

SHEDDING LIGHT ON WORDS

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A *word* over an alphabet $[k]$ can be represented by a bargraph, where the height of the i -th column is the size of the i -th part. If North is in the direction of the positive y -axis and East is in the direction of the positive x -axis, a light source projects parallel rays from the North-West direction, at an angle of 45 degrees to the y -axis. These rays strike the cells of the bargraph. We say a cell is lit if the rays strike its West facing edge or North facing edge or both. With the use of matrix algebra we find the generating function that counts the number of lit cells. From this we find the average number of lit cells in a word of length n .

1. Introduction

We define $[k] = \{1, 2, 3, \dots, k\}$ to be a (totally ordered) *alphabet* on k letters. A *word* w of length n on the alphabet $[k]$ is an element of $[k]^n$, for more information on words see [1] (for specific recent research, see [2–4]). A word can be represented by a bargraph which is a column-convex polyominoe whose lower edge lies on the x -axis and in which the height of the i -th column in the bargraph equals the number of cells in the corresponding part of that word. Thus these bargraphs have column heights less than or equal to k .

We let North be the direction of the positive y -axis and East the direction of the positive x axis. Suppose there is a light source at infinity in the North-West direction that sheds parallel light rays onto the bargraph.

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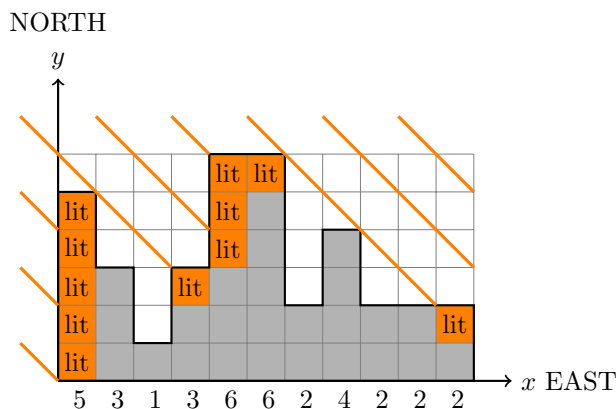


Figure 1: Word 53136624222 with 11 lit cells

A cell is defined to be lit if the ray of light hits the edge facing North or the edge facing West or both. We illustrate this in Figure 1: consider the word 53136624222, there are 11 lit cells. The unlit cells, are said to be in the shade.

In this paper, the statistic of interest is the number of lit cells in a word over the alphabet $[k]$. We shall use generating functions together with matrix algebra to investigate this statistic. Moreover, the method we use is, to our knowledge, an innovation in the literature of the Temperley [5] method of *adding a slice*. The innovation (which may well be useful in other circumstances) is that instead of adding the element a to a single following element, we consider adding a to a complete block $b_1 b_2 \cdots b_d$.

2. Generating function

Let C_k be the generating function for k -ary words (i.e., words on an alphabet $[k]$) and let $C_k(a)$ be the generating function for k -ary words, that start with a part of size a , both marking the number of lit cells. Throughout this paper whenever we mention a word with a part of size a , we will automatically mean its bargraph representation with corresponding column of height a .

Each word starting with letter a can be decomposed according to the number d of parts following the first part a which are in the shade of column a . Thus $d = 0$, represents either the first case below (a single part) or the second case where there is no part following a which is totally in the shade. The third case of Figure 2 is where $d \neq 0$; there are at least two columns, the first of which is a followed by d columns $b_1 b_2 \cdots b_d$ which are totally in the shade. The first column that is partially shaded is the one following column b_d , if it exists. So we group the word into subwords that start with the first partially lit column followed by a group of

d totally shaded columns.

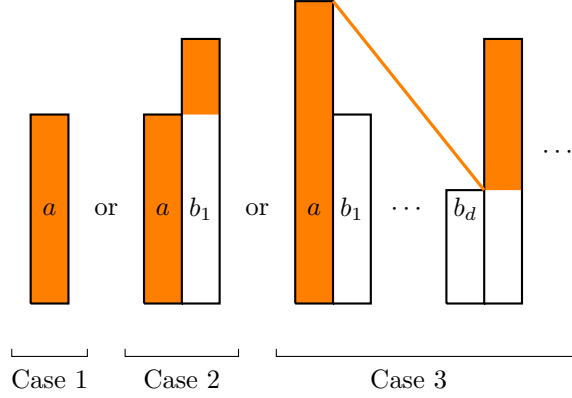


Figure 2: Decomposition of a word

In our generating function, x marks the length of the word and q the total number of lit cells. Thus

$$C_k(a) = \sum_{d=0}^a (a-1) \cdots (a-d) x^{d+1} \left[q^a + q^{d+1} C_k(a-d) + q^d \sum_{j \geq a-d+1} C_k(j) \right]$$

$$= \sum_{d=0}^a \frac{(a-1)! x^{d+1}}{(a-d-1)!} \left[q^a + q^{d+1} C_k(a-d) + q^d \sum_{j \geq a-d+1} C_k(j) \right].$$

