MERGING THE A- AND Q-SPECTRAL THEORIES

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Dedicated to Professor Dragoš Cvetković on the occasion of his 75th birthday

Let G be a graph with adjacency matrix $A(G)$, and let $D(G)$ be the diagonal matrix of the degrees of G. The signless Laplacian $Q(G)$ of G is defined as $Q(G) := A(G) + D(G)$. Cvetković called the study of the adjacency matrix the *A-spectral theory*, and the study of the signless Laplacian–the *Q-spectral theory*. To track the gradual change of $A(G)$ into $Q(G)$, in this paper it is suggested to study the convex linear combinations $A_\alpha(G)$ of $A(G)$ and $D(G)$ defined by

$$A_\alpha(G) := \alpha D(G) + (1-\alpha) A(G), \quad 0 \le \alpha \le 1.$$

This study sheds new light on $A(G)$ and $Q(G)$, and yields, in particular, a novel spectral Turán theorem. A number of open problems are discussed.

1. INTRODUCTION

Let G be a graph with adjacency matrix $A(G)$, and let $D(G)$ be the diagonal matrix of the degrees of G. In this paper we study hybrids of $A(G)$ and $D(G)$ similar to the *signless Laplacian* $Q(G) := A(G) + D(G)$, which has been put forth by Cvetković in [**5**] and extensively studied since then. For detailed coverage of this research see [**7**],[**8**],[**9**],[**4**], and their references. The study of $Q(G)$ has shown that it is a remarkable matrix, unique in many respects. Yet, $Q(G)$ is just the sum of $A(G)$ and $D(G)$, and the study of $Q(G)$ has uncovered both similarities and differences between $Q(G)$ and $A(G)$. To understand to what extent each of

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the summands $A(G)$ and $D(G)$ determines the properties of $Q(G)$, we propose to study the convex combinations $A_\alpha(G)$ of $A(G)$ and $D(G)$ defined by

$$(1) \qquad A_\alpha(G) := \alpha D(G) + (1-\alpha) A(G), \quad 0 \le \alpha \le 1.$$

Many facts suggest that the study of the family $A_\alpha(G)$ is long due. To begin with, obviously,

$$A(G) = A_0(G), \quad D(G) = A_1(G), \quad \text{and} \quad Q(G) = 2A_{1/2}(G).$$

Since $A_{1/2}(G)$ is essentially equivalent to $Q(G)$, in this paper we take $A_{1/2}(G)$ as an exact substitute for $Q(G)$. With this caveat, one sees that $A_\alpha(G)$ seamlessly joins $A(G)$ to $D(G)$, with $Q(G)$ being right in the middle of the range. In this setup, the matrices $A(G)$, $Q(G)$, and $D(G)$ can be seen in a new light, and many interesting problems arise. In particular, we are compelled to investigate the hitherto uncharted territory $\alpha > 1/2$, which holds some surprises, e.g., a novel version of the spectral Turán theorem (see Theorem 27 below).

Let us note the crucial identity

$$(2) \qquad A_\alpha(G) - A_\beta(G) = (\alpha - \beta) L(G),$$

where $L(G)$ is the well-studied Laplacian of G, defined as $L(G) := D(G) - A(G)$. This neat relation corroborates the soundness of definition (1).

It is worth pointing out that the family $A_\alpha(G)$ is just a small subset of the *generalized adjacency matrices* defined in [**10**] and the *universal adjacency matrices* defined in [**17**]. However, our restricted definition allows to prove stronger theorems, which are likely to fail for those more general classes.

The rest of the paper is structured as follows. In the next section we introduce some notation and recall basic facts about spectra of matrices. In Section 3 we present a few general results about the matrices $A_\alpha(G)$. Section 4 deals with the largest eigenvalue of $A_\alpha(G)$. Section 5 is dedicated to spectral extremal problems, which are at the heart of spectral graph theory. A miscellany of topics are covered in Section 6. Finally, in Section 7, we calculate the A_α-spectra of some specific graphs.

2. NOTATION AND PRELIMINARIES

Let $[n] := \{1, \dots, n\}$. Given a real symmetric matrix M, write $\lambda_k(M)$ for the kth largest eigenvalue of M. For short, we write $\lambda(M)$ and $\lambda_{\min}(M)$ for the largest and the smallest eigenvalues of M.

