{\bar{X}}(r)(P_{\bar{A}}(r)P_B(r) + P_A(r)P_{\bar{B}}(r) + 2P_{\bar{A}}(r)P_{\bar{B}}(r))$. By Schwenk's and Heilbronner's lemmas we get:

$$\begin{aligned} P_{\gamma(G)}(\lambda) &= P_X(\lambda)P_{G-X}(\lambda) + P_{\bar{X}}(\lambda)(\lambda P_{\gamma(G)-X}(\lambda) - \lambda P_{\bar{A}}(\lambda)P_{\bar{B}}(\lambda) - P_{\bar{A}}(\lambda)P_B(\lambda) - \\ &\quad - P_A(\lambda)P_{\bar{B}}(\lambda) + \lambda P_{\bar{A}}(\lambda)P_{\bar{B}}(\lambda) - 2P_{\bar{A}}(\lambda)P_{\bar{B}}(\lambda)) - \lambda P_{\bar{X}}(\lambda)P_{\gamma(G)-X}(\lambda) = \\ &= P_X(\lambda)P_{G-X}(\lambda) - P_{\bar{X}}(\lambda)(P_{\bar{A}}(\lambda)P_B(\lambda) + P_A(\lambda)P_{\bar{B}}(\lambda) + 2P_{\bar{A}}(\lambda)P_{\bar{B}}(\lambda)), \end{aligned}$$

and for $\lambda = r$, it follows

$$P_{\gamma(G)}(r) = -P_{\bar{X}}(r)(P_{\bar{A}}(r)P_B(r) + P_A(r)P_{\bar{B}}(r) + 2P_{\bar{A}}(r)P_{\bar{B}}(r)).$$

Therefore, $P_G(r) = P_{\gamma(G)}(r)$, which means that $\operatorname{sgn}(\lambda_2(G) - r) = \operatorname{sgn}(\lambda_2(\gamma(G)) - r)$.

(4) For $G \in \mathbf{Q}$, using Schwenks and Heilbronner's lemmas, we get

$$\begin{aligned} P_G(\lambda) &= P_A(\lambda)P_B(\lambda)P_Y(\lambda)P_X(\lambda) - \\ &\quad - (P_{\bar{Y}}(\lambda)P_X(\lambda) + P_Y(\lambda)P_{\bar{X}}(\lambda))(P_{\bar{A}}(\lambda)P_B(\lambda) + P_A(\lambda)P_{\bar{B}}(\lambda)), \\ P_{\delta(G)}(\lambda) &= \lambda(P_{\bar{A}}(\lambda)P_B(\lambda) + P_A(\lambda)P_{\bar{B}}(\lambda) - \lambda P_{\bar{A}}(\lambda)P_{\bar{B}}(\lambda))P_Y(\lambda)P_X(\lambda) - \\ &\quad - (P_{\bar{Y}}(\lambda)P_X(\lambda) + P_Y(\lambda)P_{\bar{X}}(\lambda))(P_{\bar{A}}(\lambda)P_B(\lambda) + P_A(\lambda)P_{\bar{B}}(\lambda)), \end{aligned}$$

hence, for $\lambda = r$, it follows

$$P_G(r) = P_{\delta(G)}(r) = -P_Y(r)P_{\bar{X}}(r)(P_{\bar{A}}(r)P_B(r) + P_A(r)P_{\bar{B}}(r)).$$

Let $\lambda_1(Y) < r$ and let us introduce the notation $H_1 = G - B$ and $H_2 = \delta(G) - b$. Applying GRS-theorem to the vertex a of H_1 and H_2 , we get $\lambda_2(H_i) < r$ ($i = 1, 2$); hence $\lambda_2(G - b) < r$, and finally, $\lambda_3(G), \lambda_3(\delta(G)) < r$. Since $P_G(r) = P_{\delta(G)}(r)$, it follows that $\operatorname{sgn}(\lambda_2(G) - r) = \operatorname{sgn}(\lambda_2(\delta(G)) - r)$.

Let $\lambda_1(Y) = r$. As we know, by Lemma 2, that $\lambda_4(G) < r < \lambda_1(G)$ (i.e. $\lambda_4(\delta(G)) < r < \lambda_1(\delta(G))$), it is obvious that $\lambda_2(G) = r$ or $\lambda_3(G) = r$ (i.e. $\lambda_2(\delta(G)) = r$ or $\lambda_3(\delta(G)) = r$). Let us introduce the notation: $G - X = F_1$, $\delta(G) - X = F_2$, $G - Y = K_1$ and $\delta(G) - Y = K_2$. First, by (1) we get the characteristic polynomials of these subgraphs:

$$\begin{aligned} P_{F_1}(\lambda) &= P_Y(\lambda)P_A(\lambda)P_B(\lambda) - P_{\bar{Y}}(\lambda)(P_{\bar{A}}(\lambda)P_B(\lambda) + P_A(\lambda)P_{\bar{B}}(\lambda)), \\ P_{F_2}(\lambda) &= \lambda P_Y(\lambda)(P_{\bar{A}}(\lambda)P_B(\lambda) + P_A(\lambda)P_{\bar{B}}(\lambda) - \lambda P_{\bar{A}}(\lambda)P_{\bar{B}}(\lambda)) - \\ &\quad - P_{\bar{Y}}(\lambda)(P_{\bar{A}}(\lambda)P_B(\lambda) + P_A(\lambda)P_{\bar{B}}(\lambda)), \\ P_{K_1}(\lambda) &= P_X(\lambda)P_A(\lambda)P_B(\lambda) - P_{\bar{X}}(\lambda)(P_{\bar{A}}(\lambda)P_B(\lambda) + P_A(\lambda)P_{\bar{B}}(\lambda)), \\ P_{K_2}(\lambda) &= \lambda P_X(\lambda)(P_{\bar{A}}(\lambda)P_B(\lambda) + P_A(\lambda)P_{\bar{B}}(\lambda) - \lambda P_{\bar{A}}(\lambda)P_{\bar{B}}(\lambda)) - \\ &\quad - P_{\bar{X}}(\lambda)(P_{\bar{A}}(\lambda)P_B(\lambda) + P_A(\lambda)P_{\bar{B}}(\lambda)). \end{aligned}$$