Setting $d = a - i$ for $0 \leq a \leq k$ we have

$$\frac{C_k(a)}{(a-1)!} = \sum_{i=1}^a \frac{x^{a-i+1}}{(i-1)!} \left[q^a + q^{a-i+1} C_k(i) + q^{a-i} \sum_{j \geq i+1} C_k(j) \right].$$

Hence for $1 \leq a \leq k$,

$$\frac{C_k(a)}{(a-1)!} - (a-1) \frac{xqC_k(a-1)}{(q-1)!} = \frac{1}{(a-1)!} \left[xq^a + xqC_k(a) + x \sum_{j=a+1}^k C_k(j) \right].$$

This implies

$$(1) \quad C_k(a) = xq(a-1)C_k(a-1) + xq^a + xqC_k(a) + x \sum_{j \geq a+1} C_k(j),$$

i.e.,

$$(1-xq)C_k(a) = xq^a + xq(a-1)C_k(a-1) + x \left[C_k - 1 - \sum_{j=1}^a C_k(j) \right],$$

since $C_k = 1 + \sum_{j=1}^k C_k(j)$. Thus

$$(1 + x - xq)C_k(a) = x(q^a - 1) + xC_k + xq(a-1)C_k(a-1) - x \sum_{j=1}^{a-1} C_k(j).$$

Now, consider the difference

$$(2) \quad (1 - xq)[C_k(a) - C_k(a-1)] = (a-1)xqC_k(a-1) - (a-2)xqC_k(a-2) - xC_k(a) + xq^a - xq^{a-1}.$$

Thus for $2 \leq a \leq k$,

$$(1 + x - xq)C_k(a) = [1 + (a-2)xq]C_k(a-1) - (a-2)xqC_k(a-2) + xq^{a-1}(q-1).$$

We define γ , α_a and β_a as indicated below in equation (3)

$$(3) \quad \underbrace{(1 + x - xq)}_{\gamma} C_k(a) = [1 + \underbrace{(a-2)xq}_{\alpha_a}] C_k(a-1) - \underbrace{(a-2)xq}_{\alpha_a} C_k(a-2) + \underbrace{xq^{a-1}(q-1)}_{\beta_a}.$$

We have for $2 \leq a \leq k$:

$$(4) \quad \gamma C_k(a) - (1 + \alpha_a)C_k(a-1) + \alpha_a C_k(a-2) = \beta_a.$$

Now, we write (4) in matrix form $M_k V_k = b_k$ as

$$\begin{pmatrix} \gamma & 0 & 0 & \cdots & 0 & 0 \\ -1 - \alpha_3 & \gamma & 0 & 0 & \cdots & 0 \\ \alpha_4 & -1 - \alpha_4 & \gamma & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ 0 & \cdots & \alpha_{k-1} & -1 - \alpha_{k-1} & \gamma & 0 \\ 0 & \cdots & 0 & \alpha_k & -1 - \alpha_k & \gamma \end{pmatrix} \begin{pmatrix} C_k(2) \\ C_k(3) \\ C_k(4) \\ \vdots \\ C_k(k-1) \\ C_k(k) \end{pmatrix} = \begin{pmatrix} \beta_2 + C_k(1) \\ \beta_3 - xqC_k(1) \\ \beta_4 \\ \vdots \\ \beta_{k-1} \\ \beta_k \end{pmatrix}.$$

Note that $\det M_k = \gamma^{k-1}$. By Cramer's rule, for $2 \leq a \leq k$, we have

$$C_k(a) = \frac{\begin{vmatrix} \gamma & 0 & 0 & \cdots & 0 & \beta_2 + C_k(1) \\ -1 - \alpha_3 & \gamma & 0 & \cdots & 0 & \beta_3 - xqC_k(1) \\ \alpha_4 & -1 - \alpha_4 & \ddots & 0 & \vdots & \beta_4 \\ \vdots & & \ddots & & & \vdots \\ 0 & \cdots & \alpha_{a-1} & -1 - \alpha_{a-1} & \gamma & \beta_{a-1} \\ 0 & \cdots & \cdots & \alpha_a & -1 - \alpha_a & \beta_a \end{vmatrix}}{\gamma^{k-1}}.$$

We shall represent the last expression by

$$(5) \quad C_k(a) = \frac{A_a C_k(1) + B_a x(q-1)}{\gamma^{a-1}},$$

where

$$A_a = \begin{vmatrix} \gamma & 0 & \cdots & 0 & 1 \\ -1 - \alpha_3 & \gamma & \cdots & 0 & -xq \\ \alpha_4 & 0 & \gamma & \cdots & 0 \\ 0 & 0 & \cdots & \ddots & \cdots \\ \vdots & & & & \vdots \\ 0 & \cdots & & & \gamma \\ 0 & 0 & \cdots & \alpha_a & -1 - \alpha_a \end{vmatrix}$$

and

$$B_a = \begin{vmatrix} \gamma & 0 & \cdots & 0 & q \\ -1 - \alpha_3 & \gamma & \cdots & 0 & q^2 \\ \alpha_4 & 0 & \gamma & \cdots & 0 \\ 0 & 0 & \cdots & \ddots & \cdots \\ \vdots & & & & \vdots \\ 0 & \cdots & & & \gamma \\ 0 & \cdots & & \alpha_a & -1 - \alpha_a \end{vmatrix}.$$

