ABSTRACT DEGENERATE FRACTIONAL DIFFERENTIAL INCLUSIONS IN BANACH SPACES

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In the paper under review, we investigate a class of abstract degenerate fractional differential inclusions with Caputo derivatives. We consider subordinated fractional resolvent families generated by multivalued linear operators, which do have removable singularities at the origin. Semi-linear degenerate fractional Cauchy problems are also considered in this context.

1. INTRODUCTION AND PRELIMINARIES

The main purpose of this paper is to consider a class of abstract degenerate fractional differential inclusions with multivalued linear operators satisfying the following condition

(QP): There exist finite numbers $0 < \beta \leq 1$, $0 < d \leq 1$, $M > 0$ and $0 < \eta' < \eta'' \leq 1$ such that

$$\Psi_{d,\pi\eta''/2} := \left\{ \lambda \in \mathbb{C} : |\lambda| \leq d \text{ or } \lambda \in \overline{\Sigma_{\pi\eta''/2}} \right\} \subseteq \rho(\mathcal{A})$$

and

$$\|R(\lambda : \mathcal{A})\| \leq M\big(1 + |\lambda|\big)^{-\beta}, \quad \lambda \in \Psi_{d,\pi\eta''/2}.$$

Thus we continue our previous research study [**19**], where we have recently considered fractional resolvent families subordinated to infinitely differentible semi-groups generated by the multivalued linear operators satisfying the following condition (cf. Chapter III of the monograph [**11**] by A. Favini and A. Yagi for more details):

(P) There exist finite constants c, $M > 0$ and $\beta \in (0, 1]$ such that

$$\Psi := \left\{ \lambda \in \mathbb{C} : \Re\lambda \geq -c\big(|\Im\lambda| + 1\big) \right\} \subseteq \rho(\mathcal{A})$$

and

$$\|R(\lambda : \mathcal{A})\| \leq M\big(1 + |\lambda|\big)^{-\beta}, \quad \lambda \in \Psi.$$

In both cases, Yosida approximations cannot be essentially emloyed in the analysis of abstract degenerate fractional differential inclusions under our consideration (cf. [11, p. 52]). Furthermore, a great number of results presented in [11, Section 3.2-Section 3.5] and the recent papers [8]-[10] is not attainable in fractional case.

The most important novelty of this paper lies in the fact that the resolvent set of a multivalued linear operator $\mathcal{A}$ satisfying (QP) can be strictly contained in an acute angle. In connection with this, it should be observed that the established results seem to be completely new even for non-degenerate fractional differential equations with almost sectorial operators (cf. [7], [20], [25]-[26] and [33] for the basic source of information on the abstract differential equations with almost sectorial linear operators satisfying (QP) with some number $\eta'' > 1$).

Our abstract theoretical results can be applied in the analysis of a large class of fractional differential inclusions involving the rotations of multivalued linear operators considered in [11, Chapter III, Chapter VI]. By [25, Proposition 3.6] ([21, Corollary 5.6]), fractional powers of almost sectorial operators (sectorial multivalued linear operators) satisfy, under some assumptions, the condition (QP) and can therefore be used for providing certain applications of our results, as well. Concerning purely non-degenerate case, it should be noticed that we can apply our results in the analysis of a large class of abstract time-relaxation equations with differential operators acting in Hölder spaces ([32]) as well as in the analysis of limit problems of fractional diffusion equations in complex systems on the so-called dumbbell domains ([3], [33]). Suitable translations of generators of fractionally integrated semigroups with corresponding growth order satisfy the condition (QP) with $\eta'' = 1$ and $0 < \beta < 1$, as well (cf. [26, Example 3.3]).

The organization of paper is given as follows. In Section 2, we present a brief overview of definitions and results from the theory of multivalued linear operators. The main contributions of paper are contained in Section 3, where we transfer the assertions of [11, Theorem 3.1, Theorem 3.3, Theorem 3.5; Proposition 3.4] from semigroup case to fractional relaxation case (we have faced ourselves with some serious difficulties concerning the fractional analogue of the assertion [11, Proposition 3.2], when we are no longer in a position to conclude that the operator $T_{\eta',r+\theta}(z)$ defined below is a bounded linear section of the operator $(-\mathcal{A})^\theta T_{\eta',r}(z)$). Having this done, it is almost straightforward to extend the results from [19] concerning subordinated degenerate fractional resolvent families and semilinear fractional Cauchy inclusion

$$(\text{DFP})_{f,s,\eta} : \begin{cases} \mathbf{D}_t^\eta u(t) \in \mathcal{A}u(t) + f(t, u(t)), \ t \in (0, T], \\ u(0) = u_0, \end{cases}$$

where $0 < T < \infty$, $0 < \eta < \eta'$ and $\mathbf{D}_t^\eta$ denotes the Caputo fractional derivative operator of order η ([**2**]); cf. Section 4 for more details.

We assume henceforth that $(E, \| \cdot \|)$ is a complex Banach space. In the case that X is also a complex Banach space, then we denote by $L(E, X)$ the space consisting of all continuous linear mappings from E into X; $L(E) \equiv L(E, E)$. If A is a closed linear operator acting on E, then the domain, kernel space and range of A will be denoted by $D(A)$, $N(A)$ and $R(A)$, respectively. Since no confusion seems likely, we will identify A with its graph.

Given $s \in \mathbb{R}$ and $\alpha \in (0, \pi]$ in advance, set $\lceil s \rceil := \inf\{l \in \mathbb{Z} : s \le l\}$ and $\Sigma_\alpha := \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \alpha\}$. The Gamma function is denoted by $\Gamma(\cdot)$ and the principal branch is always used to take the powers; the convolution like mapping $*$ is given by $f * g(t) := \int_0^t f(t - s) g(s) \, ds$. Set $g_\zeta(t) := t^{\zeta-1}/\Gamma(\zeta)$ and $0^\zeta := 0$ ($\zeta > 0$, $t > 0$).

Suppose that $0 < \tau \le \infty$, $m \in \mathbb{N}$ and $I = (0, \tau)$. Then we define the Sobolev space $W^{m,1}(I : E)$ in the following way (see e.g. [**2**, p. 7]):

$$W^{m,1}(I : E) := \left\{ f \mid \exists \varphi \in L^1(I : E) \ \exists c_k \in \mathbb{C} \ (0 \le k \le m-1) \right.$$
$$\left. f(t) = \sum_{k=0}^{m-1} c_k g_{k+1}(t) + \big(g_m * \varphi\big)(t) \text{ for a.e. } t \in (0, \tau) \right\}.$$

Then we have $\varphi(t) = f^{(m)}(t)$ in distributional sense, and $c_k = f^{(k)}(0)$ ($0 \le k \le m-1$).

In the sequel, we will use the following special case of weighted AM-GM inequality:

(1.1) $$\theta t + (1 - \theta)s \ge t^\theta s^{1-\theta}, \quad t, \ s \ge 0, \ \theta \in (0, 1).$$

We refer the reader to [**5**], [**11**], [**22**] and [**30**] for further information about abstract degenerate differential equations with integer order derivatives. Concerning fractional calculus and fractional differential equations, we recommend for the reader [**2**], [**6**], [**13**]-[**14**] and [**27**]-[**29**].

In this paper, we deal with the Caputo fractional derivatives of order $\gamma \in (0, 1)$. Let us recall that the Caputo fractional derivative $\mathbf{D}_t^\gamma u(t)$ is defined for those functions $u : [0, T] \to E$ satisfying that $u_{|(0,T]}(\cdot) \in C((0, T] : E)$, $u(\cdot) - u(0) \in L^1((0, T) : E)$ and $g_{1-\gamma} * (u(\cdot) - u(0)) \in W^{1,1}((0, T) : E)$, by

$$\mathbf{D}_t^\gamma u(t) = \frac{d}{dt}\left[g_{1-\gamma} * \Big(u(\cdot) - u(0)\Big) \right](t), \quad t \in (0, T].$$

The Wright function $\Phi_\gamma(z)$ is defined by

$$\Phi_\gamma(z) := \sum_{n=0}^\infty \frac{(-z)^n}{n!\Gamma(1 - \gamma - n\gamma)}, \quad z \in \mathbb{C} \ (\gamma \in (0, 1)).$$

Let us recall that $\Phi_\gamma(t) \geq 0$, $t \geq 0$ and that the following identity holds:

(a1) $\int_0^\infty t^r \Phi_\gamma(t)\, dt = \frac{\Gamma(1+r)}{\Gamma(1+\gamma r)}$, $r > -1$.

       For the theory of vector-valued Laplace transform, the reader may consult [1], [34, Chapter 1], [14, Section 1.2] and [17]. In this paper, we follow the usually considered approach from [1].

       We will use the following lemma.

Lemma 1.1. *Suppose that $\mathcal{A}$ is a closed MLO, as well as that the functions $f(\cdot)$ and $l(\cdot)$ are Laplace transformable. If $(\tilde{f}(\lambda), \tilde{l}(\lambda)) \in \mathcal{A}$, $\lambda \in \mathbb{C}$ for $\Re\lambda > \max(abs(f), abs(l))$, then $Af(t) = l(t)$ for any $t > 0$ which is a point of continuity of both functions $f(t)$ and $l(t)$. Here, $\tilde{f}(\lambda) = \mathcal{L}(f)(\lambda) = \int_0^t e^{-\lambda t} f(t)\, dt$, $\lambda > abs(f)$.*

2. MULTIVALUED LINEAR OPERATORS

       The monographs [4] and [11] contain the most important information concerning multivalued linear operators and their applications to abstract degenerate differential equations. Let us recall that a multivalued mapping $\mathcal{A} : E \to P(E)$ is said to be a multivalued linear operator (MLO in E, or simply, MLO) iff the following two conditions hold:

(i) $D(\mathcal{A}) := \{x \in X : \mathcal{A}x \neq \emptyset\}$ is a linear subspace of E;

(ii) $\mathcal{A}x + \mathcal{A}y \subseteq \mathcal{A}(x + y)$, x, $y \in D(\mathcal{A})$ and $\lambda\mathcal{A}x \subseteq \mathcal{A}(\lambda x)$, $\lambda \in \mathbb{C}$, $x \in D(\mathcal{A})$.