Using these polynomials, $P_G(\lambda)$ and $P_{\delta(G)}(\lambda)$ can be expressed in other way:

$$\begin{aligned} P_G(\lambda) &= P_X(\lambda)P_{F_1}(\lambda) + P_Y(\lambda)P_{K_1}(\lambda) - P_A(\lambda)P_B(\lambda)P_Y(\lambda)P_X(\lambda), \\ P_{\delta(G)}(\lambda) &= P_X(\lambda)P_{F_2}(\lambda) + P_Y(\lambda)P_{K_2}(\lambda) - P_A(\lambda)P_B(\lambda)P_Y(\lambda)P_X(\lambda). \end{aligned}$$

Now, we finish the proof by discussing the sign of $P_{F_1}(r)$.

(a) If $P_{F_1}(r) > 0$, then, by Lemma 1, it follows $\lambda_2(F_1) > r$, and by 1) it follows that $\lambda_2(F_2) > r$. Therefore, $\lambda_2(G) > r$ and $\lambda_2(\delta(G)) > r$.

(b) If $P_{F_1}(r) < 0$, it follows from (1) that $P_{F_2}(r) < 0$ and since $P_Y(r) > 0$, it follows that $P_{\bar{A}}(r)P_B(r) + P_A(r)P_{\bar{B}}(r) > 0$ holds, which means that $P_{K_1}(r) = P_{K_2}(r) < 0$ holds, too, considering that $P_{\bar{X}}(r) > 0$.

Let us introduce the following notation: If $\lambda = r$ is a root of a polynomial $P(\lambda)$, we denote by $S(\lambda)$ the polynomial such that $P(\lambda) = (\lambda - r)S(\lambda)$.

As r is a root of both polynomials $P_G(\lambda)$ and $P_{\delta(G)}(\lambda)$, it follows

$$\begin{aligned} S_G(\lambda) &= S_X(\lambda)P_{F_1}(\lambda) + S_Y(\lambda)P_{K_1}(\lambda) - P_A(\lambda)P_B(\lambda)S_Y(\lambda)P_X(\lambda), \\ S_{\delta(G)}(\lambda) &= S_X(\lambda)P_{F_2}(\lambda) + S_Y(\lambda)P_{K_2}(\lambda) - P_A(\lambda)P_B(\lambda)S_Y(\lambda)P_X(\lambda), \end{aligned}$$

and, therefore, $S_G(r) = S_X(r)P_{F_1}(r) + S_Y(r)P_{K_1}(r)$ and $S_{\delta(G)}(r) = S_X(r)P_{F_2}(r) + S_Y(r)P_{K_2}(r)$. Since $S_X(r), S_Y(r) > 0$, we get $S_G(r), S_{\delta(G)}(r) < 0$. This means that $\lambda_3(G), \lambda_3(\delta(G)) < r$ and $\lambda_2(G) = \lambda_2(\delta(G)) = r$, and, therefore, $\operatorname{sgn}(\lambda_2(G) - r) = \operatorname{sgn}(\lambda_2(\delta(G)) - r) = 0$.

(c) If $P_{F_1}(r) = 0$ and, consequently, $P_{F_2}(r) = P_{K_1}(r) = P_{K_2}(r) = 0$, it follows that

$$\begin{aligned} S_G(r) &= S_X(r)P_{F_1}(r) + S_Y(r)P_{K_1}(r) - P_A(r)P_B(r)S_Y(r)P_X(r) = 0, \\ S_{\delta(G)}(r) &= S_X(r)P_{F_2}(r) + S_Y(r)P_{K_2}(r) - P_A(r)P_B(r)S_Y(r)P_X(r) = 0, \end{aligned}$$

and we see that a multiplicity of the root r is at least two for both polynomials $P_G(\lambda)$ and $P_{\delta(G)}(\lambda)$. Hence, $\lambda_2(G) = \lambda_3(G) = r$ and $\lambda_2(\delta(G)) = \lambda_3(\delta(G)) = r$, and, of course, $\operatorname{sgn}(\lambda_2(G) - r) = \operatorname{sgn}(\lambda_2(\delta(G)) - r) = 0$. \square

From this theorem we have $\operatorname{sgn}(\lambda_2(G) - r) = \operatorname{sgn}(\lambda_2(\alpha(G)) - r)$, for $G \in \mathbf{V}$, and we can easily conclude that if $\alpha(G_1) = \alpha(G_2)$ for $G_1, G_2 \in \mathbf{V}$, then $\operatorname{sgn}(\lambda_2(\alpha(G_1)) - r) = \operatorname{sgn}(\lambda_2(\alpha(G_2)) - r)$. (The same holds for mappings β, γ and δ .)

Therefore, it is natural to define equivalence relations induced by these mappings:

$$(\forall G_1, G_2 \in \mathbf{V}) G_1 \sim_\alpha G_2 \Leftrightarrow \alpha(G_1) = \alpha(G_2),$$

(and similarly for β, γ and δ). We call these relations the α -, β -, γ - and δ -equivalences. It is clear that if G_1 and G_2 belong to the same equivalence class, then $\operatorname{sgn}(\lambda_2(G_1) - r) = \operatorname{sgn}(\lambda_2(G_2) - r)$.