Putting $a = 1$ into (1) gives

$$\begin{aligned} C_k(1) &= xq + xqC_k(1) + x \sum_{j \geq 2} C_k(j) \\ &= xq + xqC_k(1) + x[C_k - 1 - C_k(1)], \end{aligned}$$

which implies

$$(6) \quad C_k(1) = \frac{x(q-1)}{\gamma} + \frac{x}{\gamma} C_k.$$

Summing equation (5) over a from 2 to k we get

$$C_k - 1 - C_k(1) = \sum_{a=2}^k \frac{A_a}{\gamma^{a-1}} C_k(1) + \sum_{a=2}^k \frac{B_a}{\gamma^{a-1}} x(q-1),$$

and using (6)

$$C_k - 1 - \frac{x(q-1)}{\gamma} - \frac{x}{\gamma} C_k = \sum_{a=2}^k \frac{A_a}{\gamma^{a-1}} \left[\frac{x(q-1)}{\gamma} + \frac{x}{\gamma} C_k \right] + \sum_{a=2}^k \frac{B_a}{\gamma^{a-1}} x(q-1).$$

Therefore

$$(7) \quad C_k = \frac{1 + \frac{x(q-1)}{\gamma} + x(q-1) \sum_{a=2}^k \left[\frac{A_a}{\gamma^a} + \frac{B_a}{\gamma^{a-1}} \right]}{1 - \frac{x}{\gamma} - \frac{x}{\gamma} \sum_{a=2}^k \frac{A_a}{\gamma^{a-1}}}.$$

We now define

$$\mathcal{A}(z) := \sum_{k \geq 2} \left(\sum_{a=2}^k \frac{A_a}{\gamma^{a-1}} \right) z^k$$

and

$$\mathcal{B}(z) := \sum_{k \geq 2} \left(\sum_{a=2}^k \frac{B_a}{\gamma^{a-1}} \right) z^k.$$

Thus

$$\begin{aligned} \mathcal{A}(z) &= \frac{A_2 z^2}{\gamma} + \left(\frac{A_2 z^3}{\gamma} + \frac{A_3 z^3}{\gamma^2} \right) + \left(\frac{A_2 z^4}{\gamma} + \frac{A_3 z^4}{\gamma^2} + \frac{A_4 z^4}{\gamma^3} \right) + \dots \\ &= \frac{A_2}{\gamma} \frac{z^2}{1-z} + \frac{A_3}{\gamma^2} \frac{z^3}{1-z} + \frac{A_4}{\gamma^3} \frac{z^4}{1-z} + \dots \\ &= \frac{\gamma}{1-z} \sum_{j \geq 2} A_j \left(\frac{z}{\gamma} \right)^j \\ &= \frac{\gamma}{1-z} A \left(\frac{z}{\gamma} \right), \end{aligned}$$

where $A(z) := \sum_{a \geq 2} A_a z^a$.

Similarly for $\mathcal{B}(z)$ we have $\mathcal{B}(z) = \frac{\gamma}{1-z} B \left(\frac{z}{\gamma} \right)$, where $B(z) := \sum_{a \geq 2} B_a z^a$. We shall find expressions for A_a and B_a in the next two lemmas.

Lemma 2.1. For $a \geq 2$

$$\begin{aligned} A_a &= \begin{vmatrix} \gamma & 0 & \dots & 0 & 1 \\ -1 - \alpha_3 & \gamma & \dots & 0 & -xq \\ \alpha_4 & 0 & \gamma & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \ddots & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & \gamma & 0 \\ 0 & 0 & \dots & \alpha_a & -1 - \alpha_a & 0 \end{vmatrix} \\ &= \gamma^{a-1} \sum_{j=0}^{a-1} \left(\frac{-x}{\gamma} \right)^j \prod_{m=0}^j (1 - mq) \binom{a-1}{j}. \end{aligned}$$

Proof. Cofactor expansion using the bottom row yields

$$A_a = (1 + \alpha_a) A_{a-1} - \alpha_a \gamma A_{a-2},$$

where $A_1 := 1$ and $A_2 := 1$.

Multiplying by $\frac{z^{a-2}}{\gamma^a(a-2)!}$ and remembering that $\alpha_a = (a-2)xq$ and $\gamma = 1+x-xq$ we have

$$\frac{A_a z^{a-2}}{\gamma^a(a-2)!} = \frac{A_{a-1} z^{a-2}}{\gamma^a(a-2)!} + \frac{xq A_{a-1} z^{a-2}}{\gamma^a(a-3)!} - \frac{xq A_{a-2} z^{a-2}}{\gamma^{a-1}(a-3)!}.$$

These above expressions can be written as derivatives for $a \geq 2$

$$\left(\frac{A_a z^a}{\gamma^a a!}\right)'' = \frac{1}{\gamma} \left(\frac{A_{a-1} z^{a-1}}{\gamma^{a-1}(a-1)!}\right)' + \frac{xqz}{\gamma} \left(\frac{A_{a-1} z^{a-1}}{\gamma^{a-1}(a-1)!}\right)'' - \frac{xqz}{\gamma} \left(\frac{A_{a-2} z^{a-2}}{\gamma^{a-2}(a-2)!}\right)'$$