       It is well known that, for every x, $y \in D(\mathcal{A})$ and for every λ, $\eta \in \mathbb{C}$ with $|\lambda| + |\eta| \neq 0$, we have $\lambda\mathcal{A}x + \eta\mathcal{A}y = \mathcal{A}(\lambda x + \eta y)$. Morevoer, $\mathcal{A}0$ is a linear manifold in E and $\mathcal{A}x = f + \mathcal{A}0$ for any $x \in D(\mathcal{A})$ and $f \in \mathcal{A}x$. Define $R(\mathcal{A}) := \{\mathcal{A}x : x \in D(\mathcal{A})\}$. The set $\mathcal{A}^{-1}0 := N(\mathcal{A}) := \{x \in D(\mathcal{A}) : 0 \in \mathcal{A}x\}$ is called the kernel of $\mathcal{A}$. The inverse $\mathcal{A}^{-1}$ is given by $D(\mathcal{A}^{-1}) := R(\mathcal{A})$ and $\mathcal{A}^{-1}y := \{x \in D(\mathcal{A}) : y \in \mathcal{A}x\}$. It can be easily verified that $\mathcal{A}^{-1}$ is an MLO in E, as well as that $N(\mathcal{A}^{-1}) = \mathcal{A}0$ and $(\mathcal{A}^{-1})^{-1} = \mathcal{A}$. If $N(\mathcal{A}) = \{0\}$, i.e., if $\mathcal{A}^{-1}$ is single-valued, then $\mathcal{A}$ is said to be injective.

       Suppose that $\mathcal{A}$, $\mathcal{B}$ are two MLOs in E. Then we define its sum $\mathcal{A} + \mathcal{B}$ by $D(\mathcal{A} + \mathcal{B}) := D(\mathcal{A}) \cap D(\mathcal{B})$ and $(\mathcal{A} + \mathcal{B})x := \mathcal{A}x + \mathcal{B}x$, $x \in D(\mathcal{A} + \mathcal{B})$. It is again an MLO in E. The product of $\mathcal{A}$ and $\mathcal{B}$ is defined by $D(\mathcal{B}\mathcal{A}) := \{x \in D(\mathcal{A}) : D(\mathcal{B}) \cap \mathcal{A}x \neq \emptyset\}$ and $\mathcal{B}\mathcal{A}x := \mathcal{B}(D(\mathcal{B}) \cap \mathcal{A}x)$. We have that $\mathcal{B}\mathcal{A}$ is an MLO in E and $(\mathcal{B}\mathcal{A})^{-1} = \mathcal{A}^{-1}\mathcal{B}^{-1}$. The inclusion $\mathcal{A} \subseteq \mathcal{B}$ means that $D(\mathcal{A}) \subseteq D(\mathcal{B})$ and $\mathcal{A}x \subseteq \mathcal{B}x$ for all $x \in D(\mathcal{A})$. The scalar multiplication of an MLO $\mathcal{A}$ with the number $z \in \mathbb{C}$, $z\mathcal{A}$ for short, is defined by $D(z\mathcal{A}) := D(\mathcal{A})$ and $(z\mathcal{A})(x) := z\mathcal{A}x$, $x \in D(\mathcal{A})$.

       Suppose now that a linear single-valued operator $S : D(S) \subseteq E \to E$ has domain $D(S) = D(\mathcal{A})$ and $S \subseteq \mathcal{A}$, where $\mathcal{A}$ is an MLO in E. Then S is called a section of $\mathcal{A}$. In this case, the equalities $\mathcal{A}x = Sx + \mathcal{A}0$, $x \in D(\mathcal{A})$ and $R(\mathcal{A}) = R(S) + \mathcal{A}0$ holds good.

It is said that an MLO operator $\mathcal{A}$ is closed if for any nets (x_τ) in $D(\mathcal{A})$ and (y_τ) in E such that $y_\tau \in \mathcal{A}x_\tau$ for all $\tau \in I$ we have that the suppositions $\lim_{\tau\to\infty} x_\tau = x$ and $\lim_{\tau\to\infty} y_\tau = y$ imply $x \in D(\mathcal{A})$ and $y \in \mathcal{A}x$. If the MLO $\mathcal{A}$ is not closed, then we can simply prove that its closure $\mathcal{A}$, defined as usual, is a closed MLO in E.

We need the following lemma.

Lemma 2.1. *Suppose that $\mathcal{A}$ is a closed MLO in E, Ω is a locally compact and separable metric space, as well as that μ is a locally finite Borel measure defined on Ω. Let $f : \Omega \to E$ and $g : \Omega \to E$ be μ-integrable, and let $g(x) \in \mathcal{A}f(x)$, $x \in \Omega$. Then $\int_\Omega f \, d\mu \in D(\mathcal{A})$ and $\int_\Omega g \, d\mu \in \mathcal{A} \int_\Omega f \, d\mu$.*

The resolvent set of an MLO $\mathcal{A}$, $\rho(\mathcal{A})$ for short, is defined as the union of those complex numbers $\lambda \in \mathbb{C}$ satisfying that

(i) $R(\lambda - \mathcal{A}) = E$, and

(ii) $R(\lambda : \mathcal{A}) \equiv (\lambda - \mathcal{A})^{-1}$ is a single-valued bounded operator on E.

It is not difficult to prove that $\rho(\mathcal{A})$ is an open subset of $\mathbb{C}$. As in single-valued linear case, the operator $\lambda \mapsto R(\lambda : \mathcal{A})$ is called the resolvent of $\mathcal{A}$ ($\lambda \in \rho(\mathcal{A})$). If $\rho(\mathcal{A}) \neq \emptyset$, then $\mathcal{A}$ is closed and, for every $\lambda \in \rho(\mathcal{A})$, we have $\mathcal{A}0 = N((\lambda I - \mathcal{A})^{-1})$.

The following lemma will be of crucial importance in our further work ([**11**], [**15**]).

Lemma 2.2.

(i) *We have*

$$\left(\lambda - \mathcal{A}\right)^{-1}\mathcal{A} \subseteq \lambda\left(\lambda - \mathcal{A}\right)^{-1} - I \subseteq \mathcal{A}\left(\lambda - \mathcal{A}\right)^{-1}, \quad \lambda \in \rho(\mathcal{A}).$$

The operator $(\lambda - \mathcal{A})^{-1}\mathcal{A}$ is single-valued on $D(\mathcal{A})$ and $(\lambda - \mathcal{A})^{-1}\mathcal{A}x = (\lambda - \mathcal{A})^{-1}y$, whenever $y \in \mathcal{A}x$ and $\lambda \in \rho(\mathcal{A})$.

(ii) *Suppose that λ, $\mu \in \rho(\mathcal{A})$. Then the resolvent equation*

$$\left(\lambda - \mathcal{A}\right)^{-1}x - \left(\mu - \mathcal{A}\right)^{-1}x = (\mu - \lambda)\left(\lambda - \mathcal{A}\right)^{-1}\left(\mu - \mathcal{A}\right)^{-1}x, \quad x \in E$$

holds. In particular, $(\lambda - \mathcal{A})^{-1}(\mu - \mathcal{A})^{-1} = (\mu - \mathcal{A})^{-1}(\lambda - \mathcal{A})^{-1}$.

(iii) *Let $\emptyset \neq \Omega \subseteq \rho(\mathcal{A})$ be an open non-empty set. Then the mapping $\lambda \mapsto (\lambda - \mathcal{A})^{-1} \in L(E)$, $\lambda \in \Omega$ is analytic and*

$$\frac{d^{n-1}}{d\lambda^{n-1}}\left(\lambda - \mathcal{A}\right)^{-1} = (-1)^{n-1}(n-1)!\left(\lambda - \mathcal{A}\right)^{-n} \in L(E), \ n \in \mathbb{N}.$$

Fractional powers. In this subsection, we assume that $(-\infty, 0] \subseteq \rho(\mathcal{A})$ as well as that there exist finite numbers $M \geq 1$ and $\beta \in (0, 1]$ such that

$$\|R(\lambda : \mathcal{A})\| \leq M\left(1 + |\lambda|\right)^{-\beta}, \quad \lambda \leq 0.$$

Then there exist two positive real constants $c > 0$ and $M_1 > 0$ such that the resolvent set of $\mathcal{A}$ contains an open region $\Omega = \{\lambda \in \mathbb{C} : |\Im\lambda| \leq (2M_1)^{-1}(c - \Re\lambda)^\beta,\ \Re\lambda \leq c\}$ of complex plane around the nonpositive half-line $(-\infty, 0]$, and we have the estimate $\|R(\lambda : \mathcal{A})\| = O((1 + |\lambda|)^{-\beta})$, $\lambda \in \Omega$. Let Γ' be the upwards oriented curve $\{\xi \pm i(2M_1)^{-1}(c - \xi)^\beta : -\infty < \xi \leq c\}$. Then we define the fractional power

$$\mathcal{A}^{-\theta} := \frac{1}{2\pi i} \int_{\Gamma'} \lambda^{-\theta}(\lambda - \mathcal{A})^{-1} d\lambda \in L(E)$$

for $\theta > 1 - \beta$. Set $\mathcal{A}^\theta := (\mathcal{A}^{-\theta})^{-1}$ $(\theta > 1 - \beta)$. Then the semigroup properties $\mathcal{A}^{-\theta_1}\mathcal{A}^{-\theta_2} = \mathcal{A}^{-(\theta_1 + \theta_2)}$ and $\mathcal{A}^{\theta_1}\mathcal{A}^{\theta_2} = \mathcal{A}^{\theta_1 + \theta_2}$ hold for $\theta_1,\ \theta_2 > 1 - \beta$ (it is worth noting here that the fractional power $\mathcal{A}^\theta$ need not be injective and that the meaning of $\mathcal{A}^\theta$ is understood in the MLO sense for $\theta > 1 - \beta$).

We endow the vector space $D(\mathcal{A})$ with the norm

$$\| \cdot \|_{[D(\mathcal{A})]} := \inf_{y \in \mathcal{A}\cdot} \|y\|.$$

Then it is well known that $(D(\mathcal{A}), \| \cdot \|_{[D(\mathcal{A})]})$ is a Banach space and that the norm $\| \cdot \|_{[D(\mathcal{A})]}$ is equivalent with the graph norm $\| \cdot \| + \| \cdot \|_{[D(\mathcal{A})]}$. Similarly, $(D(\mathcal{A}^\theta), \| \cdot \|_{[D(\mathcal{A}^\theta)]})$ is a Banach space and we have the equivalence of norms $\| \cdot \|_{[D(\mathcal{A}^\theta)]}$ and $\| \cdot \| + \| \cdot \|_{[D(\mathcal{A}^\theta)]}$ for $\theta > 1 - \beta$.