4. MAPPINGS THAT PARTIALLY PRESERVE THE SIGN OF $\lambda_2 - r$

Let \mathbf{T}' be a subset of \mathbf{T} that consists of graphs $T(A \cdot Y, B, X)$, as in Fig.5, where $\lambda_1(X) = \lambda_1(Y)$. Now, let us define three new mappings ω, τ and φ , and demonstrate how they partially preserve the sign of $\lambda_2 - r$.

- (1) $\omega : \mathbf{T} \rightarrow \mathbf{V}$, $\omega(T(A, B, X)) = V(A, B, X)$,
- (2) $\tau : \mathbf{T}' \rightarrow \mathbf{V}$, $\tau(T(A \cdot Y, B, X)) = V(A, B, X)$, Fig.5,
- (3) $\varphi : \mathbf{Q} \rightarrow \mathbf{V}$, $\varphi(Q(A, B, X, Y)) = V(A, B, X)$.

Theorem 2. Let $G = T(A, B, X)$ be a graph from \mathbf{T} and let $\lambda_1(X) = r$. Then the following implication holds: if $\lambda_2(\omega(G)) \leq r$, then $\lambda_2(G) < r$.

Proof: Suppose that $\lambda_2(\omega(G)) \leq r$; it follows that graphs \bar{A} and \bar{B} cannot have indices greater than r , and by Lemma 1, $\lambda_3(G) < r < \lambda_1(G)$ holds. Therefore, we can check the sign of $P_G(r)$ again. Based on the proof of Theorem 1, part (3), we get

$$P_G(r) = -P_{\bar{X}}(r)(P_{\bar{A}}(r)P_B(r) + P_A(r)P_{\bar{B}}(r) + 2P_{\bar{A}}(r)P_{\bar{B}}(r)),$$

and by proof of Theorem 1, part (1),

$$P_{\omega(G)}(r) = -P_{\bar{X}}(r)(P_{\bar{A}}(r)P_B(r) + P_A(r)P_{\bar{B}}(r));$$

so we get $P_G(r) = P_{\omega(G)}(r) - 2P_{\bar{X}}(r)P_{\bar{A}}(r)P_{\bar{B}}(r) < P_{\omega(G)}(r)$, since $P_{\bar{X}}(r)$, $P_{\bar{A}}(r)$ and $P_{\bar{B}}(r)$ are all positive. From $\lambda_2(\omega(G)) \leq r$ it follows $P_{\omega(G)}(r) \leq 0$ and $P_G(r) < 0$ and hence $\text{sgn}(\lambda_2(G) - r) = -1$. \square

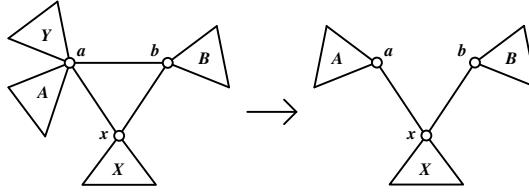


Figure 5:

Theorem 3. Let $G = T(A \cdot Y, B, X)$ be a graph from \mathbf{T}' , and let $\lambda_1(X) = \lambda_1(Y) = r$ hold. Then, depending on the value of r , the following statements hold:

- (1) If $r < 2$, then $\lambda_2(\tau(G)) \leq r \Rightarrow \lambda_2(G) < r$.
- (2) If $r > 2$, then $\lambda_2(\tau(G)) \geq r \Rightarrow \lambda_2(G) > r$.
- (3) If $r = 2$, then $\text{sgn}(\lambda_2(\tau(G)) - 2) = \text{sgn}(\lambda_2(G) - 2)$.

Proof: As in the proofs of the two previous theorems, we shall compare values of characteristic polynomials at r . By the proof of Theorem 1, part (3), $P_{\tau(G)}(r) = -P_{\bar{X}}(r)(P_{\bar{A}}(r)P_B(r) + P_A(r)P_{\bar{B}}(r))$ and we can calculate $P_G(r)$ using Schwenk's lemma and notation $M = G - X$:

$$\begin{aligned} P_G(\lambda) &= P_X(\lambda)P_M(\lambda) + P_{\bar{X}}(\lambda)[\lambda P_M(\lambda) - P_{\bar{A}}(\lambda)P_Y(\lambda)P_B(\lambda) - P_B(\lambda)(P_A(\lambda)P_Y(\lambda) + \\ &\quad + P_{\bar{A}}(\lambda)P_Y(\lambda) - \lambda P_{\bar{A}}(\lambda)P_Y(\lambda)) - 2P_{\bar{A}}(\lambda)P_{\bar{B}}(\lambda)P_Y(\lambda)] - \lambda P_{\bar{X}}(\lambda)P_M(\lambda) = \\ &= P_X(\lambda)P_M(\lambda) - P_{\bar{X}}(\lambda)\{P_Y(\lambda)P_{\bar{A}}(\lambda)P_{\bar{B}}(\lambda) + P_Y(\lambda)[P_{\bar{A}}(\lambda)P_B(\lambda) + \\ &\quad + P_A(\lambda)P_{\bar{B}}(\lambda) - (\lambda - 2)P_{\bar{A}}(\lambda)P_{\bar{B}}(\lambda)]\}. \end{aligned}$$