We proceed by defining $\tilde{A}(z) := \sum_{a \geq 1} \frac{A_a z^a}{\gamma^a a!}$, so that

$$\left(\tilde{A}(z) - \frac{z^2}{2!\gamma^2} - \frac{z}{\gamma}\right)'' = \frac{1}{\gamma} \left(\tilde{A}(z) - \frac{z}{\gamma}\right)' + \frac{xqz}{\gamma} \left(\tilde{A}(z) - \frac{z}{\gamma}\right)'' - \frac{xqz}{\gamma} \tilde{A}'(z)$$

which implies

$$\tilde{A}''(z) = \frac{1}{\gamma} \tilde{A}'(z) + \frac{xqz}{\gamma} \tilde{A}''(z) - \frac{xqz}{\gamma} \tilde{A}'(z),$$

or

$$\frac{\tilde{A}''(z)}{\tilde{A}'(z)} = \frac{1-xqz}{\gamma-xqz} = 1 + \frac{1-\gamma}{\gamma-xqz}.$$

After integrating we have

$$\ln(\tilde{A}'(z)) = z + \ln(\gamma - xqz)^{\frac{1-\gamma}{-xq}} + \ln C$$

which yields

$$\tilde{A}'(z) = C e^z (\gamma - xqz)^{\frac{1}{q}-1}.$$

In order to find C , we put $z = 0$ and we know $\tilde{A}'(0) = \frac{A_1}{\gamma 1!} = \frac{1}{\gamma}$, thus $C \gamma^{\frac{1}{q}-1} = \frac{1}{\gamma}$, which gives

$$C = \gamma^{-\frac{1}{q}}.$$

Thus

$$\begin{aligned} \tilde{A}'(z) &= \gamma^{-\frac{1}{q}} e^z (\gamma - xqz)^{\frac{1}{q}-1} \\ (8) \quad &= \frac{e^z}{\gamma} \left(1 - \frac{xqz}{\gamma}\right)^{\frac{1}{q}-1}. \end{aligned}$$

We now expand the above $\tilde{A}'(z)$ with $0 < q < 1$

$$\begin{aligned} \tilde{A}'(z) &= \frac{1}{\gamma} \sum_{i \geq 0} \frac{z^i}{i!} \sum_{j \geq 0} \binom{\frac{1}{q}-1}{j} \left(\frac{-xqz}{\gamma}\right)^j \\ (9) \quad &= \frac{1}{\gamma} \sum_{i \geq 0} \sum_{j \geq 0} (-1)^j \frac{(\frac{1}{q}-1)_j}{i! j!} \frac{x^j q^j z^{i+j}}{\gamma^j}, \end{aligned}$$

where the falling factorial is

$$(a)_j := a(a-1)(a-2)\cdots(a-j+1) \quad \text{and} \quad (a)_0 := 1.$$

Let us consider the term $q^j(\frac{1}{q}-1)_j$ separately

$$\begin{aligned} q^j\left(\frac{1}{q}-1\right)_j &= q^j \left(\frac{1}{q}-1\right) \left(\frac{1}{q}-2\right) \cdots \left(\frac{1}{q}-j\right) \\ &= (1-q)(1-2q)\cdots(1-jq) \\ &= \prod_{m=0}^j (1-mq). \end{aligned}$$

Now since $\tilde{A}(z) := \sum_{a \geq 1} \frac{A_a z^a}{\gamma^a a!}$,

$$(10) \quad \tilde{A}'(z) = \sum_{a \geq 1} \frac{A_a z^{a-1}}{\gamma^a (a-1)!}.$$

Comparing coefficient of z^{a-1} in (8) and (9) gives

$$(11) \quad \frac{A_a}{\gamma^a (a-1)!} = \frac{1}{\gamma} \sum_{j=0}^{a-1} \frac{(-1)^j x^j \prod_{m=0}^j (1-mq)}{j!(a-1-j)! \gamma^j},$$

since $i = a - 1 - j$. Hence we have the required result

$$A_a = \gamma^{a-1} \sum_{j=0}^{a-1} \left(\frac{-x}{\gamma}\right)^j \prod_{m=0}^j (1-mq) \binom{a-1}{j}.$$

□

A quick check indeed gives $A_2 = \gamma \left(1 - \frac{x(1-q)}{\gamma}\right) = 1$. Now for B_a , we have the following result.