Given a graph G, we write:

- $V(G)$ and $E(G)$ for the sets of vertices and edges of G, and $v(G)$ for $|V(G)|$;
- $\Gamma_G(u)$ for the set of neighbors of a vertex u, and $d_G(u)$ for $|\Gamma_G(u)|$;
- $\delta(G)$ and $\Delta(G)$ for the minimum and maximum degree of G;

- $w_G(u)$ for the number of walks of length 2 starting with the vertex u, i.e., $w_G(u) = \sum_{\{u,v\}\in E(G)} d_G(v)$;

- $G[X]$ for the subgraph of G induced by a set $X \subset V(G)$;

- $e(X)$ for number of edges of $G[X]$;

- $e(X,Y)$ for the number of edges between two disjoint sets $X \subset V(G)$ and $Y \subset V(G)$;

- $G - X$ for the graph obtained by deleting the vertices of a set $X \subset V(G)$.

In the above notation the subscript G will be omitted if G is understood.

A *coclique* of G is an edgeless induced subgraph of G. Further, K_n stands for the complete graph of order n, and $K_{a,b}$ stands for the complete bipartite graph with partition sets of sizes a and b. In particular, $K_{1,n-1}$ denotes the star of order n. We write $S_{n,k}$ for the graph obtained by joining each vertex of a complete graph of order k to each vertex of an independent set of order $n - k$, that is, $S_{n,k} = K_k \vee \overline{K}_{n-k}$.

On many occasions we shall use Weyl's inequalities for eigenvalues of Hermitian matrices (see, e.g. [**20**], p. 181). Although these fundamental inequalities have been known for almost a century by now, their equality case was first established by So in [**31**], based on the paper [**21**] by Ikebe, Inagaki and Miyamoto.

For convenience we state below the complete theorem of Weyl and So:

Theorem WS *Let A and B be Hermitian matrices of order n, and let $1 \leq i \leq n$ and $1 \leq j \leq n$. Then*

(3) $\qquad\qquad \lambda_i(A) + \lambda_j(B) \leq \lambda_{i+j-n}(A+B),$ if $i + j \geq n + 1,$

(4) $\qquad\qquad \lambda_i(A) + \lambda_j(B) \geq \lambda_{i+j-1}(A+B),$ if $i + j \leq n + 1.$

In either of these inequalities equality holds if and only if there exists a nonzero n-vector that is an eigenvector to each of the three eigenvalues involved.

A simplified version of (3) and (4) reads as

(5) $\qquad\qquad \lambda_k(A) + \lambda_{\min}(B) \leq \lambda_k(A+B) \leq \lambda_k(A) + \lambda(B).$

Further, we shall need the following simple properties of the Laplacian:

Proposition L *If G is a graph of order n, then*

$$\lambda(L(G)) \leq n \quad \text{and} \quad \lambda_{\min}(L(G)) = 0.$$

If G is connected, then every eigenvector of $L(G)$ to the eigenvalue 0 is constant.

Finally, recall that a real symmetric matrix M is called *positive semidefinite* if $\lambda_{\min}(M) \geq 0$. Likewise, M is called *positive definite* if $\lambda_{\min}(M) > 0$.

3. BASIC PROPERTIES OF $A_\alpha(G)$

Given a graph G of order n, it is obvious that the system of eigenequations for the matrix $A_\alpha(G)$ is

$$(6) \qquad \lambda x_k = \alpha d_G(k) x_k + (1-\alpha) \sum_{\{i,k\}\in E(G)} x_i, \quad 1 \le k \le n.$$

3.1 The quadratic form $\langle A_\alpha \mathbf{x}, \mathbf{x}\rangle$

If G is a graph of order n with $A_\alpha(G) = A_\alpha$, and $\mathbf{x} := (x_1, \ldots, x_n)$ is a real vector, the quadratic form $\langle A_\alpha \mathbf{x}, \mathbf{x}\rangle$ can be represented in several equivalent ways; for example,

$$(7) \qquad \langle A_\alpha \mathbf{x}, \mathbf{x}\rangle = \sum_{\{u,v\}\in E(G)} (\alpha x_u^2 + 2(1-\alpha)x_u x_v + \alpha x_v^2),$$

$$(8) \qquad \langle A_\alpha \mathbf{x}, \mathbf{x}\rangle = (2\alpha - 1) \sum_{u\in V(G)} x_u^2 d(u) + (1-\alpha) \sum_{\{u,v\}\in E(G)} (x_u + x_v)^2,$$

$$(9) \qquad \langle A_\alpha \mathbf{x}, \mathbf{x}\rangle = \alpha \sum_{u\in V(G)} x_u^2 d(u) + 2(1-\alpha) \sum_{\{u,v\}\in E(G)} x_u x_v.$$

Each of these representations can be useful in proofs.