From $P_X(r) = P_Y(r) = 0$ it follows that

$$\begin{aligned} P_G(r) &= -P_{\bar{X}}(r)P_{\bar{Y}}(r)[P_{\bar{A}}(r)P_B(r) + P_{\bar{B}}(r)P_A(r) - (r-2)P_{\bar{A}}(r)P_{\bar{B}}(r)] \\ &= P_{\bar{Y}}(r)[P_{\tau(G)}(r) + (r-2)P_{\bar{A}}(r)P_{\bar{B}}(r)P_{\bar{X}}(r)]. \end{aligned}$$

(1) If $\lambda_2(\tau(G)) \leq r$, then $\lambda_1(\bar{A}) < r$ and $\lambda_1(\bar{B}) < r$, and from $\lambda_1(Y) = r$ it follows that $\lambda_1(\bar{Y}) < r$. Therefore, by Lemma 1, $\lambda_3(G) < r < \lambda_1(G)$ holds and the equality $\text{sgn}(\lambda_2(G) - r) = \text{sgn}(P_G(r))$ holds, too. Since $P_{\bar{A}}(r), P_{\bar{B}}(r), P_{\bar{X}}(r), P_{\bar{Y}}(r) > 0$, $P_{\tau(G)}(r) \leq 0$ and $r < 2$, it follows that $P_G(r) < 0$. Therefore, $\lambda_2(G) < r$.

(2) If $\lambda_1(\bar{A}) \geq r$ or $\lambda_1(\bar{B}) \geq r$, then $\lambda_2(\tau(G)) > r$ and $\lambda_2(G) > r$. If $\lambda_1(\bar{A}) < r$ and $\lambda_1(\bar{B}) < r$, then by Lemma 1, $\lambda_3(\tau(G)) < r < \lambda_1(\tau(G))$ holds and also $\lambda_3(G) < r < \lambda_1(G)$ holds. From $\lambda_2(\tau(G)) \geq r$ and $r > 2$ it follows $P_G(r) > 0$ and therefore $\lambda_2(G) > r$ (since $P_{\bar{A}}(r), P_{\bar{B}}(r), P_{\bar{X}}(r), P_{\bar{Y}}(r) > 0$).

(3) Again, if $\lambda_1(\bar{A}) \geq r$ or $\lambda_1(\bar{B}) \geq r$, then $\lambda_2(\tau(G)) > r$ and $\lambda_2(G) > r$. If $\lambda_1(\bar{A}) < r$ and $\lambda_1(\bar{B}) < r$, then by Lemma 1, $\lambda_3(\tau(G)) < r < \lambda_1(\tau(G))$ and $\lambda_3(G) < r < \lambda_1(G)$ holds, and from $r = 2$ and $P_{\bar{Y}}(r) > 0$ it follows $\text{sgn}(P_{\tau(G)}(r)) = \text{sgn}(P_G(r))$. \square

Theorem 4. Let $G = Q(A, B, X, Y)$ be a graph from \mathbf{Q} , and let $\lambda_1(X) = \lambda_1(Y) = r$ hold. Then $P_G(r) = 0$ and the following equivalences hold:

- (1) $\lambda_2(\varphi(G)) \leq r \Leftrightarrow \lambda_2(G) = r \wedge \text{sgn}(\lambda_2(\varphi(G)) - r) = \text{sgn}(\lambda_3(G) - r)$
- (2) $\lambda_2(\varphi(G)) > r \Leftrightarrow \lambda_2(G) > r$.

Proof: By the proof of Theorem 1, part (4),

$$P_G(r) = -P_Y(r)P_{\bar{X}}(r)(P_{\bar{A}}(r)P_B(r) + P_A(r)P_{\bar{B}}(r)),$$

so $P_G(r) = 0$.

(1) $\boxed{\Leftarrow}$ From $\lambda_2(G) = r$, it follows $\lambda_2(\varphi(G)) \leq r$.

$\boxed{\Rightarrow}$ If $\lambda_2(\varphi(G)) < r$, then $\lambda_2(G - y) < r$ (since $\lambda_1(\bar{Y}) < r$) and $\lambda_3(G) < r$. Therefore, $\lambda_2(G) = r$ and $\text{sgn}(\lambda_2(\varphi(G)) - r) = \text{sgn}(\lambda_2(G) - r) = -1$.

If $\lambda_2(\varphi(G)) = r$, then $P_{\varphi(G)}(r) = 0$ and $P_{\bar{A}}(r)P_B(r) + P_A(r)P_{\bar{B}}(r) = 0$. From the proof of Theorem 1, part (1) and (4), it follows

$$\begin{aligned} P_G(\lambda) &= P_Y(\lambda)P_{\varphi(G)}(\lambda) - P_{\bar{Y}}(\lambda)P_X(\lambda)(P_{\bar{A}}(\lambda)P_B(\lambda) + P_A(\lambda)P_{\bar{B}}(\lambda)), \\ S_G(\lambda) &= S_Y(\lambda)P_{\varphi(G)}(\lambda) - P_{\bar{Y}}(\lambda)S_X(\lambda)(P_{\bar{A}}(\lambda)P_B(\lambda) + P_A(\lambda)P_{\bar{B}}(\lambda)), \end{aligned}$$

so $S_G(r) = 0$. Therefore, $\lambda_3(G) = \lambda_2(G) = r$.

(2) $\boxed{\Leftarrow}$ Let $\lambda_2(G) > r$ hold and suppose that $\lambda_2(\varphi(G)) \leq r$ holds. Then, by (1), it follows $\lambda_2(G) = r$, which is a contradiction. Hence, $\lambda_2(\varphi(G)) > r$ holds.

