

ON COMPARISON OF ANNULI CONTAINING ALL THE ZEROS OF A POLYNOMIAL

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There are many theorems providing annulus containing all the zeros of a polynomial, and it is known that two such theorems cannot be compared, in the sense that one can always find a polynomial for which one theorem gives a sharper bound than the other. It is natural to ask if there is a class of polynomials for which such comparison is possible and in this paper we investigate this problem and provide results which for polynomials with some condition on the degree or absolute range of coefficients, enable us to compare two such theorems.

1. INTRODUCTION

The study of polynomials and their location of zeros has been going on, since the time of Guass and Cauchy (see [17, 18, 22]), and find important applications in many areas of applied mathematics such as control theory, signal processing, communication theory, coding theory, cryptography, combinatorics, and mathematical biology. The amount of work needed to find exact zeros can be considerably reduced, if there is an accurate estimate of the annulus containing all the zeros of a polynomial, and this has been one of the reasons motivating mathematicians to look for better and better estimates for the region containing all the zeros of a polynomial.

It may be remarked that there are methods, for example, Ehrlich-Aberth's type (see [1, 13, 20]) for the simultaneous determination of the zeros of algebraic polynomials, and there are studies to accelerate convergence and increase computational efficiency of these methods (for example, see [19, 21]). These methods,

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2010 Mathematics Subject Classification. 30C15, 65F15, 30E10.

Keywords and Phrases. location of the zeros of a polynomial, Inequalities, eigenvalues.

which are of course very useful, because of their giving approximations to the zeros of a polynomial, can possibly become more efficient when combined with the results that provide annulus containing all the zeros of a polynomial.

Gauss was probably the earliest contributor, who showed that a polynomial

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0,$$

with all a_k real, has no zeros outside certain circle $|z| = R$, where

$$R = \max_{1 \leq k \leq n} (n 2^{1/2} |a_k|)^{1/k}.$$

Cauchy [4] improved the above result of Gauss by proving

Theorem 1.1. *Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$, be a complex polynomial. Then all the zeros of $p(z)$ lie in the disc*

$$\{z : |z| < \eta\} \subset \{z : |z| < 1 + A\},$$

where

$$A = \max_{0 \leq k \leq n-1} |a_k|,$$

and η is the unique positive root of the real-coefficient equation

$$z^n - |a_{n-1}|z^{n-1} - |a_{n-2}|z^{n-2} - \dots - |a_1|z - |a_0| = 0.$$

If one applies the above theorem to the polynomial $P(z) = z^n p(1/z)$ and combine it with the above theorem, one easily gets

Theorem 1.2 (Cauchy). *All the zeros of the polynomial $p(z) = a_0 + a_1 z + \dots + a_n z^n$, $a_n \neq 0$, lie in the annulus $r_1 \leq |z| \leq r_2$, where r_1 is the unique positive root of the equation*

$$(1.1) \quad |a_n|z^n + |a_{n-1}|z^{n-1} + \dots + |a_1|z - |a_0| = 0,$$

and r_2 is the unique positive root of the equation

$$(1.2) \quad |a_n| + |a_{n-1}|z + \dots + |a_1|z^{n-1} - |a_0|z^n = 0.$$

The above result of Cauchy has been sharpened among others by Affane-Aji et al. [2], Datt and Govil [7], Dehmer [8, 9], Jain [14], Joyal et al. [15], and Sun and Hsieh [24].

Although the above result of Cauchy gives an annulus containing all the zeros of a polynomial, it is implicit, in the sense, that in order to find the annulus containing all the zeros of a polynomial, one needs to compute the zeros of two other polynomials.

The following result by Diaz-Barrero [10] provides an annulus containing all the zeros of a polynomial, which is explicit in the sense that the annulus containing all the zeros can be obtained by using the polynomial coefficients only.