Suppose that $\theta \in (0, 1)$. Then the vector space

$$E_{\mathcal{A}}^\theta := \left\{ x \in E : \sup_{\xi > 0} \xi^\theta \left\| \xi(\xi + \mathcal{A})^{-1} x - x \right\| < \infty \right\}$$

equipped with the norm

$$\| \cdot \|E_{\mathcal{A}}^\theta := \| \cdot \| + \sup_{\xi > 0} \xi^\theta \left\| \xi(\xi + \mathcal{A})^{-1} \cdot - \cdot \right\|$$

forms a Banach space which is continuously embedded in E.

We refer the reader to [8], [21] and [23] for further information concerning fractional powers of multivalued linear operators.

3. SUBORDINATED FRACTIONAL RESOLVENT FAMILIES

Suppose that the condition (QP) holds. For the beginning of our work, set $\delta := \min(\pi/2(\eta'' - \eta')/\eta', \pi/2)$. Then we define the operator family $(T_{\eta',r}(z))_{z \in \Sigma_\delta} \subseteq L(E)$ as follows $(r \in \mathbb{R})$. Let $\delta' \in (0, \delta)$, let $0 < \epsilon < \delta'$ be arbitrarily chosen, and let

$$T_{\eta',r}(z)x := \frac{1}{2\pi i} \int_{\Gamma_\omega} e^{\lambda z} \lambda^r (\lambda^{\eta'} - \mathcal{A})^{-1} x\, d\lambda, \quad x \in E,\ r \in \mathbb{R},\ z \in \Sigma_{\delta' - \epsilon},$$

where Γ_ω is oriented counterclockwise and consists of $\Gamma_\pm := \{te^{i((\pi/2)+\delta')} : t \geq \omega\}$ and $\Gamma_0 := \{\omega e^{i\zeta} : |\zeta| \leq (\pi/2) + \delta'\}$. Observe that the Cauchy formula implies that the definition of $T_{\eta',r}(z)$ is independent of $\omega > 0$.

Arguing as in the proof of [**1**, Theorem 2.6.1], with $\omega = 1/|z|$, we get that for each $\delta' \in (0, \delta)$ and $r \in \mathbb{R}$,

$$(3.2) \qquad \left\| T_{\eta',r}(z) \right\| = O\big(|z|^{\eta'\beta - r - 1}\big), \quad z \in \Sigma_{\delta'},$$

as well as that

$$(3.3) \quad \int_0^\infty e^{-\lambda t} T_{\eta',r}(t) x\, dt = \lambda^r \big(\lambda^{\eta'} - \mathcal{A}\big)^{-1} x, \quad x \in E, \ \lambda > 0, \ \text{provided } \eta'\beta > r.$$

The most important for us will be the operator family $(T_{\eta',\eta'-1}(t))_{t>0}$; in some representation formulae, the operator family $(T_{\eta',0}(t))_{t>0}$ takes place, as well.

In fractional framework, the second inequality from [**11**, Proposition 3.2] reads as follows:

Proposition 3.1. *Suppose that* $0 < \theta < 1$ *and* $\theta \leq \beta$. *Then we have* $R(T_{\eta',\eta'-1}(t)) \subseteq E_{\mathcal{A}}^\theta$, $t > 0$. *Furthermore, for every* $\theta \in (0,1)$, *there exists a constant* $C_\theta > 0$ *such that*

$$\sup_{s>0} s^\theta \left\| sR(s:\mathcal{A})T_{\eta',\eta'-1}(t)x - T_{\eta',\eta'-1}(t)x \right\| \leq C_\theta t^{\eta'(\beta - \theta - 1)} \|x\|, \quad t > 0, \ x \in E.$$

Proof. Let $t > 0$ and $s > 0$ be fixed, and let $\omega > 0$ be such that $\omega^\eta < s$. By Lemma 2.2(ii) and a simple computation, we have that, for every $x \in E$,

$$
\begin{aligned}
& sR(s:\mathcal{A})T_{\eta',\eta'-1}(t)x - T_{\eta',\eta'-1}(t)x \\
(3.4) \quad & = \frac{1}{2\pi i}\int_{\Gamma_\omega} e^{\lambda t}\frac{\lambda^{\eta'}s}{\lambda^{\eta'} - s}R(s:\mathcal{A})x\, d\lambda - \frac{1}{2\pi i}\int_{\Gamma_\omega} e^{\lambda t}\frac{\lambda^{2\eta'-1}}{\lambda^{\eta'} - s}R\big(\lambda^{\eta'} : \mathcal{A}\big)x\, d\lambda.
\end{aligned}
$$

The usual contour argument shows that

$$
\begin{aligned}
& \frac{1}{2\pi i}\int_{\Gamma_\omega} e^{\lambda t}\frac{\lambda^{\eta'}s}{\lambda^{\eta'} - s}R(s:\mathcal{A})x\, d\lambda \\
& = \int_0^\infty e^{-vt}v^{\eta'-1}\left[\frac{e^{-i\pi(\eta'-1)}}{e^{-i\pi\eta'}v^{\eta'} - s} - \frac{e^{i\pi(\eta'-1)}}{e^{i\pi\eta'}v^{\eta'} - s}\right] sR(s:\mathcal{A})x\, dv, \ x \in E.
\end{aligned}
$$

It is clear that there exists a constant $a > 0$ such that $|e^{\pm i\pi\eta'}v^{\eta'} - s| \geq a(v^{\eta'} + s)$, $v > 0$. Using (1.1) and this fact, we get that there exists a constant $C_{\theta,1} > 0$,

independent of $s > 0$, such that:

$$\left\| s^\theta \int_0^\infty e^{-vt} v^{\eta'-1} \left[\frac{e^{-i\pi(\eta'-1)}}{e^{-i\pi\eta'} v^{\eta'} - s} - \frac{e^{i\pi(\eta'-1)}}{e^{i\pi\eta'} v^{\eta'} - s} \right] s R(s : \mathcal{A}) x \, dv \right\|$$

$$\leq 2Ma^{-1} \|x\| \int_0^\infty e^{-vt} v^{\eta'-1} \frac{s^{1-\beta+\theta}}{v^{\eta'} + s} \, dv$$

$$\leq C_{\theta,1} \|x\| \int_0^\infty e^{-vt} v^{\eta'-1} \frac{s^{1-\beta+\theta}}{v^{\eta'(\beta-\theta)} s^{1-\beta+\theta}} \, dv$$

(3.5)
$$= C_{\theta,1} \|x\| \Gamma(\eta'(1-\beta-\theta)) t^{\eta'(\beta-\theta-1)}, \quad t > 0.$$

Now we will estimate the second term in (3.4) multiplied with s^θ. It suffices to consider the following two cases: $s > t^{-\eta'}$ and $s < t^{-\eta'}$. Suppose first that $s > t^{-\eta'}$. Then there exists a constant $b > 0$ such that $|\lambda^{\eta'} - s| \geq b(|\lambda|^{\eta'} + s)$, $\lambda \in \Gamma_{1/t}$. Applying Cauchy theorem and (1.1), we get that there exist two constants $C_{\theta,2}, C_{\theta,3} > 0$, independent of $s > 0$, such that:

$$\left\| \frac{1}{2\pi i} \int_{\Gamma_\omega} e^{\lambda t} \frac{\lambda^{2\eta'-1} s^\theta}{\lambda^{\eta'} - s} R(\lambda^{\eta'} : \mathcal{A}) x \, d\lambda \right\|$$

$$= \left\| \frac{1}{2\pi i} \int_{\Gamma_{1/t}} e^{\lambda t} \frac{\lambda^{2\eta'-1} s^\theta}{\lambda^{\eta'} - s} R(\lambda^{\eta'} : \mathcal{A}) x \, d\lambda \right\|$$

$$\leq \frac{1}{2\pi b} \int_{\Gamma_{1/t}} e^{\Re(\lambda)t} \frac{|\lambda|^{2\eta'-1-\eta'\beta} s^\theta}{|\lambda|^{\eta'} + s} \, |d\lambda|$$

$$\leq C_{\theta,2} \int_{\Gamma_{1/t}} e^{\Re(\lambda)t} \frac{|\lambda|^{2\eta'-1-\eta'\beta} s^\theta}{|\lambda|^{\eta'(1-\theta)} s^\theta} \, |d\lambda|$$

$$= C_{\theta,2} \int_{\Gamma_{1/t}} e^{\Re(\lambda)t} |\lambda|^{\eta'(1+\theta-\beta)-1} \, |d\lambda|$$

$$\leq C_{\theta,3} t^{\eta'(\beta-\theta-1)}, \quad t > 0,$$

where the last estimate follows from the computation contained in the proof of [**1**, Theorem 2.6.1]. If $s < t^{-\eta'}$, then the equation (3.4) continues to hold with the

number ω replaced with $1/t$ therein. Then the residue theorem shows that

$$\frac{s^\theta}{2\pi i}\int_{\Gamma_\omega}e^{\lambda t}\frac{\lambda^{\eta'}s}{\lambda^{\eta'}-s}R(s:\mathcal{A})x\,d\lambda$$

$$=\int_0^\infty e^{-vt}s^\theta v^{\eta'-1}\left[\frac{e^{-i\pi(\eta'-1)}}{e^{-i\pi\eta'}v^{\eta'}-s}-\frac{e^{i\pi(\eta'-1)}}{e^{i\pi\eta'}v^{\eta'}-s}\right]sR(s:\mathcal{A})x\,dv$$

$$+2\pi i s^\theta \mathrm{Res}_{\lambda=s^{1/\eta'}}\left[\frac{e^{\lambda t}\lambda^{\eta'-1}}{\lambda^{\eta'}-s}sR(s:\mathcal{A})x\right]$$

$$=\int_0^\infty e^{-vt}s^\theta v^{\eta'-1}\left[\frac{e^{-i\pi(\eta'-1)}}{e^{-i\pi\eta'}v^{\eta'}-s}-\frac{e^{i\pi(\eta'-1)}}{e^{i\pi\eta'}v^{\eta'}-s}\right]sR(s:\mathcal{A})x\,dv$$

(3.6)
$$+\frac{s^\theta}{\eta'}e^{ts^{1/\eta'}}sR(s:\mathcal{A})x,\ \ x\in E.$$

We can estimate the first summand in (3.6) and the term

$$\left\|\frac{1}{2\pi i}\int_{\Gamma_\omega}e^{\lambda t}\frac{\lambda^{2\eta'-1}s^\theta}{\lambda^{\eta'}-s}R(\lambda^{\eta'}:\mathcal{A})x\,d\lambda\right\|=\left\|\frac{1}{2\pi i}\int_{\Gamma_{1/t}}e^{\lambda t}\frac{\lambda^{2\eta'-1}s^\theta}{\lambda^{\eta'}-s}R(\lambda^{\eta'}:\mathcal{A})x\,d\lambda\right\|$$

as in the case that $s>t^{-\eta'}$, with the same final estimate. For the second summand in (3.6), we have the following estimates:

$$\left\|s^\theta e^{ts^{1/\eta'}}sR(s:\mathcal{A})x/\eta'\right\|\le M\|x\|s^{1+\theta-\beta}e^1/\eta'\le M\|x\|t^{-\eta'(1+\theta-\beta)}e/\eta',\quad t>0.$$

The proof of the theorem is thereby complete. $\qquad\square$

Let Γ' be the integral contour used in the definition of fractional power $(-\mathcal{A})^\theta$, $\theta>1-\beta$. Denote by Φ the open region on the right of Γ'.