\Rightarrow From $\lambda_2(\varphi(G)) > r$, it follows $\lambda_2(G) > r$. \square

It is interesting to notice that we can define a mapping, based on φ , between the family $Q(A, B, X, Y)$ and a family of disconnected graphs, that completely preserves the sign of $\lambda_2 - r$. Let \mathbf{V}' consist of disconnected graphs of type $V(A, B, X) \cup Y$, and let us define the mapping $\varphi_1 : \mathbf{Q} \rightarrow \mathbf{V}'$, $\varphi_1(Q(A, B, X, Y)) = V(A, B, X) \cup Y$. The next theorem shows that this mapping preserves the sign of $\lambda_2 - r$.

Theorem 5. Let $G = Q(A, B, X, Y)$ be a graph from the family \mathbf{Q} and let $\lambda_1(X) = r$ hold. Then $\text{sgn}(\lambda_2(G) - r) = \text{sgn}(\lambda_2(\varphi_1(G)) - r)$.

Proof: First, if $\lambda_1(\bar{A}) \geq r$ or $\lambda_1(\bar{B}) \geq r$ or $\lambda_1(Y) > r$, then $\text{sgn}(\lambda_2(G) - r) = \text{sgn}(\lambda_2(\varphi_1(G)) - r) = 1$. We further assume that $\lambda_1(\bar{A}) < r$, $\lambda_1(\bar{B}) < r$ and $\lambda_1(Y) \leq r$. Notice that $\varphi_1(G) - Y = \varphi(G)$. By Lemma 1 $\lambda_3(\varphi_1(G) - Y) < r < \lambda_1(\varphi_1(G))$ holds, hence $\lambda_4(\varphi_1(G)) < r < \lambda_1(\varphi_1(G))$, and by Lemma 2 $\lambda_4(G) < r < \lambda_1(G)$. As we know, by proof of the Theorem 1, part (4) and part (1), the following equations hold:

$$\begin{aligned} P_G(\lambda) &= P_A(\lambda)P_B(\lambda)P_Y(\lambda)P_X(\lambda) - \\ &\quad -(P_{\bar{Y}}(\lambda)P_X(\lambda) + P_Y(\lambda)P_{\bar{X}}(\lambda))(P_{\bar{A}}(\lambda)P_B(\lambda) + P_A(\lambda)P_{\bar{B}}(\lambda)) \\ P_{\varphi_1(G)}(\lambda) &= P_Y(\lambda)[P_X(\lambda)P_A(\lambda)P_B(\lambda) - P_{\bar{X}}(\lambda)(P_{\bar{A}}(\lambda)P_B(\lambda) + P_A(\lambda)P_{\bar{B}}(\lambda))]. \end{aligned}$$

Therefore, we see that

$$P_G(r) = P_{\varphi_1(G)}(r) = -P_Y(r)P_{\bar{X}}(r)(P_{\bar{A}}(r)P_B(r) + P_A(r)P_{\bar{B}}(r))$$

holds and that $P_{\varphi_1(G)}(\lambda) = P_Y(\lambda)P_{\varphi(G)}(\lambda)$ and

$$P_G(\lambda) = P_{\varphi_1(G)}(\lambda) - P_X(\lambda)P_{\bar{Y}}(\lambda)(P_{\bar{A}}(\lambda)P_B(\lambda) + P_A(\lambda)P_{\bar{B}}(\lambda))$$

hold. Notice that $\varphi(G)$ is a proper subgraph of both G and $\varphi_1(G)$, so we continue the proof by discussing the sign of $\lambda_2(\varphi(G) - r)$.

(1) If $\lambda_2(\varphi(G)) > r$, then $\lambda_2(G), \lambda_2(\varphi_1(G)) > r$.

(2) Suppose that $\lambda_2(\varphi(G)) = r$. If $\lambda_1(Y) = r$, then obviously $\lambda_2(\varphi_1(G)) = \lambda_3(\varphi_1(G)) = r$, and by Theorem 4, $\lambda_2(G) = \lambda_3(G) = r$. Now suppose that $\lambda_1(Y) < r$. We see that $\lambda_2(\varphi_1(G)) = r$ and $\lambda_3(\varphi_1(G)) < r$. Also, from $P_{\varphi(G)}(r) = 0$ it follows $P_{\bar{A}}(r)P_B(r) + P_A(r)P_{\bar{B}}(r) = 0$ and hence $P_G(r) = P_{\varphi_1(G)}(r) = 0$. From

$$S_G(r) = S_{\varphi_1(G)}(r) - P_{\bar{Y}}(r)S_X(r)(P_{\bar{A}}(r)P_B(r) + P_A(r)P_{\bar{B}}(r)) < 0$$

(since $S_{\varphi_1(G)}(r) < 0$) it follows $\lambda_3(G) < r$ and therefore $\lambda_2(G) = r$.

(3) Suppose that $\lambda_2(\varphi(G)) < r$. Then $\lambda_3(G), \lambda_3(\varphi_1(G)) < r$ and from $P_G(r) = P_{\varphi_1(G)}(r)$ it follows $\text{sgn}(\lambda_2(G) - r) = \text{sgn}(\lambda_2(\varphi_1(G)) - r)$. \square

5. APPLICATIONS OF GIVEN MAPPINGS TO SOME CLASSES OF REFLEXIVE GRAPHS

Using the presented mappings, applications of GRS-theorem can be extended to some GRS-undecidable graphs. Especially, if there exists a GRS-decidable graph in an equivalence class (induced by some of the mappings mentioned above), then the conclusion about $\text{sgn}(\lambda_2 - r)$ for that graph holds for any graph from the same class. To show actual applications, we shall revisit some previously determined classes of reflexive cacti. Reflexive graphs are graphs whose second largest eigenvalue does not exceed 2. We shall use the RS-theorem, the special case of the GRS-theorem, for $r = 2$. Before stating the RS-theorem, let us recall Smith graphs. Connected graphs with the property $\lambda_1 = 2$ are called Smith graphs [24], [3, p. 79], Fig. 6. They have an essential role in investigations on reflexive graphs. It is important to notice that every connected graph is either a Smith graph or it is a proper subgraph or a proper supergraph of a Smith graph.