Lemma 2.2. For $a \geq 2$

$$\begin{aligned} B_a &= \begin{vmatrix} \gamma & 0 & \cdots & 0 & q \\ -1 - \alpha_3 & \gamma & \cdots & 0 & q^2 \\ \alpha_4 & 0 & \gamma & \cdots & 0 & q^3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \ddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & & & & \gamma & q^{a-2} \\ 0 & \cdots & & \alpha_a & -1 - \alpha_a & q^{a-1} \end{vmatrix} \\ &= q\gamma^{a-1} \sum_{l=0}^{a-1} \sum_{j=0}^{a-1-l} \frac{l}{a-l} \binom{a-1}{l, j, a-1-j-l} A_l (q-1)^{a-1-j-l} \prod_{m=0}^{j-1} (1+mq) \frac{x^j}{\gamma^j}, \end{aligned}$$

where $\binom{n}{a,b,c} := \frac{n!}{a!b!c!}$ and $a + b + c = n$.

Proof. We expand this determinant by cofactor expansion along the bottom row. Thus

$$B_a = q^{a-1}\gamma^{a-2} + (1 + \alpha_a)B_{a-1} - \alpha_a\gamma B_{a-2}$$

with $B_0 := 0$ and $B_1 := 0$ and as before $\alpha_a = (a-2)xq$ and $\gamma = 1 + x - xq$.

We proceed in a similar way to the proof for A_a , by expressing the above equation in terms of derivatives.

$$\begin{aligned} \left(\frac{B_a z^a}{\gamma^a a!}\right)'' &= \frac{q}{\gamma^2} \frac{q^{a-2} z^{a-2}}{(a-2)!} + \frac{1}{\gamma} \left(\frac{B_{a-1} z^{a-1}}{\gamma^{a-1} (a-1)!}\right)' \\ &+ \frac{xqz}{\gamma} \left(\frac{B_{a-1} z^{a-1}}{\gamma^{a-1} (a-1)!}\right)'' - \frac{xqz}{\gamma} \left(\frac{B_{a-2} z^{a-2}}{\gamma^{a-2} (a-2)!}\right)' . \end{aligned}$$

We proceed by defining $\tilde{B}(z) := \sum_{a \geq 0} \frac{B_a z^a}{\gamma^a a!}$, so that, summing over $a \geq 2$ we obtain

$$\tilde{B}''(z) = \frac{q}{\gamma^2} e^{qz} + \frac{1}{\gamma} \tilde{B}'(z) + \frac{xqz}{\gamma} \tilde{B}''(z) - \frac{xqz}{\gamma} \tilde{B}'(z)$$

which simplifies to

$$\left(1 - \frac{xqz}{\gamma}\right) \tilde{B}''(z) - \frac{1 - xqz}{\gamma} \tilde{B}'(z) = \frac{qe^{qz}}{\gamma^2}.$$

Considering this equation as a first order differential equation, the solution $\tilde{B}'(z)$ is

$$\begin{aligned} \tilde{B}'(z) &= \frac{q(-1 - x + xq + xqz)^{\frac{1}{q}-1}}{-1 - x + xq} e^z \int_0^z (-1 - x + xq + xqt)^{-\frac{1}{q}} e^{t(q-1)} dt \\ &= \frac{q(-\gamma + xqz)^{\frac{1}{q}-1}}{-\gamma} e^z \int_0^z (-\gamma + xqt)^{-\frac{1}{q}} e^{t(q-1)} dt \\ (12) \quad &= \frac{q}{\gamma^2} \left(1 - \frac{xqz}{\gamma}\right)^{\frac{1}{q}-1} e^z \int_0^z \left(1 - \frac{xqt}{\gamma}\right)^{-\frac{1}{q}} e^{t(q-1)} dt. \end{aligned}$$

We proceed by focusing on the integral

$$\begin{aligned} \int_0^z \left(1 - \frac{xqt}{\gamma}\right)^{-\frac{1}{q}} e^{t(q-1)} dt &= \int_0^z \sum_{i \geq 0} \frac{t^i (q-1)^i}{i!} \sum_{j \geq 0} \binom{\frac{1}{q} - 1 + j}{j} \left(\frac{xqt}{\gamma}\right)^j dt \\ &= \int_0^z \sum_{i, j \geq 0} \frac{(q-1)^i \binom{\frac{1}{q} - 1 + j}{j} q^j x^j t^{i+j}}{\gamma^j i! j!} dt \\ &= \sum_{i, j \geq 0} \frac{(q-1)^i \prod_{m=0}^{j-1} (1 + mq) x^j z^{i+j+1}}{\gamma^j i! j! (i+j+1)}, \end{aligned}$$

since

$$\begin{aligned} q^j \left(\frac{1}{q} - 1 + j\right)_j &= q^j \left(\frac{1}{q} - 1 + j\right) \left(\frac{1}{q} - 2 + j\right) \cdots \frac{1}{q} \\ &= 1(1+q)(1+2q) \cdots (1+(j-1)q) \\ &= \prod_{m=0}^{j-1} (1+mq). \end{aligned}$$

Now by (8) and (10), we have

$$\frac{1}{\gamma} \left(1 - \frac{xqz}{\gamma}\right)^{\frac{1}{q}-1} e^z = \tilde{A}'(z) = \sum_{a \geq 1} \frac{A_a z^{a-1}}{\gamma^a (a-1)!}.$$