For the sequel, we need the following useful lemma.

Lemma 3.2.

(i) *Suppose that $1-\beta<\theta\le 1$. Then there exists a constant $C_\theta>0$ such that*

(3.7)
$$\left\|\lambda R(\lambda:\mathcal{A})x-x\right\|\le C_\theta|\lambda|^{(1-\beta-\theta)}\|x\|_{[D((-\mathcal{A})^\theta)]},\ \lambda\in\overline{\Sigma_{\eta''\pi/2}},\ x\in D\big((-\mathcal{A})^\theta\big).$$

(ii) *Suppose that $1-\beta<\theta<1$. Then there exists a constant $C_\theta>0$ such that*

(3.8)
$$\left\|\lambda R(\lambda:\mathcal{A})x-x\right\|\le C_\theta|\lambda|^{(1-\beta-\theta)}\|x\|_{E_\mathcal{A}^\theta},\ \lambda\in\overline{\Sigma_{\eta''\pi/2}},\ x\in E_\mathcal{A}^\theta.$$

Proof. Suppose first that $\theta=1$. Then Lemma 2.2(i) implies that for any $(x,y)\in\mathcal{A}$ one has

$$\left\|\lambda R(\lambda:\mathcal{A})x-x\right\|=\left\|R(\lambda:\mathcal{A})y\right\|\le M\big(1+|\lambda|\big)^{-\beta},\quad\lambda\in\overline{\Sigma_{\eta''\pi/2}}.$$

Taking the infimum, we immediately obtain (3.7). Let $1 - \beta < \theta < 1$. Then the function $\lambda \mapsto H(\lambda) := \lambda^{(\beta+\theta-1)}[\lambda R(\lambda : \mathcal{A})x - x]$, $\lambda \in \overline{\Sigma_{\eta''\pi/2}}$ is continuous on $\overline{\Sigma_{\eta''\pi/2}}$, holomorphic on $\Sigma_{\eta''\pi/2}$ and $\|H(\lambda)\| \leq M|\lambda|^{(\beta+\theta-1)}[M|\lambda|^{1-\beta} + 1]\|x\|$, $\lambda \in \overline{\Sigma_{\eta''\pi/2}}$, $x \in E$. Let $R_1 > 0$ be sufficiently large, obeying the properties that $|z + \lambda| \geq 1$ for $z \in \Gamma', \lambda = Re^{\pm i\pi\eta''/2}$ and $-\lambda = -Re^{\pm i\pi\eta''/2} \in \Phi$ $(R \geq R_1)$. Put $\Gamma_{R,\pm} := \{Re^{\pm i\pi\eta''/2} : R \geq R_1\}$. Then it is clear that there exists a constant $a > 0$ such that $a^{-1}(|z| + |\lambda|) \geq |z + \lambda|$, $z \in \Gamma'$, $\lambda \in \Gamma_{R,\pm}$. By the Phragmén-Lindelöf type theorem [**1**, Theorem 3.9.8, p. 179], it suffices to show that the estimates (3.7) and (3.8) hold for $\lambda \in \Gamma_{R,\pm}$, with an appropriately chosen constant $C_\theta > 0$ independent of $\lambda \in \overline{\Sigma_{\eta''\pi/2}}$ and $x \in D((-\mathcal{A})^\theta))$ $(x \in E_{\mathcal{A}}^\theta)$. Suppose first that $x \in D((-\mathcal{A})^\theta)$ and $y \in (-\mathcal{A})^\theta x$ is arbitrarily chosen. Then $x = (-\mathcal{A})^{-\theta}y = \frac{1}{2\pi i}\int_{\Gamma'} z^{-\theta}R(z : -\mathcal{A})y\,dz$ and it is not difficult to prove with the help of Lemma 2.2(ii) and the residue theorem that

(3.9)
$$\lambda^{\eta'} R(\lambda^{\eta'} : \mathcal{A})x - x = \frac{(-1)}{2\pi i}\int_{\Gamma'} \frac{z^{1-\theta}}{z + \lambda}R(z : -\mathcal{A})y\,dz + (-\lambda)^{-\theta}y, \quad \lambda \in \Gamma_{R,\pm}.$$

Keeping in mind the parametrization of Γ', (3.9) and the fact that y was arbitrary, we get that, for every $\lambda \in \Gamma_{R,\pm}$, the following holds:

$$\left\|\lambda R(\lambda : \mathcal{A})x - x\right\| - |\lambda|^{-\theta}\|x\|_{[D((-\mathcal{A})^\theta)]}$$

$$(3.10) \qquad \leq a\|x\|_{[D((-\mathcal{A})^\theta)]}\int_{-\infty}^c \frac{|v|^{1-\theta-\beta}}{|v| + |\lambda|}\left(1 + \beta^2(4M^2)^{-1}(c - v)^{2\beta-2}\right)^{1/2} dv.$$

For the estimation of this integral, we divide the path of integration into three segments: $(-\infty, 0]$, $[0, c/2]$ and $[c/2, c]$. We have

$$\left(\int_{-\infty}^0 + \int_0^{c/2}\right)\frac{|v|^{1-\theta-\beta}}{|v| + |\lambda|}\left(1 + \beta^2(4M^2)^{-1}(c - v)^{2\beta-2}\right)^{1/2} dv$$

$$\leq 2\left(1 + \beta^2(4M^2)^{-1}(c/2)^{2\beta-2}\right)^{1/2}\int_0^\infty \frac{v^{1-\theta-\beta}}{v + |\lambda|}\,dv$$

$$= 2\left(1 + \beta^2(4M^2)^{-1}(c/2)^{2\beta-2}\right)^{1/2}|\lambda|^{1-\theta-\beta}\int_0^\infty \frac{v^{1-\theta-\beta}}{v + 1}\,dv$$

$$= 2\left(1 + \beta^2(4M^2)^{-1}(c/2)^{2\beta-2}\right)^{1/2}|\lambda|^{1-\theta-\beta}\frac{(-\pi)}{\sin\pi(\theta + \beta)}, \quad \lambda \in \Gamma_{R,\pm}.$$

The integral over segment $[c/2, c]$ can be majorized by using the following consequence of the inequality (1.1):

$$\frac{|v|^{1-\theta-\beta}}{|v| + |\lambda|} \leq C_\theta' \frac{|v|^{1-\theta-\beta}}{|v|^{2-\theta-\beta}|\lambda|^{\theta+\beta-1}}, \quad \lambda \in \Gamma_{R,\pm}, \ v \in [c/2, c],$$

giving the same final estimate as above. This completes the proof of (i). In order to prove (ii), fix an element $x \in E_{\mathcal{A}}^\theta$. Let us observe that there exists a finite constant

$c_\theta > 0$, independent of $x \in E_{\mathcal{A}}^\theta$, such that, for every $\lambda \in \Gamma_{R,\pm}$,

$$c_\theta |\lambda|^{1-\beta-\theta} \|x\|_{E_{\mathcal{A}}^\theta} \geq \left\| \left(|\lambda| - \lambda \right) \left(\lambda - \mathcal{A} \right)^{-1} \left[\left(|\lambda| - \mathcal{A} \right)^{-1} x - x \right] \right\|$$

$$= \left\| \left(1 - \frac{|\lambda|}{\lambda} \right) \left[\left(|\lambda| - \mathcal{A} \right)^{-1} x - x \right] + \left(\lambda - \mathcal{A} \right)^{-1} x \right.$$

$$\left. + \left[\left(1 - \frac{|\lambda|}{\lambda} \right) x - \left(|\lambda| - \mathcal{A} \right)^{-1} x \right] \right\|,$$

where the last equality follows from Lemma 2.2(ii) and a simple computation. This implies that, for every $\lambda \in \Gamma_{R,\pm}$,

$$\left| 1 - \frac{|\lambda|}{\lambda} \right| \left\| \left(|\lambda| - \mathcal{A} \right)^{-1} x - x \right\|$$

$$\leq c_\theta |\lambda|^{1-\beta-\theta} \|x\|_{E_{\mathcal{A}}^\theta} + \left\| \left(\lambda - \mathcal{A} \right)^{-1} x \right\| + \left\| \left[\left(1 - \frac{|\lambda|}{\lambda} \right) x - \left(|\lambda| - \mathcal{A} \right)^{-1} x \right] \right\|$$

$$\leq c_\theta |\lambda|^{(1-\beta-\theta)} \|x\|_{E_{\mathcal{A}}^\theta} + M|\lambda|^{-\beta} \|x\|$$

$$+ \left\| \left(1 - \frac{|\lambda|}{\lambda} \right) x - \left(|\lambda| - \mathcal{A} \right)^{-1} \left(1 - \frac{|\lambda|}{\lambda} \right) x \right\| + \left\| \left(|\lambda| - \mathcal{A} \right)^{-1} x \right\|$$

$$\leq c_\theta |\lambda|^{1-\beta-\theta} \|x\|_{E_{\mathcal{A}}^\theta} + M|\lambda|^{-\beta} \|x\| + 2|\lambda|^{-\theta} \|x\|_{E_{\mathcal{A}}^\theta} + M|\lambda|^{-\beta} \|x\|.$$

Taking into account this estimate, the proof of (ii) is completed through a routine argument. $\square$

REMARK 3.3. ([8])

(i) The operator $(-\mathcal{A})^n$, defined as fractional power, coincides with the usually considered power $(-\mathcal{A})^n$ ($n \in \mathbb{N}$).

(ii) The space $[D(\mathcal{A})]$ is continuously embedded in $[D((-\mathcal{A})^\theta)]$ provided that $\beta > 1/2$ and $1 - \beta < \theta < \beta$.

Now we are ready to prove the following generalization of [11, Theorem 3.5] for degenerate fractional differential equations.

Theorem 3.4. *Let* $\delta' \in (0, \delta)$.