Equation (2) implies the following characteristic property of $\langle A_\alpha(G)\mathbf{x}, \mathbf{x}\rangle$:

Proposition 1. *If $1 \ge \alpha > \beta \ge 0$ and G is a graph of order n, then*

$$\langle A_\alpha(G)\mathbf{x}, \mathbf{x}\rangle \ge \langle A_\beta(G)\mathbf{x}, \mathbf{x}\rangle$$

for any n-vector $\mathbf{x}$.

Further, since $A_\alpha(G)$ is a real symmetric matrix, Rayleigh's principle implies the following assertion:

Proposition 2. *If $\alpha \in [0,1]$ and G is a graph of order n with $A_\alpha(G) = A_\alpha$, then*

$$(10) \qquad \lambda(A_\alpha) = \max_{\|\mathbf{x}\|_2 = 1} \langle A_\alpha \mathbf{x}, \mathbf{x}\rangle \quad \text{and} \quad \lambda_{\min}(A_\alpha) = \min_{\|\mathbf{x}\|_2 = 1} \langle A_\alpha \mathbf{x}, \mathbf{x}\rangle.$$

Moreover, if $\mathbf{x}$ is a unit n-vector, then $\lambda(A_\alpha) = \langle A_\alpha \mathbf{x}, \mathbf{x}\rangle$ if and only if $\mathbf{x}$ is an eigenvector to $\lambda(A_\alpha)$, and $\lambda_{\min}(A_\alpha) = \langle A_\alpha \mathbf{x}, \mathbf{x}\rangle$ if and only if $\mathbf{x}$ is an eigenvector to $\lambda_{\min}(A_\alpha)$.

In turn, relations (10) yield the following statement:

Proposition 3. *If $\alpha \in [0,1]$ and G is a graph with $A_\alpha(G) = A_\alpha$, then*

$$\lambda(A_\alpha) = \max\{\lambda(A_\alpha(H)) : H \text{ is a component of } G\},$$
$$\lambda_{\min}(A_\alpha) = \min\{\lambda_{\min}(A_\alpha(H)) : H \text{ is a component of } G\}.$$

Caution: *If G is disconnected, $\lambda\left(A_\alpha\right)$ can be attained on different components of G, depending on α. For example, let $k \geq 2$ be an integer and let G be the disjoint union of $K_{3k+1,3k+1}$, $K_{3,3k^2}$, and $K_{1,3k^2+1}$. Calculating the largest eigenvalues of A_0, $A_{1/2}$, and A_1 for each of the three components of G, we get the following table:*

	$K_{3k+1,3k+1}$	$K_{3,3k^2}$	$K_{1,3k^2+1}$	G
$\lambda\left(A_0\right)$	$3k+1$	$3k$	$\sqrt{3k^2+1}$	$3k+1$
$\lambda\left(A_{1/2}\right)$	$3k+1$	$\left(3k^2+3\right)/2$	$\left(3k^2+1\right)/2$	$\left(3k^2+3\right)/2$
$\lambda\left(A_1\right)$	$3k+1$	$3k^2$	$3k^2+1$	$3k^2+1$

Thus, $\lambda\left(A_\alpha\left(G\right)\right)$ may be attained on any components of G, depending on α.

3.2 Monotonicity of $\lambda_k\left(A_\alpha\left(G\right)\right)$ in α

In this subsection, we shall show that $\lambda_k\left(A_\alpha\left(G\right)\right)$ is nondecreasing in α for any k. For a start, note that if G is a d-regular graph of order n, then

$$A_\alpha\left(G\right) = \alpha d I_n + \left(1 - \alpha\right) A\left(G\right);$$

hence, there is a linear correspondence between the spectra of $A_\alpha\left(G\right)$ and of $A\left(G\right)$

$$(11) \qquad \lambda_k\left(A_\alpha\left(G\right)\right) = \alpha d + \left(1 - \alpha\right)\lambda_k\left(A\left(G\right)\right), \quad 1 \leq k \leq n.$$

In particular, if G is a d-regular graph, then $\lambda\left(A_\alpha\left(G\right)\right) = d$ for any $\alpha \in [0,1]$. Moreover, if G is a regular and connected graph of order n, equations (11) imply that $\lambda_k\left(A\left(G\right)\right)$ is increasing in α for any $2 \leq k \leq n$. It turns out that the latter property is essentially valid for any graph:

Proposition 4. *Let $1 \geq \alpha > \beta \geq 0$. If G is a graph of order n with $A_\alpha\left(G\right) = A_\alpha$ and $A_\beta\left(G\right) = A_\beta$, then*

$$(12) \qquad \lambda_k\left(A_\alpha\right) - \lambda_k\left(A_\beta\right) \geq 0$$

for any $k \in [n]$. If G is connected, then inequality (12) is strict, unless $k = 1$ and G is regular.