So that the expression for $\tilde{B}'(z)$ in equation (12)

$$\tilde{B}'(z) = \frac{q}{\gamma^2} \left(1 - \frac{xqz}{\gamma}\right)^{\frac{1}{q}-1} e^z \int_0^z \left(1 - \frac{xqt}{\gamma}\right)^{-\frac{1}{q}} e^{t(q-1)} dt$$

becomes

$$\begin{aligned} \sum_{a \geq 1} \frac{B_a z^{a-1}}{\gamma^a (a-1)!} &= \frac{q}{\gamma} \sum_{a \geq 1} \frac{A_a z^{a-1}}{\gamma^a (a-1)!} \sum_{i,j \geq 0} \frac{(q-1)^i \prod_{m=0}^{j-1} (1+mq) x^j z^{i+j+1}}{\gamma^j i! j! (i+j+1)} \\ &= \frac{q}{\gamma} \sum_{l,i,j \geq 0} \frac{A_l (q-1)^i \prod_{m=0}^{j-1} (1+mq) x^j}{\gamma^{j+l} (l-1)! i! j! (i+j+1)} z^{i+j+l}, \end{aligned}$$

where we define $A_0 := 0$. Setting $i + j + l = a - 1$ and comparing the coefficient of z^{a-1} we have,

$$\begin{aligned} B_a &= q\gamma^{a-1} \sum_{l=0}^{a-1} \sum_{j=0}^{a-1-l} \frac{A_l (q-1)^{a-1-j-l} (a-1)! \prod_{m=0}^{j-1} (1+mq) x^j}{\gamma^{j+l} (l-1)! j! (a-1-j-l)! (a-l)} \\ &= q\gamma^{a-1} \sum_{l=0}^{a-1} \sum_{j=0}^{a-1-l} \frac{l}{a-l} \binom{a-1}{l, j, a-1-j-l} \frac{A_l x^j (q-1)^{a-1-j-l}}{\gamma^{l+j}} \prod_{m=0}^{j-1} (1+mq), \end{aligned}$$

where $\binom{n}{a,b,c} := \frac{n!}{a!b!c!}$ and $a + b + c = n$. □

Now recall, equation (7),

$$C_k = \frac{1 + \frac{x(q-1)}{\gamma} + x(q-1) \sum_{a=2}^k \left[\frac{A_a}{\gamma^a} + \frac{B_a}{\gamma^{a-1}} \right]}{1 - \frac{x}{\gamma} \sum_{a=1}^k \frac{A_a}{\gamma^{a-1}}},$$

which can be formulated knowing the results for A_a and B_a from the two Lemmas 2.1 and 2.2. Thus we have the following result.

Theorem 2.3. *The generating function for words on an alphabet $[k]$ counting the number of lit cells where x marks the size of the word and q marks the number of lit cells is*

$$C_k = \frac{1 + \frac{x(q-1)}{\gamma} + x(q-1) \sum_{a=2}^k \left[\frac{A_a}{\gamma^a} + \frac{B_a}{\gamma^{a-1}} \right]}{1 - \frac{x}{\gamma} \sum_{a=1}^k \frac{A_a}{\gamma^{a-1}}},$$

where

$$A_a = \gamma^{a-1} \sum_{j=0}^{a-1} \left(\frac{-x}{\gamma} \right)^j \prod_{m=0}^j (1 - mq) \binom{a-1}{j},$$

$$B_a = q\gamma^{a-1} \sum_{l=0}^{a-1} \sum_{j=0}^{a-1-l} \frac{l}{a-l} \binom{a-1}{l, j, a-1-j-l} \frac{A_l x^j (q-1)^{a-1-j-l}}{\gamma^{l+j}} \prod_{m=0}^{j-1} (1 + mq).$$

and $\gamma = 1 + x - xq$.

3. Expected number of lit squares

To find the expected number of lit cells in a word of size n , we use the generating function C_k for k -ary words (i.e., words on an alphabet $[k]$) found in Section 2. Using the standard technique, differentiating (7) and putting $q = 1$ gives

$$\begin{aligned} & \left. \frac{\partial C_k}{\partial q} \right|_{q=1} \\ (13) \quad &= \frac{\left[x + x \sum_{a=2}^k \left(\frac{A_a}{\gamma^a} + \frac{B_a}{\gamma^{a-1}} \right) \right] \Big|_{q=1} \left(1 - \frac{x}{\gamma} \sum_{a=1}^k \frac{A_a}{\gamma^{a-1}} \right) \Big|_{q=1} + x \left(\sum_{a=1}^k \frac{A_a}{\gamma^a} \right)' \Big|_{q=1}}{\left(1 - \frac{x}{\gamma} \sum_{a=1}^k \frac{A_a}{\gamma^{a-1}} \right)^2 \Big|_{q=1}}. \end{aligned}$$

In order to find $A_a|_{q=1}$ we use Lemma 2.1 and put $q = 1$ which implies $\gamma = 1 + x - xq = 1$. Thus

$$\tilde{A}'(z) \Big|_{q=1} = \frac{1}{\gamma} \left(1 - \frac{xqz}{\gamma} \right)^{\frac{1}{q}-1} e^z \Big|_{q=1} = e^z.$$

Since $\tilde{A}'(z) := \sum_{a \geq 1} \frac{A_a z^{a-1}}{\gamma^a (a-1)!}$, and $e^z = \sum_{j \geq 0} \frac{z^j}{j!}$, this implies that $A_a|_{q=1} = 1$.