Theorem 1.3. Let $p(z) = \sum_{k=0}^n a_k z^k$ ($a_k \neq 0, 0 \leq k \leq n$) be a non-constant complex polynomial. Then all its zeros lie in the annulus $\mathcal{C} = \{z : r_1 \leq |z| \leq r_2\}$, where

$$(1.3) \quad r_1 = \frac{3}{2} \min_{1 \leq k \leq n} \left\{ \frac{\binom{n}{k} 2^n F_k}{F_{4n}} \left| \frac{a_0}{a_k} \right| \right\}^{1/k}$$

and

$$(1.4) \quad r_2 = \frac{2}{3} \max_{1 \leq k \leq n} \left\{ \frac{F_{4n}}{\binom{n}{k} 2^n F_k} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{1/k}.$$

Here F_k is the k^{th} Fibonacci number, namely, $F_0 = 0, F_1 = 1$ and for $k \geq 2, F_k = F_{k-1} + F_{k-2}$.

Another result in this direction providing annulus containing all the zeros of a polynomial is the following, and is due to Kim [16].

Theorem 1.4. Let $p(z) = \sum_{k=0}^n a_k z^k$ ($a_k \neq 0, 1 \leq k \leq n$) be a non constant polynomial with complex coefficients. Then all the zeros of $p(z)$ lie in the annulus $A = \{z : r_1 \leq |z| \leq r_2\}$ where,

$$(1.5) \quad r_1 = \min_{1 \leq k \leq n} \left\{ \frac{C(n, k)}{2^n - 1} \left| \frac{a_0}{a_k} \right| \right\}^{1/k}$$

and

$$(1.6) \quad r_2 = \max_{1 \leq k \leq n} \left\{ \frac{2^n - 1}{C(n, k)} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{1/k}.$$

Here, $C(n, k)$ are the binomial coefficients.

There are many results, including the above Theorems 1.3 and 1.4, available in this direction (for example, see [3, 5, 10, 11, 12, 16, 23]).

Dalal and Govil [5] proved the following which generalizes all the above results.

Theorem 1.5. Let $\sum_{k=1}^n A_k = 1$ be an identity with $A_k > 0$ for $1 \leq k \leq n$, and let $p(z) = \sum_{k=0}^n a_k z^k$ ($a_k \neq 0, 1 \leq k \leq n$) be a non constant polynomial with complex coefficients. Then all the zeros of $p(z)$ lie in the annulus $\mathcal{C} = \{z : r_1 \leq |z| \leq r_2\}$, where

$$r_1 = \min_{1 \leq k \leq n} \left\{ A_k \left| \frac{a_0}{a_k} \right| \right\}^{1/k}$$

and

$$r_2 = \max_{1 \leq k \leq n} \left\{ \frac{1}{A_k} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{1/k}.$$

Theorem 1.5 is capable of generating infinitely many results providing annulus containing all the zeros of a polynomial including Theorems 1.3 and 1.4, and over the years, mathematicians have compared their bounds with the existing bounds in the literature by generating some polynomials and showing that for those polynomials their bound is better than the bound obtained from some of the known results. In this regard, Dalal and Govil [6] by proving following two theorems showed that no-matter what results one obtains as a corollary of Theorem 1.5, one can always generate polynomials for which ones bound is better than the existing ones

Theorem 1.6. *Let $\{A_k\}_{k=1}^n$ and $\{B_k\}_{k=1}^n$ be sequences of positive numbers such that $\sum_{k=1}^n A_k = 1$ and $\sum_{k=1}^n B_k = 1$. Then, there always exists a polynomial for which $r_1^A > r_1^B$ and vice-versa, where r_1^A and r_1^B are the inner radii of the annulus obtained from the Theorem 1.5 by using the sequences $\{A_k\}_{k=1}^n$ and $\{B_k\}_{k=1}^n$ respectively.*