(i) *Suppose that* $1 - \beta < \theta \leq 1$. *Then there exists a constant* $C_{\theta,\delta'} > 0$ *such that*

(3.11)
$$\left\| T_{\eta',\eta'-1}(z)x - x \right\| \leq C_{\theta,\delta'} |z|^{\eta'(\beta+\theta-1)} \|x\|_{[D((-\mathcal{A})^\theta)]}, \ z \in \Sigma_{\delta'}, \ x \in D\left((-\mathcal{A})^\theta \right).$$

(ii) *Suppose that* $1 - \beta < \theta < 1$. *Then there exists a constant* $C_{\theta,\delta'} > 0$ *such that*

$$\left\| T_{\eta',\eta'-1}(z)x - x \right\| \leq C_{\theta,\delta'} |z|^{\eta'(\beta+\theta-1)} \|x\|_{E_{\mathcal{A}}^\theta}, \ z \in \Sigma_{\delta'}, \ x \in E_{\mathcal{A}}^\theta.$$

Proof. Let $\delta'' \in (\delta', \delta)$, and let $0 < \epsilon < \delta - \delta''$ be arbitrarily chosen. Then it is clear that

$$T_{\eta', \eta'-1}(z)x - x = \frac{1}{2\pi i} \int_{\Gamma_{1/|z|}} \frac{e^{\lambda z}}{\lambda} \left[\lambda^{\eta'} R(\lambda^{\eta'} : \mathcal{A})x - x \right] d\lambda, \quad z \in \Sigma_{\delta'-\epsilon}, \ x \in E.$$

Now the result follows from Lemma 3.2 and the computation contained in the proof of [**1**, Theorem 2.6.1].   □

Suppose now that $\theta > 1 - \beta$. Then it is clear that there exists a sufficiently small number $t_0 > 0$ such that, for every $t \in (0, t_0]$ and $\lambda \in \Gamma_{1/t}$, we have $\lambda^{\eta'} \in \Phi$. Making use of Lemma 2.2(ii) and Fubini theorem, we get that, for every $t \in (0, t_0]$,

$$(-\mathcal{A})^{-\theta} T_{\eta', \eta'-1}(t)x = \frac{(-1)}{(2\pi i)^2} \int_{\Gamma_{1/t}} e^{\lambda t} \lambda^{\eta'-1} (\lambda^{\eta'} - \mathcal{A})^{-1} \left[\int_{\Gamma'} \frac{dz}{z^\theta(\lambda^{\eta'} - z)} \right] d\lambda$$

$$+ \frac{1}{(2\pi i)^2} \int_{\Gamma_{1/t}} e^{\lambda t} \lambda^{\eta'-1} (\lambda^{\eta'} - \mathcal{A})^{-1} \left[\int_{\Gamma'} \frac{(z - \mathcal{A})^{-1} x}{z^\theta(\lambda^{\eta'} - z)} \, dz \right] d\lambda.$$

Applying the residue theorem on the first integral and Fubini theorem on the second one, we get from the above and definition of $(-\mathcal{A})^{-\theta}$ that, for every $t \in (0, t_0]$,

$$(-\mathcal{A})^{-\theta} T_{\eta', \eta'-1}(t)x = \frac{1}{2\pi i} \int_{\Gamma_{1/t}} e^{\lambda t} \lambda^{\eta'-1-\eta'\theta} (\lambda^{\eta'} - \mathcal{A})^{-1} x \, d\lambda + (-\mathcal{A})^{-\theta} x$$

$$+ \frac{1}{2\pi i} \int_{\Gamma'} z^{1-\theta} (z - \mathcal{A})^{-1} \left[\int_{\Gamma_{1/t}} \frac{e^{\lambda t}}{\lambda(\lambda^{\eta'} - z)} \, d\lambda \right] dz$$

$$:= I_1(t) + (-\mathcal{A})^{-\theta} x + I_3(t).$$

Making use of dominated convergence theorem, we immediately obtain that $I_1(t) \to 0$ as $t \to 0 +$. Let $1 > \zeta > 2 - \theta - \beta$. Then (1.1) implies

$$(3.12) \qquad \int_{\Gamma_{1/t}} \frac{|d\lambda|}{|\lambda||\lambda^{\eta'} - z|} \leq a \int_{\Gamma_{1/t}} \frac{|d\lambda|}{|\lambda||\lambda|^{\eta'} + |z|} \leq a \int_{\Gamma_{1/t}} \frac{|d\lambda|}{|\lambda||\lambda|^{\eta'(1-\zeta)}|z|^\zeta},$$

where $a > 0$ is a constant independent of $t \in (0, t_0]$, $\lambda \in \Gamma_{1/t}$ and $z \in \Gamma'$. Taking into account (3.12), we may apply dominated convergence theorem in order to see that $I_3(t) \to 0$ as $t \to 0 +$. Keeping in mind Theorem 3.4(ii) and the commutation of operators $(-\mathcal{A})^{-\theta}$ and $T_{\eta', \eta'-1}(t)$, we obtain from the above that the following holds:

(CS) $T_{\eta', \eta'-1}(t)x \to x$, $t \to 0+$ for any $x \in E$ belonging to the space $D((-\mathcal{A})^\theta)$ with $\theta > 1 - \beta$ ($x \in E_\mathcal{A}^\theta$ with $1 > \theta > 1 - \beta$).

4. SUBORDINATION PRINCIPLES

In this section, we investigate degenerate fractional resolvent families that are subordinated to $(T_{\eta',\eta'-1}(t))_{t>0}$. For the simplicity of notation, set $\mathcal{T}_{\eta'}(t) := T_{\eta',\eta'-1}(t)$, $t > 0$.

Henceforth, we assume that $0 < \gamma < 1$. Set $\eta := \gamma\eta'$ and, for every $\nu > -1 - \eta'(\beta - 1)$,

$$\mathcal{T}_{\eta',\gamma}^{\nu}(t)x := t^{-\gamma} \int_0^{\infty} s^{\nu}\Phi_{\gamma}\big(st^{-\gamma}\big)\mathcal{T}_{\eta'}(s)x\,ds, \quad t > 0, \ x \in E \text{ and } \mathcal{T}_{\eta',\gamma}^0(0) := I.$$

Since

$$(4.13) \qquad \mathcal{T}_{\eta',\gamma}^{\nu}(t)x = t^{\gamma\nu} \int_0^{\infty} s^{\nu}\Phi_{\gamma}(s)\mathcal{T}_{\eta'}\big(st^{\gamma}\big)x\,ds, \quad t > 0, \ x \in E,$$

the estimate (3.2) and (a1) together imply that the integral which defines the operator $\mathcal{T}_{\eta',\gamma}^{\nu}(t)$ is absolutely convergent as well as that

$$\big\|\mathcal{T}_{\eta',\gamma}^{\nu}(t)\big\| = O\Big(t^{\gamma(\nu+\eta'(\beta-1))}\Big), \quad t > 0.$$

Due to Proposition 3.1, we have that, for every $\theta \in (0,1)$, there exists a constant $C_{\theta} > 0$ such that, for every $\nu > -1 - \eta'(\beta - \theta - 1)$,

$$\sup_{s>0} s^{\theta}\Big\|sR(s:\mathcal{A})\mathcal{T}_{\eta',\gamma}^{\nu}(t)x - \mathcal{T}_{\eta',\gamma}^{\nu}(t)x\Big\| \leq C_{\theta}t^{\gamma(\nu+\eta'(\beta-\theta-1))}, \quad t > 0.$$

Taking into account (4.13) and (a1), we get that, for every $\nu > -1 - \eta'(\beta - 1)$,

$$(4.14) \quad \frac{\mathcal{T}_{\eta',\gamma}^{\nu}(t)}{t^{\gamma\nu}}x - \frac{\Gamma(1+\nu)}{\Gamma(1+\gamma\nu)}x = \int_0^{\infty} s^{\nu}\Phi_{\gamma}(s)\Big[\mathcal{T}_{\eta'}\big(st^{\gamma}\big)x - x\Big]\,ds, \quad t > 0, \ x \in E.$$

Keeping in mind (a1), (4.14) and (CS), we can apply the dominated convergence theorem in order to see that:

(B1) $\frac{\mathcal{T}_{\eta',\gamma}^{\nu}(t)}{t^{\gamma\nu}}x \to \frac{\Gamma(1+\nu)}{\Gamma(1+\gamma\nu)}x$, $t \to 0+$ provided that $\theta > 1 - \beta$ and $x \in D((-\mathcal{A})^{\theta})$, or that $1 > \theta > 1 - \beta$ and $x \in E_{\mathcal{A}}^{\theta}$ $(\nu > -1 - \eta'(\beta - 1))$.

Using the proof of [**2**, Theorem 3.1] and (3.3), we get that:

(B2) $\int_0^{\infty} e^{-\lambda t}\mathcal{T}_{\eta',\gamma}^0(t)x\,dt = \lambda^{\gamma-1}\int_0^{\infty} e^{-\lambda^{\gamma}t}\mathcal{T}_{\eta'}(t)x\,dt = \lambda^{\eta-1}(\lambda^{\eta} - \mathcal{A})^{-1}x$, $\Re\lambda > 0$, $x \in E$.

Further on, applying Theorem 3.4, (a1) and (4.14), we obtain that:

(B3) $\|\frac{\mathcal{T}_{\eta',\gamma}^{\nu}(t)}{t^{\gamma\nu}}x - \frac{\Gamma(1+\nu)}{\Gamma(1+\gamma\nu)}x\| = O(t^{\eta(\beta+\theta-1)}\|x\|_{[D((-\mathcal{A})^{\theta})]})$, $t > 0$, provided $1 > \theta > 1 - \beta$, $x \in D((-\mathcal{A})^{\theta})$ and $\|\frac{\mathcal{T}_{\eta',\gamma}^{\nu}}{t^{\gamma\nu}}x - \frac{\Gamma(1+\nu)}{\Gamma(1+\gamma\nu)}x\| = O(t^{\eta(\beta+\theta-1)}\|x\|_{E_{\mathcal{A}}^{\theta}})$, $t > 0$, provided $1 > \theta > 1 - \beta$, $x \in E_{\mathcal{A}}^{\theta}$ $(\nu > -1 - \eta'(\beta - 1))$.