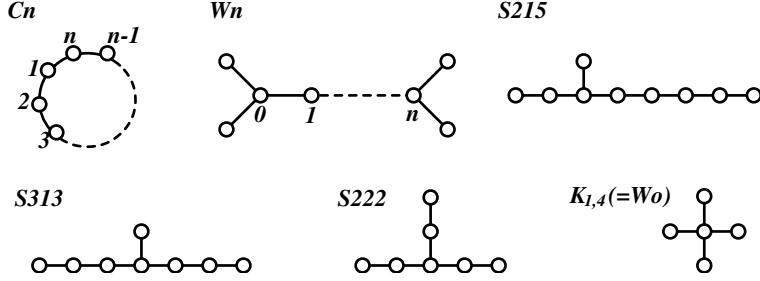


Figure 6:

RS-Theorem. [19] Let G be a graph with a cut-vertex v .

- (1) If at least two components of $G - v$ are supergraphs of Smith graphs, and if at least one of them is a proper supergraph, then $\lambda_2(G) > 2$ holds.
- (2) If at least two components of $G - v$ are Smith graphs, and the rest are subgraphs of Smith graphs, then $\lambda_2(G) = 2$ holds.
- (3) If at most one component of $G - v$ is Smith graph, and the rest are proper subgraphs of Smith graphs, then $\lambda_2(G) < 2$ holds.

This theorem does not give the answer whether G is reflexive or not only in the case when, after removing the cut-vertex, one of the components is a proper supergraph of a Smith graph and all others are subgraphs of some Smith graphs. Now we show that using the aforementioned mappings we can indirectly apply the RS-theorem to RS-undecidable graphs. Notice that for $r = 2$, mappings $\alpha, \beta, \gamma, \delta, \tau, \varphi$ and φ_1 preserve the property $\lambda_2 \leq 2$, i.e.

$$(\forall G \in \mathbf{V}) \lambda_2(G) \leq 2 \Leftrightarrow \lambda_2(\alpha(G)) \leq 2,$$

and the same holds for β , γ , δ , τ , φ and φ_1 .

5.1 Class of bicyclic graphs with the bridge between the cycles

In [19] all RS-undecidable maximal reflexive bicyclic graphs with the bridge between the cycles are determined and classified into four families: $A_1 - A_{14}$, $B_1 - B_{11}$, $C_1 - C_{41}$ and $D_1 - D_{36}$. In many cases, it is possible to show that these graphs are equivalent (in terms of the same $\text{sgn}(\lambda_2 - 2)$) to some RS-decidable graphs, using the mappings α and β . Because of the structural properties, graphs

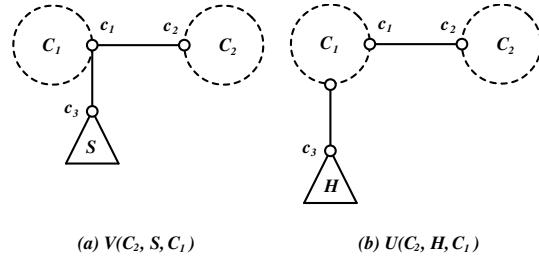


Figure 7:

$A_1 - A_{14}$ and D_{36} are neither α -equivalent nor β -equivalent to any RS-decidable graph (though $A_3 \sim_\beta A_7$, $A_4 \sim_\beta A_9$ and $A_6 \sim_\beta A_8$ hold). On the other hand, every maximal graph from the family $D_1 - D_{35}$ is α -equivalent to some RS-decidable reflexive graph $V(C_2, S, C_1)$ (Fig.7(a)), where C_1 and C_2 are cycles of an arbitrary length, and S is some Smith tree. In Fig.8 this is shown, as an example, for D_8 (Fig.8(a)). Graph D_8 is α -equivalent to graph $V(C_2, S222, C_1)$ (Fig.8(b)), because both of them can be mapped by α into graph $V(C_2 \cdot S222, c_3, C_1)$ (Fig.8(c)).

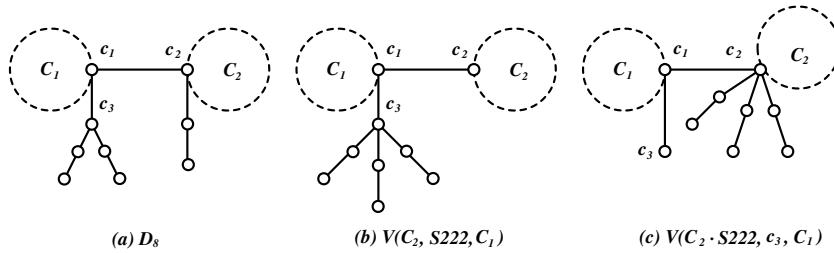


Figure 8:

Every maximal reflexive graph from the family $B_1 - B_{11}$ or $C_1 - C_{41}$ is β -equivalent or $\alpha \circ \beta$ -equivalent to some RS-decidable graph $U(C_2, H, C_1)$ (Fig.7(b)), where C_1 and C_2 are cycles and $G - C_2 - c_1$ is a subgraph of some Smith tree (including the case when it is equal to some Smith tree). Such β -equivalence is

shown, for example, for B_6 (Fig.9(a)), where $B_6 \sim_{\beta} U(C_1, P_2 \cdot P_2, C_2)$ (Fig.9(b)) and C_2 is a cycle of length 4 (with two relevant vertices labeled c_2 and c_3 as in Fig.9), C_1 is a cycle of an arbitrary length with the root c_1 , $P_2 \cdot P_2$ is the coalescence of two paths of the length 1 leaned in the vertex in which coalescence is formed, and $G - C_1 - c_2$ is Smith graph W_1 . In this example β -equivalence holds since $\beta(B_6) = \beta(U(C_1, P_2 \cdot P_2, C_2)) = U(C_1 \cdot P_2 \cdot P_2, c_2, C_2)$ (Fig.9(c)). Graph $U(C_1, P_2 \cdot P_2, C_2)$ is RS-decidable and reflexive (after removing vertex c_2 from the graph, the remaining two components are Smith graphs). Notice that β preserves the sign of $\lambda_2 - 2$ since $P_{C_2 - c_2}(2) = P_{C_2 - c_3}(2)$.

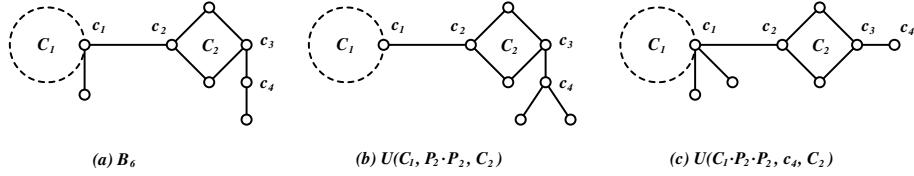


Figure 9:

For illustrating an $\alpha \circ \beta$ -equivalence we start from graph C_{26} (Fig.10(a)). It is α -equivalent to graph $V(C_1 \cdot P_3, P_2, C_2)$ (Fig.10(b)), considering the cycle C_2 as X -graph, where coalescence $C_1 \cdot P_3$ is formed at the vertex c_3 of the cycle C_1 and end-vertex of the path of the length 3. Next we consider the cycle C_1 as X -graph and $V(C_1 \cdot P_3, P_2, C_2)$ as $U(C_2 \cdot P_3, P_2, C_1)$, where coalescence $C_2 \cdot P_3$ is formed at the vertex c_2 of the cycle C_2 and the end-vertex of the path of the length 3. Now β -equivalence can be established between $U(C_2 \cdot P_3, P_2, C_1)$ and $U(C_2, P_2 \cdot P_3, C_1)$ (Fig.10(c)), since both of them can be mapped by β into graph $U(C_2 \cdot P_2 \cdot P_3, c_4, C_1)$ (Fig.10(d)). Therefore, C_{26} is $\alpha \circ \beta$ -equivalent to RS-decidable and reflexive graph $U(C_2, P_2 \cdot P_3, C_1)$.

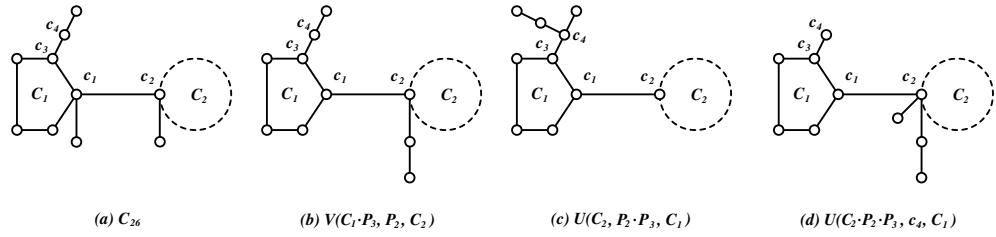


Figure 10:

Notice that $\lambda_2(C_{26}) < 2$, and because of the preserving properties of α and β , $\lambda_2(U(C_2, P_2 \cdot P_3, C_1)) < 2$ holds, too. This property of $U(C_2, P_2 \cdot P_3, C_1)$ follows also from RS-theorem: after removing the vertex c_1 , one of the remaining components is a cycle, and the other is a proper subgraph of Smith tree S215. Suppose now that this component is extended by a pendent edge to S215. Notice that neither

the corresponding extension of $U(C_2, P_2 \cdot P_3, C_1)$ is reflexive, nor β -equivalence can be established (because of the structural properties of the extension).

As a special case, in [19] a tricyclic graph $T_0 = V(C_1, C_2 \cdot C_3, c_3)$ shown in Fig.11(a) is introduced, where C_1, C_2 and C_3 are cycles of an arbitrary length. We see that it is α -equivalent to the graph $G_0 = V(C_1, C_2, C_3)$ (Fig.11(b)).

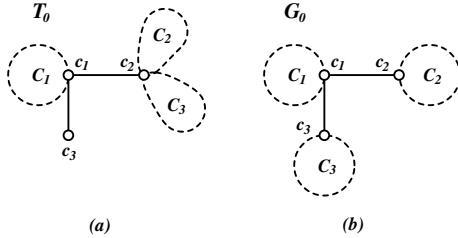


Figure 11:

5.2 Reflexive cacti with five and four cycles

RS-undecidable multicyclic reflexive graphs whose cycles do not form a bundle have at most five cycles [17]. Let us now consider all four families of maximal reflexive cacti with 5 cycles: Q_1, Q_2, T_1 and T_2 [17] (Fig.12). It is obvious that the corresponding graphs from Q_1 and Q_2 are δ -equivalent and the corresponding graphs from T_1 and T_2 are γ -equivalent. It is also obvious that, if a graph G belongs to Q_1 , then $\varphi(G) = T_0$; if G belongs to Q_2 , then $\varphi(G) = G_0$; if G belongs to T_1 , then $\tau(G) = T_0$; and for G from T_2 , $\tau(G) = G_0$ holds.