Proof. Identity (2), inequality (5), and Proposition L imply that

$$(13) \qquad \lambda_k\left(A_\alpha\right) - \lambda_k\left(A_\beta\right) \geq \left(\alpha - \beta\right)\lambda_{\min}\left(L\left(G\right)\right) = 0.$$

If G is connected and equality holds in (13), Theorem WS implies that $\lambda_k\left(A_\beta\right)$, $\lambda_k\left(A_\alpha\right)$, and $\lambda_{\min}\left(L\left(G\right)\right)$ have a common eigenvector, which by Proposition L must be constant, say, the all-ones vector $\mathbf{j}_n$. Now, Proposition 14 implies that $k = 1$, and the eigenequations (6) imply that G is regular. $\qquad\square$

The premises of Proposition 4 also imply that

$$\lambda_k\left(A_\alpha\right) - \lambda_k\left(A_\beta\right) \leq \left(\alpha - \beta\right)n,$$

which leads us to

Proposition 5. *If G is a graph with $A_\alpha(G) = A_\alpha$, then the function $\lambda_k(A_\alpha)$ is Lipschitz continuous in α for any $k \in [n]$. Furthermore, $\lambda(A_\alpha)$ is convex in α, and $\lambda_{\min}(A_\alpha)$ is concave in α.*

Let us note that the convexity of $\lambda(A_\alpha)$ and the concavity of $\lambda_{\min}(A_\alpha)$ follow from Weyl's inequalities (5).

Question 6. *If $n \geq k \geq 1$, is the function $f(\alpha) := \lambda_k(A_\alpha)$ differentiable in α?*

3.3 Positive semidefiniteness of $A_\alpha(G)$

An important property of the signless Laplacian $Q(G)$ is that it is positive semidefinite matrix. This is certainly not true for $A_\alpha(G)$ if α is sufficiently small, but if $\alpha \geq 1/2$, then $A_\alpha(G)$ is similar to $Q(G)$.

Proposition 7. *If $\alpha > 1/2$, and G is a graph, then $A_\alpha(G)$ is positive semidefinite. If G has no isolated vertices, then $A_\alpha(G)$ is positive definite.*

Proof. Let $\mathbf{x} := (x_1, \ldots, x_n)$ be a nonzero vector. If $\alpha > 1/2$, then for any edge $\{u, v\} \in E$, we see that

$$(14) \qquad \langle A_\alpha(G)\mathbf{x}, \mathbf{x} \rangle \geq (1-\alpha)(x_u + x_v)^2 + (2\alpha - 1)x_u^2 + (2\alpha - 1)x_v{}^2 \geq 0.$$

Hence $A_\alpha(G)$ is positive semidefinite. Now, suppose that G has no isolated vertices. Select a vertex u with $x_u \neq 0$ and let $\{u, v\} \in E$. Then we have strict inequality in (14), implying that $A_\alpha(G)$ is positive definite. $\qquad\square$

Obviously, Proposition 4 implies that if $A_\alpha(G)$ is positive (semi)definite for some α, then $A_\beta(G)$ is positive (semi)definite for any $\beta > \alpha$. This observation leads to the following problem:

Problem 8. *Given a graph G, find the smallest α for which $A_\alpha(G)$ is positive semidefinite.*

For example, if G is the complete graph K_n, we have $\lambda_{\min}(A_\alpha(K_n)) = n\alpha - 1$; hence, $A_\alpha(K_n)$ is positive semidefinite if and only if $\alpha \geq 1/n$. This example can be generalized as follows:[1]

Proposition 9. *Let G be a regular graph with chromatic number r. If $\alpha < 1/r$, then $A_\alpha(G)$ is not positive semidefinite.*

Proof. Let G be a d-regular graph and let A be its adjacency matrix. Hoffman's bound [**19**] implies that

$$\lambda_{\min}(A) \leq -\frac{\lambda(A)}{r-1} = -\frac{d}{r-1}.$$

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[1]Some progress with Problem 8 is reported in [**28**], along with an extension of Proposition 9.

Hence, (11) implies that

$$\lambda_{\min}\left(A_\alpha\left(G\right)\right) \leq \alpha d - \left(1-\alpha\right)\frac{d}{r-1} = \left(\alpha - \frac{1}{r}\right)\frac{rd}{r-1} < 0,$$

completing the proof.   □

3.4 Some degree based bounds

It is not an exaggeration to say that degree based bounds are the most usable bounds in spectral graph theory. We give a few such bounds for $A_\alpha\left(G\right)$, the first of which follows from Proposition 4.