Theorem 1.7. *Let $\{A_k\}_{k=1}^n$ and $\{B_k\}_{k=1}^n$ be sequences of positive numbers such that $\sum_{k=1}^n A_k = 1$ and $\sum_{k=1}^n B_k = 1$. Then, there always exists a polynomial for which $r_2^A < r_2^B$ and vice-versa, where r_2^A and r_2^B are the outer radii of the annulus obtained from the Theorem 1.5 by using sequences $\{A_k\}_{k=1}^n$ and $\{B_k\}_{k=1}^n$ respectively.*

Since the above theorems of Dalal and Govil [6] imply that the bounds computed by substituting different $\{A_k\}_{k=1}^n$'s in the Theorem 1.5 cannot be in general compared hence a natural question arises if there is a class of polynomials for which the bounds obtained by two different theorems can be compared.

Consider $p(z) = \sum_{k=0}^n a_k z^k$ ($a_k \neq 0, 1 \leq k \leq n$), a non-constant polynomial with complex coefficients, such that $1 \leq |a_k| \leq c(n)$ for $0 \leq k \leq n$, and for some constant $c(n)$ which will depend on the degree n . We took $c(n) = 1000$ and compared the performance of Theorem 1.3 and Theorem 1.4 over polynomials of varying degree. The methodology was to compute the inner radii over 1 million random polynomials and compute the percentage of polynomials for which bound obtained by Theorem 1.3 is better than the bound obtained by Theorem 1.4. See Figure 1 for the results.

Now in the Figure 1, when $n \geq 6$ then for the polynomials considered, the bound of Theorem 1.4 is always better than the bound of Theorem 1.3. We got similar plots when we compared the bounds given by other theorems in the literature (We omit to put other such plots here). This gives a motivation to search for a result that can compare bounds for polynomials over varying degrees and absolute range of coefficients.

In the next section we provide results which will help to compare the bounds such that, with some conditions on degree or absolute range of coefficients, the bound obtained by one theorem is always better than the bound obtained by the other. By using some computational analysis, in Section 3 we describe some applications of the results obtained in Section 2.

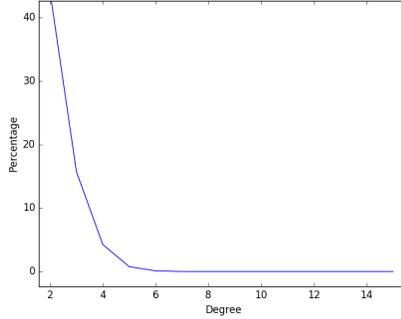


Figure 1: Comparison of Theorem 1.3 with Theorem 1.4

2. MAIN RESULTS

We begin with the following

Theorem 2.8. *Let $\{A_k\}_{k=1}^n$ and $\{B_k\}_{k=1}^n$ be sequences of positive numbers, such that $\sum_{k=1}^n A_k = 1$ and $\sum_{k=1}^n B_k = 1$. Let r_A and r'_A be inner and outer radii of the annulus in Theorem A obtained by substituting $\{A_k\}_{k=1}^n$ in Theorem 1.5. Let r_B and r'_B be inner and outer radii of the annulus in Theorem B obtained by substituting $\{B_k\}_{k=1}^n$ in Theorem 1.5. Let n be a fixed positive integer, and let $f_A(n) = \min_{1 \leq k \leq n} \{A_k\}^{1/k}$ and $f_B(n) = \min_{1 \leq k \leq n} \{B_k\}^{1/k}$. If $f_B(n) > f_A(n)$, then for all non-constant polynomials $p(z) = \sum_{k=0}^n a_k z^k$ ($a_k \neq 0, 0 \leq k \leq n$), with complex coefficients $1 \leq |a_k| < c$ where $c = \left(\frac{f_B(n)}{f_A(n)}\right)^{1/2}$, Theorem B always gives better bound than Theorem A. In other words, for any polynomial satisfying this condition $r_B > r_A$ and $r'_B < r'_A$.*

Proof. For a fixed n , let i be the value of k for which $\min_{1 \leq k \leq n} \{A_k\}^{1/k} = \{A_i\}^{1/i}$, and this we denote by $f_A(n)$. Similarly, let j be the value of k for which $\min_{1 \leq k \leq n} \{B_k\}^{1/k} = \{B_j\}^{1/j}$, and this we denote by $f_B(n)$, where by hypothesis $f_B(n) > f_A(n)$.