Set $\xi := \min((1/\gamma - 1)\pi/2, \pi)$. By the proof of [2, Theorem 3.3(i)-(ii)], we have that, for every $\nu > -1 - \eta'(\beta - 1)$, the mapping $t \mapsto \mathcal{T}_{\eta',\gamma}^{\nu}(t)x$, $t > 0$ can be analytically extended to the sector Σ_{ξ} (we will denote this extension by the same symbol) and that, for every $\theta \in (0, 1)$, $\epsilon \in (0, \xi)$ and $\nu > -1 - \eta'(\beta - 1)$,

(B4) $\|\mathcal{T}_{\eta',\gamma}^{\nu}(z)\| = O(|z|^{\gamma(\nu + \eta'(\beta - 1))})$, $z \in \Sigma_{\xi - \epsilon}$,

as well as that, for every $\theta \in (0, 1)$, $\epsilon \in (0, \xi)$ and $\nu > -1 - \eta'(\beta - \theta - 1)$,

(B5) $\sup_{s>0} s^{\theta} \left\| sR(s : \mathcal{A})\mathcal{T}_{\eta',\gamma}^{\nu}(z)x - \mathcal{T}_{\eta',\gamma}^{\nu}(z)x \right\| = O(|z|^{\gamma(\nu + \eta'(\beta - \theta - 1))})$, $z \in \Sigma_{\xi - \epsilon}$.

REMARK 4.1. The angle of analyticity ξ of operator family $(\mathcal{T}_{\eta',\gamma}^{\nu}(t))_{t>0}$ is not optimal and can be improved by using the consideration from the proof of [16, Theorem 3.10]. We will not go into further details concerning this question here.

Owing to (B4)-(B5), the Cauchy integral formula and an elementary argumentation, we can deduce that, for every $\theta \in (0, 1)$, $\epsilon \in (0, \xi)$, $\nu > -1 - \eta'(\beta - 1)$ and $n \in \mathbb{N}$,

(B4)' $\|(d^n/dz^n)\mathcal{T}_{\eta',\gamma}^{\nu}(z)\| = O(|z|^{\gamma(\nu + \eta'(\beta - 1)) - n})$, $z \in \Sigma_{\xi - \epsilon}$,

as well as that, for every $\theta \in (0, 1)$, $\epsilon \in (0, \xi)$, $\nu > -1 - \eta'(\beta - \theta - 1)$ and $n \in \mathbb{N}$,

(B5)' $\sup_{s>0} s^{\theta} \left\| sR(s : \mathcal{A})(d^n/dz^n)\mathcal{T}_{\eta',\gamma}^{\nu}(z)x - (d^n/dz^n)\mathcal{T}_{\eta',\gamma}^{\nu}(z)x \right\|$
$= O(|z|^{\gamma(\nu + \eta'(\beta - \theta - 1)) - n})$, $z \in \Sigma_{\xi - \epsilon}$.

Since $\int_0^{\infty} z^{-\gamma(1+\nu)} s^{\nu} \Phi_{\gamma}(sz^{-\gamma}) \, ds = \frac{\Gamma(1+r)}{\Gamma(1+\gamma r)}$, $r > -1$, we have

$$\frac{\mathcal{T}_{\eta',\gamma}^{\nu}(z)}{z^{\gamma\nu}}x - \frac{\Gamma(1+\nu)}{\Gamma(1+\gamma\nu)}x = \int_0^{\infty} s^{\nu} \Phi_{\gamma}(se^{i\varphi}) \left[\mathcal{T}_{\eta'}(s|z|^{\gamma})x - x \right] ds,$$

where $\varphi = -\gamma \arg(z)$. Keeping in mind this identity and the foregoing argumentation, we can prove the following extension of properties (B1), (B3):

(B1)' Suppose that $\epsilon \in (0, \xi)$ and $\delta = \xi - \epsilon$. Then $\lim_{z \to 0, z \in \Sigma_{\delta}} \frac{\mathcal{T}_{\eta',\gamma}^{\nu}(z)}{z^{\gamma\nu}}x = \frac{\Gamma(1+\nu)}{\Gamma(1+\gamma\nu)}x$, provided that $\theta > 1 - \beta$ and $x \in D((-\mathcal{A})^{\theta})$, or that $1 > \theta > 1 - \beta$ and $x \in E_{\mathcal{A}}^{\theta}$ ($\nu > -\beta$).

(B3)' Suppose that $\epsilon \in (0, \xi)$, $\delta = \xi - \epsilon$ and $\nu > -1 - \eta'(\beta - 1)$. Then $\|\frac{\mathcal{T}_{\eta',\gamma}^{\nu}(z)}{z^{\gamma\nu}}x - \frac{\Gamma(1+\nu)}{\Gamma(1+\gamma\nu)}x\| = O(|z|^{\eta(\beta+\theta-1)}\|x\|_{[D((-\mathcal{A})^{\theta})]})$, $z \in \Sigma_{\delta}$, provided $1 > \theta > 1 - \beta$, $x \in D((-\mathcal{A})^{\theta})$, and $\|\frac{\mathcal{T}_{\eta',\gamma}^{\nu}(z)}{z^{\gamma\nu}}x - \frac{\Gamma(1+\nu)}{\Gamma(1+\gamma\nu)}x\| = O(|z|^{\eta(\beta+\theta-1)}\|x\|_{E_{\mathcal{A}}^{\theta}})$, $z \in \Sigma_{\delta}$, provided $1 > \theta > 1 - \beta$, $x \in E_{\mathcal{A}}^{\theta}$.

For our further work, we need to single out the following operator families:

$$\mathcal{S}_{\eta}(z) := \mathcal{T}_{\eta',\gamma}^{0}(z), \ z \in \Sigma_{\xi} \quad \text{and} \quad \mathcal{P}_{\eta}(z) := z^{1-\eta} \frac{d}{dz} \int_0^z g_{\eta}(z - s)\mathcal{S}_{\eta}(s) \, ds, \ z \in \Sigma_{\xi}.$$

Clearly, $\mathcal{S}_\eta(z)$ and $\mathcal{P}_\eta(z)$ depend analytically on the parameter z in the uniform operator topology. It is not difficult to prove that, for every $\epsilon \in (0, \xi)$, we have

$$(4.15) \qquad \left\| \mathcal{S}_\eta(z) \right\| + \left\| \mathcal{P}_\eta(z) \right\| = O\left(|z|^{\eta(\beta-1)} \right), \quad z \in \Sigma_{\xi-\epsilon},$$

as well as that $\|(d/dz)\mathcal{P}_\eta(z)\| = O(|z|^{\eta(\beta-1)-1})$, $z \in \Sigma_{\xi-\epsilon}$. Using this fact, we can simply prove that, for every $R > 0$, the mappings $z \mapsto \mathcal{S}_\eta(z) \in L(E)$, $z \in \Sigma_{\xi-\epsilon} \setminus B_R$ and $z \mapsto \mathcal{P}_\eta(z) \in L(E)$, $z \in \Sigma_{\xi-\epsilon} \setminus B_R$ are uniformly continuous. Almost immediately from definition of $\mathcal{P}_\eta(\cdot)$, we have that:

$$\mathcal{S}_\eta(z)x = \left(g_{1-\eta} * \left[\cdot^{\eta-1} \mathcal{P}_\gamma(\cdot)x \right] \right)(z), \quad z \in \Sigma_\xi, \ x \in E.$$

Taking the Laplace transform, we get that

$$(4.16) \qquad \int_0^\infty e^{-\lambda t} t^{\eta-1} P_\eta(t)x \, dt = \left(\lambda^\eta - \mathcal{A} \right)^{-1} x, \quad \lambda > 0, \ x \in E.$$

On the other hand, employing the identity $[\mathbf{2}, (3.10)]$ we deduce that

$$\int_0^\infty e^{-\lambda t} t^{-\gamma-1} \Phi_\gamma\left(st^{-\gamma} \right) dt = \frac{1}{\gamma s} e^{-\lambda^\gamma s}, \quad s > 0, \ \lambda > 0.$$

Making use of this equality and (3.3) with $r = 0$, we get that

$$(4.17)$$
$$\int_0^\infty \int_0^\infty \gamma s T_{\eta',0}(s)x \left[e^{-\lambda t} t^{-\gamma-1} \Phi_\gamma\left(st^{-\gamma} \right) dt \right] ds = \left(\lambda^\eta - \mathcal{A} \right)^{-1} x, \quad \lambda > 0, \ x \in E.$$

By (4.16)-(4.17) and the uniqueness theorem for the Laplace transform, we obtain the following representation formula

$$(4.18) \qquad \mathcal{P}_\gamma(t)x = t^{-\eta} \int_0^\infty \gamma s \Phi_\gamma\left(st^{-\gamma} \right) T_{\eta',0}(s)x \, ds, \quad t > 0, \ x \in E,$$

which continue to hold on subsectors of Σ_ξ.

Applying (B2), Lemma 1.1 and Lemma 2.2(i), it is very simple to prove the expected inclusion $((g_\eta * \mathcal{S}_\eta)(t), \mathcal{S}_\eta(t)x - x) \in \mathcal{A}$, $t > 0$, $x \in E$. By the closedness of $\mathcal{A}$, we get that

$$(4.19) \qquad \left(\frac{d}{dt}(g_\eta * \mathcal{S}_\eta)(t), \frac{d}{dt}\mathcal{S}_\eta(t)x \right) \in \mathcal{A}, \quad t > 0, \ x \in E.$$

Similarly, we can prove that

$$(4.20) \qquad \left(\frac{d}{dt}(g_{\eta'} * \mathcal{T}_{\eta'})(t), \frac{d}{dt}\mathcal{T}_{\eta'}(t)x \right) \in \mathcal{A}, \quad t > 0, \ x \in E.$$

On the other hand, differentiating (4.13) we get

$$(4.21) \qquad \frac{d}{dz}\mathcal{S}_\eta(z)x = \gamma z^{\gamma-1} \int_0^\infty s\Phi_\gamma(s)\mathcal{T}'_{\eta'}(sz^\gamma)x\,ds, \quad z \in \Sigma_\xi,\ x \in E.$$

Keeping in mind (4.21) and Lemma 2.1, it readily follows that

(4.22)
$$\frac{d}{dz}\mathcal{S}_\eta(z)x \in \mathcal{A}\left[\gamma z^{\gamma-1} \int_0^\infty s\Phi_\gamma(s)\frac{d}{ds}\big(g_{\eta'} * \mathcal{T}_{\eta'}\big)(sz^\gamma)x\,ds\right], \quad z \in \Sigma_\xi,\ x \in E.$$

Due to (4.19)- (4.21) and Lemma 2.1, we have

$$\frac{d}{dt}\big(g_\eta * \mathcal{S}_\eta\big)(t)x = \mathcal{A}^{-1}\frac{d}{dt}\mathcal{S}_\eta(t)x$$

$$= \mathcal{A}^{-1}\left[\gamma t^{\gamma-1} \int_0^\infty s\Phi_\gamma(s)\mathcal{T}'_{\eta'}(st^\gamma)x\,ds\right]$$

$$= \gamma t^{\gamma-1} \int_0^\infty s\Phi_\gamma(s)\frac{d}{ds}\big(g_{\eta'} * \mathcal{T}_{\eta'}\big)(st^\gamma)x\,ds, \quad t > 0,\ x \in E.$$