Proposition 10. *Let G be a graph of order n with degrees $d\left(1\right) \geq \cdots \geq d\left(n\right)$ and with $A_\alpha\left(G\right) = A_\alpha$. If $k \in [n]$, then*

$$\lambda_k\left(A_\alpha\right) \leq d\left(k\right).$$

In particular, $\lambda\left(A_\alpha\right) \leq \Delta\left(G\right).$

Using an idea of Das [11], the bound $\lambda_{\min}\left(A_\alpha\right) \leq \delta\left(G\right)$ can be improved further: let u be a vertex with minimum degree and define the n-vector $\mathbf{x} := \left(x_1, \ldots, x_n\right)$ by letting $x_u := 1$ and zeroing the other entries. Then Proposition 2 and equation (8) imply that

$$\lambda_{\min}\left(A_\alpha\right) \leq \langle A_\alpha \mathbf{x}, \mathbf{x} \rangle = \left(2\alpha - 1\right)\delta + \left(1-\alpha\right)\delta = \alpha\delta.$$

But, if $\alpha \in [0,1)$, the vector $\mathbf{x}$ does not satisfy the eigenequations for $\lambda_{\min}\left(A_\alpha\right)$; hence, we see that

$$\lambda_{\min}\left(A_\alpha\right) < \alpha\delta.$$

Further, Weyl's inequality (5) immediately implies the following bounds:

Proposition 11. *If $\alpha \in [0,1]$ and G is a graph with $A\left(G\right) = A$ and $A_\alpha\left(G\right) = A_\alpha$, then*

$$\alpha\delta + \left(1-\alpha\right)\lambda_k\left(A\right) \leq \lambda_k\left(A_\alpha\right) \leq \alpha\Delta + \left(1-\alpha\right)\lambda_k\left(A\right)$$

Next, we give a tight lower bound on $\lambda\left(A_\alpha\right)$, which generalizes a result of Lovász ([24], Problem 11.14):

Proposition 12. *If G is a graph with $\Delta\left(G\right) = \Delta$, then*

$$\lambda\left(A_\alpha\right) \geq \frac{1}{2}\left(\alpha\left(\Delta+1\right) + \sqrt{\alpha^2\left(\Delta+1\right)^2 + 4\Delta\left(1-2\alpha\right)}\right)$$

If $\alpha \in [0,1)$ and G is connected, equality holds if and only if $G = K_{1,\Delta}$.

Proof. Proposition 39 gives the spectral radius of A_α of a star. This result, combined with Proposition 14, yields

$$\lambda\left(A_\alpha\left(G\right)\right) \geq \lambda\left(A_\alpha\left(K_{1,\Delta}\right)\right) = \frac{1}{2}\left(\alpha\left(\Delta+1\right) + \sqrt{\alpha^2\left(\Delta+1\right)^2 + 4\Delta\left(1-2\alpha\right)}\right).$$

The case of equality also follows from Proposition 14.   □

Some algebra can be used to deduce a simpler lower bound:

Corollary 13. *Let G be a graph with $\Delta\left(G\right) = \Delta$. If $\alpha \in \left[0, 1/2\right]$, then*

$$\lambda\left(A_\alpha\left(G\right)\right) \geq \alpha\left(\Delta+1\right).$$

If $\alpha \in \left[1/2, 1\right)$, then

$$\lambda\left(A_\alpha\left(G\right)\right) \geq \alpha\Delta + \left(1-\alpha\right)^2/\alpha.$$

4. THE LARGEST EIGENVALUE $\lambda\left(A_\alpha\left(G\right)\right)$

As for the adjacency matrix and the signless Laplacian, the spectral radius $\lambda\left(A_\alpha\left(G\right)\right)$ of $A_\alpha\left(G\right)$ is its most important eigenvalue, owing to the fact that $A_\alpha\left(G\right)$ is nonnegative, so $\lambda\left(A_\alpha\left(G\right)\right)$ has maximal modulus among all eigenvalues of $A_\alpha\left(G\right)$.