Thus $r_i = \{A_i\}^{1/i} \left| \frac{a_0}{a_i} \right|^{1/i} = f_A(n) \left| \frac{a_0}{a_i} \right|^{1/i}$, because $\{A_i\}^{1/i} = \min_{1 \leq k \leq n} \{A_k\}^{1/k} = f_A(n)$.

Also, the hypothesis $1 \leq |a_k| < c$ for $0 \leq k \leq n$, implies $\frac{1}{c} < \left| \frac{a_0}{a_k} \right| \leq |a_0| < c$, for $0 \leq k \leq n$, which in particular implies $\left| \frac{a_0}{a_i} \right|^{1/i} < c^{1/i}$. Therefore

$$r_i = \{A_i\}^{1/i} \left| \frac{a_0}{a_i} \right|^{1/i} < f_A(n) c^{1/i} < f_A(n) c, \text{ because } c > 1.$$

Since the inner radius $r_A = \min_{1 \leq k \leq n} \{A_k\}^{1/k} \left| \frac{a_0}{a_k} \right|^{1/k} \leq r_i$, the above gives

$$(2.7) \quad r_A < f_A(n) c.$$

Next we show that $r_B > f_B(n)\frac{1}{c}$, and for this note that $1 \leq |a_k| < c$ for $0 \leq k \leq n$, implies $\left|\frac{a_0}{a_k}\right| > \frac{1}{c}$ for $0 \leq k \leq n$. Therefore, $\{B_j\}^{1/j} \left|\frac{a_0}{a_j}\right|^{1/j} > f_B(n) \left(\frac{1}{c}\right)^{1/j} > f_B(n)\frac{1}{c}$, because $\frac{1}{c} < 1$.

Now, since $B_k^{1/k} \geq B_j^{1/j} = f_B(n)$ and $\left|\frac{a_0}{a_k}\right|^{1/k} > \frac{1}{c}$ for $1 \leq k \leq n$, we get $\{B_k\}^{1/k} \left|\frac{a_0}{a_k}\right|^{1/k} > f_B(n)\frac{1}{c}$ for $1 \leq k \leq n$. Hence,

$$(2.8) \quad r_B = \min_{1 \leq k \leq n} \{B_k\}^{1/k} \left|\frac{a_0}{a_k}\right|^{1/k} > f_B(n)\frac{1}{c}.$$

Note that the hypothesis $f_B(n) > f_A(n)$ implies $c = \left(\frac{f_B(n)}{f_A(n)}\right)^{1/2} > 1$ and $f_B(n)\frac{1}{c} = f_A(n)c$, and this when combined with (2.7) and (2.8) gives

$$(2.9) \quad r_B > f_B(n)\frac{1}{c} = f_A(n)c > r_A.$$

The above shows that if $f_B(n) > f_A(n)$, then $r_B > r_A$, and the proof of the first part of Theorem 2.8 is thus complete.

Now we proceed to show that $r'_B < r'_A$, where r'_A is the outer radii of the annulus in Theorem A obtained by substituting $\{A_k\}_{k=1}^n$ in Theorem 1.5, and r'_B the outer radii of the annulus in Theorem B obtained by substituting $\{B_k\}_{k=1}^n$ in Theorem 1.5.