Now for $B_a|_{q=1}$, we use (12), where $\tilde{B}(z) := \sum_{a \geq 0} \frac{B_a z^a}{\gamma^a a!}$.

$$\begin{aligned} \tilde{B}'(z) \Big|_{q=1} &= e^z \int_0^z \frac{1}{1-xt} dt = -\frac{e^z}{x} [\ln(1-xt)]_0^z \\ &= -\frac{e^z}{x} \ln(1-xz) = \frac{1}{x} \sum_{i \geq 0} \sum_{j \geq 1} \frac{z^i}{i! j} (xz)^j = \sum_{i, j \geq 0} \frac{x^j}{(j+1)!} z^{i+j+1}. \end{aligned}$$

Comparing the coefficient of z^{a-1} yields

$$(14) \quad B_a|_{q=1} = (a-1)! \sum_{j=0}^{a-2} \frac{x^j}{(j+1)(a-2-j)!} = \sum_{j=0}^{a-2} j! \binom{a-1}{j+1} x^j$$

with

$$x \sum_{a=2}^k \frac{A_a}{\gamma^a} \Big|_{q=1} = (k-1)x$$

and

$$\left(1 - \frac{x}{\gamma} \sum_{a=1}^k \frac{A_a}{\gamma^{a-1}}\right)^2 \Big|_{q=1} = (1-kx)^2.$$

Now we know from (8),

$$(15) \quad \sum_{a \geq 1} \frac{A_a z^{a-1}}{\gamma^a (a-1)!} = \frac{1}{\gamma} \left(1 - \frac{xqz}{\gamma}\right)^{\frac{1}{q}-1} e^z,$$

and therefore differentiating (15) with respect to q and putting $q = 1$ gives

$$\begin{aligned} \sum_{a \geq 1} \left(\frac{A_a}{\gamma^a}\right)' \Big|_{q=1} \frac{z^{a-1}}{(a-1)!} &= x e^z + e^z \left[\left(1 - \frac{xqz}{\gamma}\right)^{\frac{1}{q}-1} \right]' \Big|_{q=1} \\ &= x e^z - \ln(1-xz) e^z. \end{aligned}$$

So

$$[x - \ln(1-xz)] e^z = \left(x + \sum_{j \geq 0} \frac{x^{j+1}}{j+1}\right) \sum_{i \geq 0} \frac{z^i}{i!}.$$

We also need $\sum_{a=1}^k \left(\frac{A_a}{\gamma^a}\right)' \Big|_{q=1}$. We use (11), which yields

$$(16) \quad \frac{A_a}{\gamma^a} = \frac{(a-1)!}{\gamma} \sum_{j=0}^{a-1} \frac{(-1)^j x^j \prod_{m=0}^j (1-mq)}{j!(a-1-j)! \gamma^j}.$$

We first differentiate the terms that contain any q , (recall that $\gamma = 1+x-xq$) and then put $q = 1$ to obtain

$$\frac{\partial \prod_{m=0}^j (1-mq)}{(1+x-xq)^{j+1}} \Big|_{q=1} = \begin{cases} (-1)^j (j-1)!, & \text{for } j \geq 1 \\ x, & \text{for } j = 0 \end{cases}.$$

Thus

$$\begin{aligned} \sum_{a=1}^k \left(\frac{A_a}{\gamma^a}\right)' \Big|_{q=1} &= \sum_{a=1}^k x + \sum_{a=1}^k (a-1)! \sum_{j=1}^{a-1} \frac{x^j (j-1)!}{j!(a-1-j)!} \\ &= kx + \sum_{a=1}^k \sum_{j=1}^{a-1} \binom{a-1}{j} (j-1)! x^j. \end{aligned}$$

Now finally we need

$$\begin{aligned}
 \sum_{a=2}^k \frac{B_a}{\gamma^{a-1}} \Big|_{q=1} &= \sum_{a=2}^k \sum_{j=0}^{a-2} j! \binom{a-1}{j+1} x^j \text{ from (14)} \\
 &= 0! \binom{1}{1} x^0 \\
 &\quad + 0! \binom{2}{1} x^0 + 1! \binom{2}{2} x^1 + \dots \\
 &\quad + 0! \binom{k-1}{1} x^0 + 1! \binom{k-1}{2} x^1 + \dots + (k-2)! \binom{k-1}{k-1} x^{k-2} \\
 &= 0! \binom{k}{2} x^0 + 1! \binom{k}{3} x^1 + 2! \binom{k}{4} x^2 + \dots + (k-2)! \binom{k}{k} x^{k-2} \\
 &= \sum_{i=2}^k \binom{k}{i} (i-2)! x^{i-2}.
 \end{aligned}$$

Thus using (13) we have the following result.