Using this equality, the uniqueness theorem for analytic functions and the fact that the mapping $z \mapsto \gamma z^{\gamma-1} \int_0^\infty s\Phi_\gamma(s)\frac{d}{ds}(g_{\eta'} * \mathcal{T}_{\eta'})(sz^\gamma)\,ds$, $z \in \Sigma_\xi$ is strongly analytic, we may conclude that the term appearing in brackets of (4.22) is equal to $\frac{d}{dz}(g_\eta * \mathcal{S}_\eta)(z)x$, $z \in \Sigma_\xi$, $x \in E$. Hence,

$$\frac{d}{dz}\mathcal{S}_\eta(z)x \in z^{\eta-1}\mathcal{A}\mathcal{P}_\eta(z)x, \quad z \in \Sigma_\xi,\ x \in E.$$

Suppose now that $(x, y) \in \mathcal{A}$. Then $\mathcal{S}_\eta(z)y \in \mathcal{A}\mathcal{S}_\eta(z)x$ and $\mathcal{P}_\eta(z)y \in \mathcal{A}\mathcal{P}_\eta(z)x$ $(z \in \Sigma_\xi)$. Further on, performing the Laplace transform, we can simply prove with the help of Lemma 2.2(i) that $\mathcal{S}_\eta(t)x - x = (g_\eta * \mathcal{S}_\eta)(t)y$, $t > 0$. Hence, $(d/dt)\mathcal{S}_\eta(t)x = (d/dt)(g_\eta * \mathcal{S}_\eta)(t)y$, $t > 0$ and, by the uniqueness theorem for analytic functions, $(d/dz)\mathcal{S}_\eta(z)x = (d/dz)(g_\eta * \mathcal{S}_\eta)(z)y$, $z \in \Sigma_\xi$. Therefore, $\mathcal{S}_\eta(z)x - x = \int_0^z \lambda^{\eta-1}\mathcal{P}_\eta(\lambda)y\,d\lambda$, $z \in \Sigma_\xi$ and

$$\frac{d}{dz}\mathcal{S}_\eta(z)x \in z^{\eta-1}\mathcal{P}_\eta(z)y, \quad z \in \Sigma_\xi,$$

which clearly implies in combination with (4.15) that the mapping $t \mapsto \frac{d}{dt}\mathcal{S}_\eta(t)x$, $t > 0$ is locally integrable. Furthermore, the identity $(g_{1-\eta} * [\mathcal{S}_\eta(\cdot)x - x])(t) = \int_0^t \mathcal{S}_\eta(s)y\,ds$, $t \geq 0$ almost immediately implies that $\mathbf{D}_t^\eta\mathcal{S}_\eta(t)x = \mathcal{S}_\eta(t)y \in \mathcal{A}\mathcal{S}_\eta(t)x$, $t > 0$. As observed in [**19**], this result is not optimal. Let $1 > \theta > 1-\beta$, and let $x \in D((-\mathcal{A})^\theta) \cap E_\mathcal{A}^\theta$. Then the mapping $t \mapsto F(t) := (g_{1-\eta} * [\mathcal{S}_\eta(\cdot)x - x])(t)$ is continuous for $t \geq 0$ and can be analytically extended from the positive real axis to the sector Σ_ξ, with the estimate $\|F(z)\| = O(|z|^{\eta(\beta+\theta-2)+1})$ on any proper subsector of Σ_ξ (see (B3)'). By the Cauchy integral formula, we deduce that $\|F'(z)\| = O(|z|^{\eta(\beta+\theta-2)})$

on proper subsectors of Σ_ξ and, consequently, the Caputo fractional derivative $\mathbf{D}_t^\eta \mathcal{S}_\eta(t)x$ is well-defined. Since $\mathcal{S}_\eta(t)x - x \in \mathcal{A}(g_\eta * \mathcal{S}_\eta)(t)x$, $t > 0$, Lemma 2.1 yields that $F(t) \in \mathcal{A}(g_1 * \mathcal{S}_\eta)(t)x$, $t \geq 0$. By the closedness of $\mathcal{A}$, we get that $\mathbf{D}_t^\eta \mathcal{S}_\eta(t)x = F'(t) \in \mathcal{A}\mathcal{S}_\eta(t)x$, $t > 0$.

Suppose now that $1 > \theta > 1 - \beta$. Then, for every $x \in D((-\mathcal{A})^\theta) \cap E_\mathcal{A}^\theta$, the mapping $z \mapsto \frac{d}{dz}\mathcal{S}_\eta(z)x = \frac{d}{dz}[\mathcal{S}_\eta(z)x - x]$ is bounded by $|z|^{\eta(\beta+\theta-1)-1}$ on subsectors of Σ_ξ; this follows from the Cauchy integral formula and the property (B3)' with $\nu = 0$. Therefore, the mapping $t \mapsto \frac{d}{dt}\mathcal{S}_\eta(t)x$, $t > 0$ is locally integrable for any $x \in D((-\mathcal{A})^\theta) \cap E_\mathcal{A}^\theta$.

The following definition has been recently introduced in [**19**].

Definition 4.2. *Let $T \in (0, \infty)$ and $f \in L^1((0,T) : E)$. Consider the following abstract degenerate fractional inclusion:*

$$(DFP)_{f,\eta} : \begin{cases} \mathbf{D}_t^\eta u(t) \in \mathcal{A}u(t) + f(t), \ t \in (0,T], \\ u(0) = u_0. \end{cases}$$

(i) *By a mild solution of $(DFP)_{f,\eta}$, we mean a function*

$$u(t) = \mathcal{S}_\eta(t)u_0 + \int_0^t (t-s)^{\eta-1} \mathcal{P}_\eta(t-s)f(s)\,ds, \quad t \in (0,T].$$

(ii) *By a classical solution of $(DFP)_{f,\eta}$, we mean any function $u \in C([0,T] : E)$ satisfying that the function $\mathbf{D}_t^\eta u(t)$ is well-defined and belongs to the space $C((0,T] : E)$, as well as that $u(0) = u_0$ and $\mathbf{D}_t^\eta u(t) - f(t) \in \mathcal{A}u(t)$ for $t \in (0,T]$.*

A mild solution $u(t)$ of problem $(DFP)_{f,\eta}$ is automatically continuous on $(0,T]$. If $x \in D((-\mathcal{A})^\theta) \cap E_\mathcal{A}^\theta$, where $1 > \theta > 1-\beta$, then the function $u(t) = \mathcal{S}_\eta(t)x$ is a classical solution of $(DFP)_{f,\eta}$, with $f \equiv 0$. The same holds if $x \in D(\mathcal{A})$.

The following extension of [**19**, Theorem 3.1] holds true:

Theorem 4.3. *Suppose that $T \in (0, \infty)$, $1 \geq \theta > 1 - \beta$ and $x \in D((-\mathcal{A})^\theta)$, resp. $1 > \theta > 1 - \beta$ and $x \in E_\mathcal{A}^\theta$, as well as that there exist constants $\sigma > \eta(1-\beta)$ and $M \geq 1$ such that*

$$(4.23) \qquad \|f(t) - f(s)\| \leq M|t-s|^\sigma, \quad 0 < t, \ s \leq T.$$

Let $1 \geq \theta > 1 - \beta$, resp. $1 > \theta > 1 - \beta$, and let

$$f \in L^\infty\Big((0,T) : [D((-\mathcal{A})^\theta)]\Big), \ resp. \ f \in L^\infty\Big((0,T) : E_\mathcal{A}^\theta\Big).$$

Then there exists a unique classical solution of problem $(DFP)_{f,\eta}$.

Proof. We will only provide the most relevant points of proof provided that $1 \geq \theta > 1 - \beta$ and $x \in D((-\mathcal{A})^\theta)$. Let $\delta' \in (0, \delta)$ be fixed. Then (3.11) and the Cauchy

integral formula together imply:

$$\|(d/dz)\mathcal{T}_{\eta'}(z)x\|$$
$$= \|(d/dz)[\mathcal{T}_{\eta'}(z)x - x]\|$$
$$\leq C_{\theta,\delta'}|z|^{\eta'(\beta+\theta-1)-1}\|x\|_{[D((-\mathcal{A})^{\theta})]}, \quad z \in \Sigma_{\delta'},\ x \in D\big((-\mathcal{A})^{\theta}\big).$$

Using (4.21) and this estimate, it readily follows that there exists a constant $C_{\theta} > 0$ such that, for every $0 < s \leq T$ and $0 < \omega \leq T$,

$$\left\|S'_{\gamma}(\omega)f(s)\right\| \leq C_{\theta}\big\|f(s)\big\|_{[D((-\mathcal{A})^{\theta})]}\omega^{\eta(\beta+\theta-1)-1}.$$

The remaining part of proof follows by repeating literally the argumentation contained in the proof of [**19**, Theorem 3.1]. □

Semilinear degenerate Cauchy inclusions. In this subsection, we will present the most important details concerning the existence and uniqueness of mild solutions of the following semilinear degenerate fractional Cauchy inclusion:

$$(\mathrm{DFP})_{f,s,\eta} : \begin{cases} \mathbf{D}_t^{\eta}u(t) \in \mathcal{A}u(t) + f(t,u(t)),\ t \in (0,T], \\ u(0) = u_0, \end{cases}$$

where $T \in (0,\infty)$ (cf. [**12**], [**19**] and references cited there for further information on the subject). Let us recall ([**33**], [**19**]) that a mild solution $u(t) := u(t;u_0)$ of problem $(\mathrm{DFP})_{f,s,\eta}$ is any function $u \in C((0,T] : E)$ such that

$$(4.24) \qquad u(t) = \mathcal{S}_{\eta}(t)u_0 + \int_0^t (t-s)^{\eta-1}\mathcal{P}_{\eta}(t-s)f(s,u(s))\,ds, \quad t \in (0,T].$$

As in linear case, a classical solution of $(\mathrm{DFP})_f$ is any function $u \in C([0,T] : E)$ satisfying that the function $\mathbf{D}_t^{\eta}u(t)$ is well-defined and belongs to the space $C((0,T] : E)$, as well as that $u(0) = u_0$ and $\mathbf{D}_t^{\eta}u(t) - f(t,u(t)) \in \mathcal{A}u(t)$ for $t \in (0,T]$. Conditions under which (4.24) defines a classical solution of problem $(\mathrm{DFP})_f$ will not be explored in this paper.