In order to prove this, for any polynomial $p(z) = \sum_{k=0}^n a_k z^k$ satisfying the hypothesis of Theorem 2.8 we consider the polynomial $P(z) = z^n p(1/z) = \sum_{k=0}^n a_{n-k} z^k$, and apply to this polynomial the result proved in (2.9). Let r_C be the inner bound obtained for the polynomial $P(z)$ obtained by Theorem A and r_D the inner bound obtained for the polynomial $P(z)$ from Theorem B. Note that by hypothesis $1 \leq |a_k| < c$ for $0 \leq k \leq n$ therefore the coefficients a_{n-k} of the polynomial $P(z) = z^n p(1/z) = \sum_{k=0}^n a_{n-k} z^k$ also satisfy this condition, that is $1 \leq |a_{n-k}| < c$ as the coefficients of $P(z)$ are just the rearrangement of coefficients of polynomials $p(z)$, which satisfy this condition. Also, $f_B(n) > f_A(n)$ because $f_A(n)$ and $f_B(n)$ depend only on $\{A_k\}_{k=1}^n$ and $\{B_k\}_{k=1}^n$ respectively and not on the polynomial coefficients. Thus the polynomial $P(z)$ satisfies the hypotheses of Theorem 2.8 and therefore by the above proved part of this theorem we get

$$(2.10) \quad r_D > r_C.$$

Now, note that

$$(2.11) \quad r_C = \min_{1 \leq k \leq n} \{A_k\}^{1/k} \left|\frac{a_n}{a_{n-k}}\right|^{1/k} = \frac{1}{\max_{1 \leq k \leq n} \left\{\frac{1}{A_k}\right\}^{1/k} \left|\frac{a_{n-k}}{a_n}\right|^{1/k}} = \frac{1}{r'_A}$$

and

(2.12)

$$r_D = \min_{1 \leq k \leq n} \{B_k\}^{1/k} \left| \frac{a_n}{a_{n-k}} \right|^{1/k} = \frac{1}{\max_{1 \leq k \leq n} \left\{ \frac{1}{B_k} \right\}^{1/k} \left| \frac{a_{n-k}}{a_n} \right|^{1/k}} = \frac{1}{r'_B}.$$

If we combine (2.10), (2.11) and (2.12) we get $r'_B < r'_A$, and this when combined with (2.9) completes the proof of Theorem 2.8. \square

Remark: From the above proof, one can see that a more precise expression for $C(n)$ would be $\left(\frac{f_B(n)}{f_A(n)} \right)^{\frac{ij}{i+j}}$, where i and j are as defined in the proof of Theorem 2.1. It is clear that $\left(\frac{f_B(n)}{f_A(n)} \right)^{\frac{ij}{i+j}} \geq \left(\frac{f_B(n)}{f_A(n)} \right)^{\frac{1}{2}}$, and therefore using $\left(\frac{f_B(n)}{f_A(n)} \right)^{\frac{ij}{i+j}}$ instead of $\left(\frac{f_B(n)}{f_A(n)} \right)^{\frac{1}{2}}$ will obviously give a result that would be better than Theorem 2.8. We did not use this expression in order to keep the statement and proof of our theorem simple.

As a corollary of the above result we get

Corollary 2.9. *Let $\{A_k\}_{k=1}^n$ and $\{B_k\}_{k=1}^n$ be sequences of positive numbers such that $\sum_{k=1}^n A_k = 1$ and $\sum_{k=1}^n B_k = 1$. Let r_A and r'_A be inner and outer bound of Theorem A obtained by substituting $\{A_k\}_{k=1}^n$ in Theorem 1.5, and let r_B and r'_B be inner and outer bound of Theorem B obtained by substituting $\{B_k\}_{k=1}^n$ in Theorem 1.5. Let n be a fixed positive integer, and $c > 1$. If $f_A(n) = \min_{1 \leq k \leq n} \{A_k\}^{1/k}$, $f_B(n) = \min_{1 \leq k \leq n} \{B_k\}^{1/k}$, and if $\frac{f_B(n)}{f_A(n)}$ is an unbounded and increasing function of n , then there exists an integer n_0 such that for all $p(z) = \sum_{k=0}^n a_k z^k$ ($a_k \neq 0, 0 \leq k \leq n$) with degree $n \geq n_0$ and coefficients a_k satisfying $1 \leq |a_k| < c$, Theorem B always gives better bound than Theorem A.*