Theorem 2.4. *The generating function $\frac{\partial C_k}{\partial q} \Big|_{q=1}$ is given by*

$$\frac{x}{1-kx} \left(k + \sum_{i=2}^k \binom{k}{i} (i-2)! x^{i-2} \right) + \frac{x}{(1-kx)^2} \sum_{a=1}^k \left(x + \sum_{j=1}^{a-1} \binom{a-1}{j} (j-1)! x^j \right).$$

We write expressions for $\frac{\partial C_k}{\partial q} \Big|_{q=1}$ for $k = 1$ to 8 in the table below:

k	$\frac{\partial C_k}{\partial q} \Big _{q=1}$
1	$\frac{x}{(1-x)^2}$
2	$\frac{3x(1-x)}{(1-2x)^2}$
3	$\frac{x(6-11x-2x^2)}{(1-3x)^2}$
4	$\frac{2x(5-13x-5x^2-3x^3)}{(1-4x)^2}$
5	$\frac{x(15-50x-30x^2-34x^3-24x^4)}{(1-5x)^2}$
6	$\frac{x(21-85x-70x^2-114x^3-156x^4-120x^5)}{(1-6x)^2}$
7	$\frac{x(28-133x-140x^2-294x^3-588x^4-888x^5-720x^6)}{(1-7x)^2}$
8	$\frac{4x(9-49x-63x^2-161x^3-420x^4-936x^5-1500x^6-1260x^7)}{(1-8x)^2}$

We now extract the coefficients $[x^n]$ for $n \geq k - 1$ from $\frac{\partial C_k}{\partial q} \Big|_{q=1}$.

$$\begin{aligned} [x^n] \frac{\partial C_k}{\partial q} \Big|_{q=1} &= \sum_{i=2}^k \binom{k}{i} (i-2)! k^{n+1-i} + (n+k-1)k^{n-1} \\ &\quad + \sum_{a=1}^k \sum_{i=1}^{a-1} (n-i) \binom{a-1}{i} (i-1)! k^{n-1-i}. \end{aligned}$$

Interchanging the sum in the second term gives

$$\begin{aligned} [x^n] \frac{\partial C_k}{\partial q} \Big|_{q=1} &= \sum_{i=2}^k \binom{k}{i} (i-2)! k^{n+1-i} + (n+k-1)k^{n-1} \\ &\quad + \sum_{i=1}^{k-1} \sum_{a=i+1}^k (n-i) \binom{a-1}{i} (i-1)! k^{n-1-i} \\ &= k^n \left(\sum_{i=2}^k \binom{k}{i} (i-2)! k^{1-i} + (n+k-1)k^{-1} + \sum_{i=1}^{k-1} \frac{(i-k)(i-n) \binom{k}{i} (-1+i)!}{k^{1+i}(1+i)} \right). \end{aligned}$$

This finally yields the following result.

Theorem 2.5. *The total number of lit cells in words of size n is*

$$\begin{aligned} [x^n] \frac{\partial C_k}{\partial q} \Big|_{q=1} &= k^n \left(\sum_{i=2}^k \binom{k}{i} (i-2)! k^{1-i} + (n+k-1)k^{-1} + \sum_{i=1}^{k-1} \frac{(i-k)(i-n) \binom{k}{i} (-1+i)!}{k^{1+i}(1+i)} \right). \end{aligned}$$

After dividing by k^n , we obtain the average number of lit cells to be

Theorem 2.6. *As $n \rightarrow \infty$, the average number of lit cells is asymptotic to $c_k n$ where*

$$c_k = \frac{1}{k} + \frac{1}{k} \sum_{j=1}^{k-1} \frac{(k-j) \binom{k}{j} (-1+j)!}{k^j (1+j)}.$$

Values for c_k are shown below in an exact form and in a decimal form:

k	c_k	c_k in decimal form
1	1	1
2	$\frac{3}{4}$	0.75
3	$\frac{19}{27}$	0.7037
4	$\frac{89}{128}$	0.6953
5	$\frac{2181}{3125}$	0.6979
10	$\frac{11468413}{15625000}$	0.7340
100	—	0.8879
1000	—	0.9617

Corollary 2.7. *As $k \rightarrow \infty$ we have $c_k \rightarrow 1$, thus we have on average one lit cell per column.*

Proof. The terms in the sum of the expression for c_k in Theorem 2.6 simplify to

$$(17) \quad \frac{1}{k(k+1)} \frac{n!}{(n-k-1)!n^{k+1}}.$$

For $k = O(n^{2/3})$ we apply Stirling's formula $j! = \sqrt{2\pi j} \frac{j^j}{e^j} \left(1 + O\left(\frac{1}{j}\right)\right)$, as $j \rightarrow \infty$ with $j = n$ and $j = n - k$ to obtain for (17)

$$\frac{e^{-1-k} n^{-\frac{1}{2}-k+n} (n-k-1)^{\frac{1}{2}+k-n}}{k(k+1)} \left(1 + O\left(\frac{1}{n}\right)\right) = \frac{1}{k(k+1)} + O\left(\frac{1}{n}\right).$$

The contribution of the terms in (17) where $k > N^{\frac{2}{3}}$ are exponentially small. Therefore the sum is asymptotic to $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} + o(1) = 1 + o(1)$. \square

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