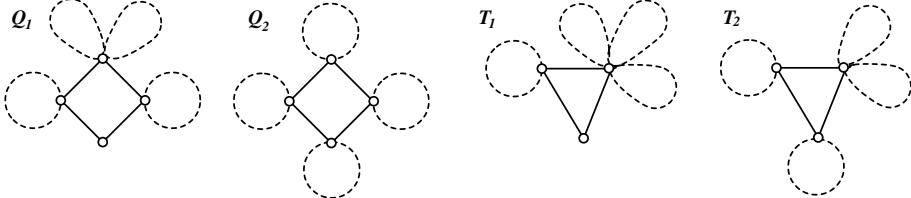


Figure 12:

There are four families of RS-undecidable reflexive cacti with four cycles: Q_1^- , Q_2^- , T_1^- and T_2^- (their cyclic structure is shown in Fig.13). All reflexive graphs from these families are determined. Maximal reflexive cacti whose at least one vertex different from the vertices of the central cycle is additionally loaded (i.e. incident to some edge different from cycle edges) are shown in [18]. In Q_1^- there are no such graphs. Graphs from Q_2^- with such properties are $H_1 - H_{48}$. They belong to \mathbf{Q} and, using φ , they can be mapped into graphs $A_2, B_1 - B_{11}, C_1 - C_{10}$

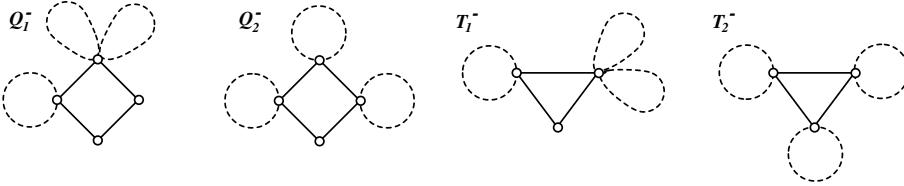


Figure 13:

and $C_{16} - C_{41}$ ($C_{11} - C_{15}$ are not from the family \mathbf{V}). Graphs from T_1^- with such properties are classified into three families: $I_1 - I_9$, $J_1 - J_{11}$ and $K_1 - K_{36}$. Using τ , graphs $J_1 - J_{11}$ can be mapped into $B_1 - B_{11}$ and graphs $K_1 - K_{36}$ into $C_1 - C_{10}$ and $C_{16} - C_{41}$, while $I_1 - I_9$ are not from \mathbf{T}' .

Graphs from T_2^- with such properties are also classified into three families: $L_1 - L_{12}$, $M_1 - M_{12}$ and $N_1 - N_{42}$. By τ , graph L_7 is mapped into A_2 , graphs M_1 and $M_4 - M_{12}$ are mapped into $B_1 - B_4$ and $B_6 - B_{11}$ and graphs $N_1 - N_{34}$ and N_{36} into $C_1 - C_{10}$ and $C_{17} - C_{41}$ (all remaining graphs from the families $L_1 - L_{12}$, $M_1 - M_{12}$ and $N_1 - N_{42}$ are not from the family \mathbf{T}').

So, whenever it is possible all graphs with mentioned properties can be mapped by φ or τ into maximal bicyclic reflexive graphs with the bridge between the cycles, described above, which are equivalent to some RS-decidable graphs (with the exception of the graph A_2).

Maximal reflexive cacti whose no vertices different from the vertices of the central cycle are additionally loaded are shown in [17]. Such graphs from Q_1^- and Q_2^- can be mapped, using φ , into graphs $D_1 - D_{35}$ or T_0 (which are all equivalent to RS-decidable graphs, as shown above). By τ , graphs from T_1^- and those from T_2^- that do belong to \mathbf{T}' are also mapped into $D_1 - D_{35}$.

6. CONCLUSION

Establishing bounds for the second largest eigenvalue of a graph is an important and interesting topic, both in theoretical investigations and in applications across many fields of research. Here we presented some new methods for establishing whether $\lambda_2 < r$, $\lambda_2 > r$, or $\lambda_2 = r$ (for $r > 0$) for some classes of graphs. We introduced 5 mappings that preserve the sign of $\lambda_2 - r$ and 3 mappings that partially preserve the sign of $\lambda_2 - r$.

These mappings naturally led to equivalence relations among the corresponding graphs, inducing the equivalence classes. This enables us to indirectly apply the GRS-theorem (the generalization of the RS-theorem) to some GRS-undecidable graphs via their GRS-decidable representatives in the corresponding equivalence classes. We illustrated possible applications by relating maximal reflexive graphs from various families, determined in previous work, to some RS-decidable graphs using the introduced equivalence relations.

To conclude, the described mappings give examples of modifications of considered graphs that preserve a prescribed spectral property. Equivalence relations induced by such transformations can naturally be understood as a sort of generalization of the notion of cospectral graphs. Those mappings appear to be very efficient in the quest for maximal reflexive graphs by modifying RS-undecidable graphs into equivalent RS-decidable graphs. Obviously, detecting of such modifications may be a powerful device in investigations directed towards finding or describing all graphs with a given spectral constraint.

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We used the expert system *newGRAPH* [26] for computations of eigenvalues of graphs.

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