Proof. We define a function $C(n) = \left(\frac{f_B(n)}{f_A(n)} \right)^{1/2}$ for each n . Since it is given that $C(n)$ is an unbounded and increasing function of n , so we can find n_0 such that $C(n) > c > 1$, if $n \geq n_0$. Since $c > 1$ hence if the degree n of the polynomial satisfies $n \geq n_0$ then $f_B(n) > f_A(n)$, and also if the coefficients of the polynomial satisfy $1 \leq |a_k| < c$ then by Theorem 2.8, Theorem B will always give a better bound than Theorem A. \square

Theorem 2.8 and Corollary 2.9 empower us to compare the bounds of Theorem A and Theorem B in the following manner:

1. If S is the class of polynomials of degree n , then Theorem 2.8 finds a constant c , such that for all polynomials in S , the absolute value of whose coefficients are less than c , the Theorem B will always give a bound better than obtainable from Theorem A.

2. If S' is the class of polynomials, the absolute value of whose coefficients are less than a constant c , then Corollary 2.9 gives an integer n_0 such that for all polynomials in S' with degree $n \geq n_0$, Theorem B will always give better bound than Theorem A.

3. COMPUTATIONS

In this section, we delineate the utility of Theorem 2.8 and Corollary 2.9 by comparing the bounds obtained from Theorem 1.3 and Theorem 1.4.

If in Theorem 2.8, for Theorem A we take Theorem 1.3, and for Theorem B we take Theorem 1.4, then clearly $A_k = \frac{C(n,k)2^n (\frac{3}{2})^k}{F_{4n}}$ and $B_k = \frac{C(n,k)}{2^n - 1}$, which easily gives $f_A(n) = \frac{1.5n2^n}{F_{4n}}$ for $n \geq 1$, and $f_B(n) = \frac{n}{2^n - 1}$ for $n \geq 3$. The above implies that $C(n) = \left(\frac{f_B(n)}{f_A(n)}\right)^{1/2} = \left(\frac{F_{4n}}{1.52^n(2^n - 1)}\right)^{1/2}$ for $n \geq 3$, which is an increasing and unbounded function of n . Therefore for any c there is a positive integer n_0 such that for all polynomials of degree $n \geq n_0$ and coefficients satisfying $1 \leq |a_k| < c$ for $0 \leq k \leq n$, the Theorem 1.4 will give a sharper bound than obtainable from Theorem 1.3. In Table 1 given below we have calculated values of n_0 for the corresponding value of $c = C(n)$.

n	$C(n)$
5	2.13
10	8.07
15	31.00
20	119.15
25	457.97
30	1760.23
35	6756.45
40	26002.98
45	99942.67
50	384128.12
55	1476394.87
60	5674517.75

Now, to compare Theorem 1.3 and Theorem 1.4 we make use of the Table 1, given below and get the following:

1. One can see in Table 1 that for $n = 20$ we have $C(n) = 119.15$. Therefore, if S denotes the class of polynomials of degree 20, then by Theorem 2.8, for all polynomials belonging to S with coefficients a_k satisfying $1 \leq |a_k| < 119.15$, the Theorem 1.4 will always give better bound than Theorem 1.3.

2. If S' denotes the class of polynomials with coefficients a_k satisfying $1 \leq |a_k| < 1760$ then because $C(30) = 1760.23$, using Corollary 2.9, we get that for all polynomials of class S' and degree $n \geq 30$, Theorem 1.4 will always give a bound that is better than Theorem 1.3.

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(Received 16. 11. 2016.)

(Revised 28. 03. 2017.)

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