The first thing we would like to observe is that the assertions of [**33**, Theorem 5.1, Corollary 5.1, Remark 5.1], [**18**, Theorem 2.1] and [**20**, Theorem 3.1] can be simply reformulated in our framework. As explained in [**19**], this is not the case with the assertions of [**33**, Theorem 5.2, Theorem 5.4].

The situation is slightly different with the assertion of [**33**, Theorem 5.3], where the authors have considered the existence of mild solutions of semilinear degenerate fractional Cauchy inclusion $(\mathrm{DFP})_{f,s,\eta}$, provided that the resolvent of $\mathcal{A}$ is compact. The operator familiy $(\mathcal{S}_{\eta}(t))_{t>0}$ is then subordinated to a semigroup $(T(t))_{t>0}$ which do have a removable singularity at zero, and the compactness of operators $\mathcal{S}_{\eta}(t)$ and $\mathcal{P}_{\eta}(t)$ for $t > 0$ (cf. [**33**, Lemma 3.1, Theorem 3.5]) has been proved by following a method based on the use of semigroup property of $(T(t))_{t>0}$.

In purely fractional case, we can argue as follows. Recall that the set consisting of all compact operators on E is a closed linear subspace of $L(E)$ forming a two-sided ideal in $L(E)$. Since (3.3) and (B2) hold in the uniform operator topology, we can apply Lemma 2.2(iii), the Post-Widder inversion formula [1, Theorem 1.7.7] and the formulae [2, (2.16)-(2.17)] in order to see that the operator $T_{\eta',r}(t)$ is compact for $\eta'\beta > r$, $t > 0$ and that the operator $\mathcal{S}_\eta(t)$ is compact for $t > 0$. Applying (4.18), we obtain that the operator $\mathcal{P}_\eta(t)$ is compact for $t > 0$, as well. Now we can reformulate [33, Theorem 5.3] by using the following approximation in Step 3 of its proof:

$$\Gamma^\eta_{\epsilon,\delta}(t) := \mathcal{S}_\eta(t) + \int_0^{t-\epsilon} \left(t - s\right)^{2\gamma-1} \int_\delta^\infty \gamma\tau\Phi_\gamma(\tau)T_{\eta',0}\left(\tau(t - s)^\gamma\right) d\tau\, ds,$$

for $t \in (0, T]$, $\delta > 0$, $0 < \epsilon < t$ and $u \in \Omega_r$; cf. [33] for the notion. Assuming the validity of a condition like [33, (H2)], we can prove the estimate

$$\left\| \mathcal{S}_\eta(t)u_0 + \int_0^t \left(t - s\right)^{\eta-1}\mathcal{P}_\eta(t - s)f(s, u(s))\, ds - \Gamma^\eta_{\epsilon,\delta}(t) \right\|$$

$$\leq \mathrm{Const.}\left(\int_0^{t-\epsilon} \left(t - s\right)^{q[\beta\eta+\gamma-1]}\, ds\right)^{1/q} \|m_r\|_{L^p(0,T)} \int_0^\infty \tau^{\eta'\beta}\Phi_\gamma(\tau)\, d\tau$$

$$+ \mathrm{Const.}\left(\int_{t-\epsilon}^t \left(t - s\right)^{q[\beta\eta+\gamma-1]}\, ds\right)^{1/q} \|m_r\|_{L^p(0,T)} \int_0^\infty \tau^{\eta'\beta}\Phi_\gamma(\tau)\, d\tau$$

for $p > 1$ and $q = p/p - 1$. Now it is quite simple to reformulate [33, Theorem 5.3] in our context.

In [26, Theorem 3.1, Theorem 3.2], F. Periago has considered semilinear Cauchy inclusions of first order associated with the use of almost sectorial operators. In [19], we have extended these results to degenerate differential inclusions of first order and remark what can be done in fractional relaxation case. Here we would like to observe that it is not clear how we can prove an extension of [19, Theorem 4.3] in the case that the operator family $(\mathcal{S}_\eta(t))_{t>0}$ is not subordinated to a degenerate semigroup.

Before we provide some illustrative applications of our results to abstract (non-)degenerate fractional differential equations, it would be worthwhile to mention that some of theorems considered in this subsection require the condition $\beta > 1/2$, which seems to be restrictive in degenerate case (cf. [19] for more details).

EXAMPLE 4.4.

(i) ([32]) Suppose that $\alpha \in (0, 1)$, $m \in \mathbb{N}$, Ω is a bounded domain in $\mathbb{R}^n$ with boundary of class C^{4m} and $E := C^\alpha(\overline{\Omega})$. Let us consider the operator $A : D(A) \subseteq C^\alpha(\overline{\Omega}) \to C^\alpha(\overline{\Omega})$ given by

$$Au(x) := \sum_{|\beta|\leq 2m} a_\beta(x)D^\beta u(x) \text{ for all } x \in \overline{\Omega}$$

with domain $D(A) := \{u \in C^{2m+\alpha}(\overline{\Omega}) : D^\beta u_{|\partial\Omega} = 0 \text{ for all } |\beta| \leq m - 1\}$. Here, $\beta \in \mathbb{N}_0^n$, $|\beta| = \sum_{i=1}^n \beta_j$, $D^\beta = \prod_{i=1}^n (\frac{1}{i}\frac{\partial}{\partial x_i})^{\beta_i}$, and we assume that $a_\beta : \overline{\Omega} \to \mathbb{C}$ satisfy the following:

(i) $a_\beta(x) \in \mathbb{R}$ for all $x \in \overline{\Omega}$ and $|\beta| = 2m$.

(ii) $a_\beta \in C^\alpha(\overline{\Omega})$ for all $|\beta| \leq 2m$, and

(iii) there is a constant $M > 0$ such that

$$M^{-1}|\xi|^{2m} \leq \sum_{|\beta|=2m} a_\beta(x)\xi^\beta \leq M|\xi|^{2m} \text{ for all } \xi \in \mathbb{R}^n \text{ and } x \in \overline{\Omega}.$$

Then it is well known that there exists a sufficiently large number $\sigma > 0$ such that the operator $-A_\sigma \equiv -(A + \sigma)$ satisfies $\Sigma_\omega \cup \{0\} \subseteq \rho(-A_\sigma)$ with some $\omega \in (\frac{\pi}{2}, \pi)$ and

$$(4.25) \qquad \left\| R(\lambda : -A_\sigma) \right\| = O\big(|\lambda|^{\frac{\alpha}{2m}-1}\big), \ \lambda \in \Sigma_\omega.$$

Let us recall that A is not densely defined and that the exponent $\frac{\alpha}{2m} - 1$ in (4.25) is sharp. Define $A_{\sigma,\delta} := e^{i(\pi/2\pm\delta)}A_\sigma$. Suppose that $\omega - (\pi/2) < \delta < \omega - \eta(\pi/2)$, $1 \geq \theta > \alpha/2m$, $u_0 \in D((-A_{\sigma,\delta})^\theta)$, $\sigma > \eta\alpha/2m$, (4.23) holds and $f \in L^\infty((0,T) : [D((-\mathcal{A})^\theta)])$. Then the condition (QP) holds for each number $\eta'' \in (\eta, 1)$ such that $\omega - (\pi/2) < \delta < \omega - \eta''(\pi/2)$. Applying Theorem 4.3, we obtain that the abstract fractional Cauchy problem

$$\left\{ \begin{array}{l} \mathbf{D}_t^\eta u(t,x) = A_{\sigma,\delta}u(t,x) + f(t,x), \ t \in (0,T], \\ \quad u(0) = u_0, \end{array} \right.$$

has a unique classical solution, which is analytically extendable to the sector Σ_ϑ provided that $f(t,x) \equiv 0$ ($\vartheta \equiv \min((\frac{(2/\pi)(\omega-\delta)}{\eta} - 1)\pi/2, \pi)$).

(ii) ([11]) Consider now the following modification of inhomogeneous fractional Poisson heat equation in the space $L^p(\Omega)$:

$$(P)_\eta^\delta : \left\{ \begin{array}{l} \mathbf{D}_t^\eta[m(x)v(t,x)] = e^{\pm i\delta}(\Delta - b)v(t,x) + f(t,x), \quad t \geq 0, \ x \in \Omega; \\ v(t,x) = 0, \quad (t,x) \in [0,\infty) \times \partial\Omega, \\ m(x)v(0,x) = u_0(x), \quad x \in \Omega, \end{array} \right.$$

where Ω is a bounded domain in $\mathbb{R}^n$, $b > 0$, $m(x) \geq 0$ a.e. $x \in \Omega$, $m \in L^\infty(\Omega)$, $1 < p < \infty$ and $0 < \eta < 1$. Let the operator $A := \Delta - b$ act on E with the Dirichlet boundary conditions, and let B be the multiplication operator by the function $m(x)$. As it has been proved in [11, Example 3.6], there exist an appropriate angle $\omega \in (\frac{\pi}{2}, \pi)$ and a number $d > 0$ such that the multivalued linear operator $\mathcal{A} := AB^{-1}$ satisfies $\Psi_{d,\omega} = \{\lambda \in \mathbb{C} : |\lambda| \leq d \text{ or } \lambda \in \overline{\Sigma_\omega}\} \subseteq \rho(\mathcal{A})$ and $\|R(\lambda : \mathcal{A})\| \leq M(1+|\lambda|)^{-1/p}$, $\lambda \in \Psi_{d,\omega}$; here it is worth noting that the validity of additional condition [11, (3.42)] on the function $m(x)$ enables us to get the better exponent β in (QP), provided that $p > 2$. Henceforth

we consider the general case. Suppose, as in the part (i), that $\omega - (\pi/2) < \delta < \omega - \eta(\pi/2)$, $1 \geq \theta > 1 - 1/p$, $u_0 \in D((-e^{\pm i\delta}\mathcal{A})^\theta)$, $\sigma > \eta(1 - 1/p)$, (4.23) holds and $f \in L^\infty((0,T) : [D((-e^{\pm i\delta}\mathcal{A})^\theta)])$. Then Theorem 4.3 implies that the abstract Cauchy problem $(P)_\eta^\delta$ has a unique solution $t \mapsto v(t,\cdot)$, $t \in (0,T]$, i.e., any function $v(t,\cdot)$ satisfying that $Bv(t,\cdot) \in C([0,T] : E)$, the Caputo fractional derivative $\mathbf{D}_t^\eta Bv(t,\cdot)$ is well-defined and belongs to the space $C((0,T] : E)$, $Bv(t,\cdot) \in C((0,T] : E)$, $m(x)v(0,x) = u_0(x)$, $x \in \Omega$ and $(P)_\eta^\delta$ holds identically.

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