4.1 Perron-Frobenius properties of $A_\alpha\left(G\right)$

In this subsection, we spell out some properties of $\lambda\left(A_\alpha\left(G\right)\right)$, which follow from the Perron-Frobenius theory of nonnegative matrices. Observe that if $0 \leq \alpha < 1$ and G is a graph, then G is connected if an only if $A_\alpha\left(G\right)$ is irreducible, because irreducibility is not affected by the diagonal entries of $A_\alpha\left(G\right)$. Hence, the Perron-Frobenius theory of nonnegative matrices implies the following properties of $A_\alpha\left(G\right)$:

Proposition 14. *Let $\alpha \in \left[0, 1\right)$, let G be a graph, and let $\mathbf{x}$ be a nonnegative eigenvector to $\lambda\left(A_\alpha\left(G\right)\right)$:*

(a) If G is connected, then $\mathbf{x}$ is positive and is unique up to scaling;

(b) If G is not connected and P is the set of vertices with positive entries in $\mathbf{x}$, then the subgraph induced by P is a union of components H of G with $\lambda\left(A_\alpha\left(H\right)\right) = \lambda\left(A_\alpha\left(G\right)\right)$;

(c) If G is connected and μ is an eigenvalue of $A_\alpha\left(G\right)$ with a nonnegative eigenvector, then $\mu = \lambda\left(A_\alpha\left(G\right)\right)$;

(d) If G is connected, and H is a proper subgraph of G, then $\lambda\left(A_\alpha\left(H\right)\right) < \lambda\left(A_\alpha\left(G\right)\right)$.

A practical consequence of Proposition 14 reads as:

Proposition 15. *Let $\alpha \in [0,1)$ and let G be a graph with $A_\alpha (G) = A_\alpha$. Let $u,v,w \in V(G)$ and suppose that $\{u,v\} \in E(G)$ and $\{u,w\} \notin E(G)$. Let H be the graph obtained from G by deleting the edge $\{u,v\}$ and adding the edge $\{u,w\}$. If $\mathbf{x} := (x_1, \ldots, x_n)$ is a unit eigenvector to $\lambda (A_\alpha)$ such that $x_u > 0$ and*

$$\langle A_\alpha (H) \mathbf{x}, \mathbf{x} \rangle \geq \langle A_\alpha \mathbf{x}, \mathbf{x} \rangle,$$

then $\lambda (A_\alpha (H)) > \lambda (A_\alpha)$.

Proof. Proposition 2 implies that $\lambda (A_\alpha (H)) \geq \lambda (A_\alpha)$, so our goal is to show that equality cannot hold. Assume for a contradiction that $\lambda (A_\alpha (H)) = \lambda (A_\alpha)$ and set $\lambda = \lambda (A_\alpha)$. Proposition 2 implies that $\mathbf{x}$ is an eigenvector to H, and therefore

$$\lambda x_w = \alpha d_H (w) x_w + (1 - \alpha) \sum_{\{i,w\} \in E(H)} x_i$$

$$= \alpha (d_G (w) + 1) x_w + (1 - \alpha) x_u + \sum_{\{i,w\} \in E(G)} x_i$$

$$> \alpha d_G (w) x_w + \sum_{\{i,w\} \in E(G)} x_i,$$

contradicting the fact that $\mathbf{x}$ is an eigenvector to $\lambda (A_\alpha)$ in G. □

4.2 Eigenvectors to $\lambda (A_\alpha (G))$ and automorphisms

Knowing the symmetries of a graph G can be quite useful to find the spectral radius of $\lambda (A_\alpha (G))$. Thus, we say that u and v are *equivalent in G*, if there exists an automorphism $p : G \to G$ such that $p(u) = v$. Vertex equivalence implies useful properties of eigenvectors to $\lambda (A_\alpha (G))$:

Proposition 16. *Let G be a connected graph of order n, and let u and v be equivalent vertices in G. If $(x_1, \ldots, x_n)$ is an eigenvector to $\lambda (A_\alpha (G))$, then $x_u = x_v$.*

Proof. Let G be a connected graph with $A_\alpha (G) = A_\alpha$; let $\lambda := \lambda (A_\alpha)$ and $\mathbf{x} := (x_1, \ldots, x_n)$ be a unit nonnegative eigenvector to λ. Let $p : G \to G$ be an automorphism of G such that $p(u) = v$. Note that p is a permutation of $V(G)$, and let P be the permutation matrix corresponding to p. Since p is an automorphism, we have $P^{-1} A_\alpha P = A_\alpha$; hence,

$$P^{-1} A_\alpha P \mathbf{x} = \lambda \mathbf{x},$$

so $P\mathbf{x}$ is an eigenvector to A_α. Since A_α is irreducible, $\mathbf{x}$ is unique, implying that $P\mathbf{x} = \mathbf{x}$, and so $x_u = x_v$. □

Note that eigenvector entries corresponding to equivalent vertices need not be equal for disconnected graphs; for example, if G is a union of two disjoint copies of an r-regular graph. However, Proposition 16 implies the following practical statement:

Corollary 17. *If G is a connected graph and $V(G)$ is partitioned into equivalence classes by the relation "u is equivalent to v", then every eigenvector to $\lambda(A_\alpha)$ is constant within each equivalence class.*

4.3 A few general bounds on $\lambda(A_\alpha(G))$

In this section, we give a few additional bounds on $\lambda(A_\alpha)$.

Proposition 18. *Let G be a graph, with $\Delta(G) = \Delta$, $A(G) = A$, and $A_\alpha(G) = A_\alpha$. The following inequalities hold for $\lambda(A_\alpha(G))$:*

$$\text{(15)} \qquad \lambda(A_\alpha) \geq \lambda(A),$$

$$\text{(16)} \qquad \lambda(A_\alpha) \leq \alpha\Delta + (1-\alpha)\lambda(A).$$

If equality holds in (15), then G has a $\lambda(A)$-regular component. Equality in (16) holds if an only if G has a Δ-regular component.

Proof. Note that inequality (15) follows from Proposition 4, but we shall give another proof in order to deduce the case of equality. Let H be a component of G such that $\lambda(A) = \lambda(A(H))$. Write h for the order of H, and let $(x_1, \ldots, x_h)$ be a positive unit vector to $\lambda(A(H))$. For every edge $\{u, v\}$ of H, the AM-GM inequality implies that

$$\text{(17)} \qquad 2x_u x_v = 2\alpha x_u x_v + 2(1-\alpha)x_u x_v \leq \alpha x_u^2 + 2(1-\alpha)x_u x_v + \alpha x_v^2.$$

Summing this inequality over all edges $\{u, v\} \in E(H)$, and using (7), we get

$$\lambda(A) = \lambda(A(H)) = \langle A(H)\mathbf{x}, \mathbf{x}\rangle \leq \langle A_\alpha(H)\mathbf{x}, \mathbf{x}\rangle \leq \lambda(A_\alpha),$$

so (15) is proved. If equality holds in (15), then $x_1 = \cdots = x_h$, hence H is $\lambda(A)$-regular.

Inequality (16) follows by Weyl's inequalities (5), because

$$\lambda(A_\alpha) \leq \lambda(\alpha D(G)) + \lambda((1-\alpha)(A)) = (1-\alpha)\lambda(A) + \alpha\Delta,$$

but we shall give a direct proof based on identity (9) that is more appropriate for the case of equality. Let H be a component of G such that $\lambda(A_\alpha) = \lambda(A_\alpha(H))$ and let h be the order of H. Let $\mathbf{x} := (x_1, \ldots, x_h)$ be a positive unit eigenvector to $\lambda(A_\alpha(H))$. We have

$$\begin{aligned}
\lambda(A_\alpha) &= \alpha \sum_{u \in V(H)} x_u^2 d_G(u) + 2(1-\alpha) \sum_{\{u,v\} \in E(H)} x_u x_v \\
&\leq \alpha\Delta(H) \sum_{u \in V(H)} x_u^2 + (1-\alpha)\lambda(A(H)) \\
&\leq \alpha\Delta + (1-\alpha)\lambda(A),
\end{aligned}$$

proving (16). If equality holds in (16), then H is Δ-regular.

It is not hard to see that if G has a Δ-regular component, then $\lambda(A) = \Delta = \lambda(A_\alpha)$, so equality holds in (16). $\qquad\square$

Having inequality (15) in hand, every lower bound of $\lambda(A)$ gives a lower bound on $\lambda(A_\alpha)$, which, however, is never better than (15). We mention just two such bounds.

Corollary 19. *Let G be a graph with $A_\alpha(G) = A_\alpha$. If G is of order n and has m edges, then*

$$\lambda(A_\alpha) \geq \sqrt{\frac{1}{n} \sum_{u \in V(G)} d_G^2(u)} \text{ and } \lambda(A_\alpha) \geq \frac{2m}{n}.$$

Equality holds in the second inequality if and only if G is regular. If $\alpha > 0$, equality holds in the first inequality if and only G is regular.

Proof. The only difficulty is to prove that if $\alpha > 0$, then the equality

$$(18) \qquad \lambda(A_\alpha) = \sqrt{\frac{1}{n} \sum_{u \in V(G)} d_G^2(u)}$$

implies that G is regular. Indeed, suppose that (18) holds, which implies also that

$$\lambda(A(G)) = \sqrt{\frac{1}{n} \sum_{u \in V(G)} d_G^2(u)}.$$

Let $G_1, \ldots, G_k$ be the components of G and let $n_1, \ldots, n_k$ be their orders. We see that

$$\sum_{u \in V(G)} d_G^2(u) = \lambda^2(A(G)) n \geq \lambda^2(A(G_1)) n_1 + \cdots + \lambda^2(A(G_1)) n_k$$

$$\geq \sum_{u \in V(G_1)} d_{G_1}^2(u) + \cdots + \sum_{u \in V(G_k)} d_{G_k}^2(u) = \sum_{u \in V(G)} d_G^2(u).$$

Hence,

$$\lambda(A(G_1)) = \cdots = \lambda(A(G_k)) = \lambda(A(G)),$$

and likewise,

$$\lambda(A_\alpha(G_1)) = \cdots = \lambda(A_\alpha(G_k)) = \lambda(A_\alpha(G)).$$

Now, Proposition 4 implies that all components of G are regular, completing the proof. $\qquad\square$

A very useful bound in extremal problems about $\lambda(Q)$ is the following one

$$(19) \qquad \lambda(Q) \leq \max_{u \in V(G)} \left\{ d(u) + \frac{1}{d(u)} \sum_{\{u,v\} \in E(G)} d(v) \right\},$$

with equality if and only if G is regular or semiregular. Bound (19) goes back to Merris [25], whereas the case of equality has been established by Feng and Yu in [13]. It is not hard to modify (19) for the matrices $A_\alpha(G)$:

Proposition 20. *If G is a graph with no isolated vertices, then*

$$(20) \qquad \lambda\left(A_\alpha\left(G\right)\right) \leq \max_{u \in V(G)} \left\{ \alpha d\left(u\right) + \frac{1-\alpha}{d\left(u\right)} \sum_{\{u,v\} \in E(G)} d\left(v\right) \right\},$$

and

$$(21) \qquad \lambda\left(A_\alpha\left(G\right)\right) \geq \min_{u \in V(G)} \left\{ \alpha d\left(u\right) + \frac{1-\alpha}{d\left(u\right)} \sum_{\{u,v\} \in E(G)} d\left(v\right) \right\}.$$

If $\alpha \in (1/2, 1)$ and G is connected, equality in (20) and (21) holds if and only if G is regular.

Proof. Let $A_\alpha\left(G\right) = A_\alpha$. Our proof of (20) and (21) uses the idea of Merris. The matrix $D^{-1}A_\alpha D$ is similar to A_α, so $\lambda\left(A_\alpha\right) = \lambda\left(D^{-1}A_\alpha D\right)$. Since $D^{-1}A_\alpha D$ is nonnegative, $\lambda\left(D^{-1}A_\alpha D\right)$ is between the smallest and the largest rowsums of $D^{-1}A_\alpha D$, implying both (20) and (21).

If G is connected, then A_α is irreducible and so is $D^{-1}A_\alpha D$. Hence, if equality holds in either (20) and (21), then all rowsums of $D^{-1}A_\alpha D$ are equal. The remaining part of the proof uses an idea borrowed from [**13**]. For any vertex $v \in V\left(G\right)$, set

$$m\left(u\right) := \frac{1}{d\left(u\right)} \sum_{\{u,v\} \in E(G)} d\left(v\right).$$

Fix a vertex u and let v be any neighbor of u. Now, from

$$\alpha d\left(u\right) + \left(1-\alpha\right) m\left(u\right) = \alpha d\left(v\right) + \left(1-\alpha\right) m\left(v\right)$$

we see that

$$\sum_{\{u,v\} \in E(G)} \alpha d\left(u\right) + \left(1-\alpha\right) m\left(u\right) = \sum_{\{u,v\} \in E(G)} \alpha d\left(v\right) + \left(1-\alpha\right) m\left(v\right).$$

Hence,

$$\alpha d^2\left(u\right) + \left(1-\alpha\right) d\left(u\right) m\left(u\right) = \alpha d\left(u\right) m\left(u\right) + \left(1-\alpha\right) \sum_{\{u,v\} \in E(G)} m\left(v\right).$$

Taking u to be a vertex with maximum degree, we find that

$$\alpha d^2\left(u\right) + \left(1-2\alpha\right) d\left(u\right) m\left(u\right) = \left(1-\alpha\right) \sum_{\{u,v\} \in E(G)} m\left(v\right) \leq \left(1-\alpha\right) d^2\left(u\right).$$

Hence $m\left(u\right) \geq d\left(u\right)$, which is possible only if all neighbors of u have maximal degree as well. Since G is connected, it turns out that G is regular. $\qquad \square$

Corollary 21. *For any graph G,*

(22)
$$\lambda\left(A_{\alpha}\right) \le \max_{\{u,v\}\in E(G)} \left\{\alpha d\left(u\right) + \left(1-\alpha\right) d\left(v\right)\right\}$$

and

(23)
$$\lambda\left(A_{\alpha}\right) \ge \min_{\{u,v\}\in E(G)} \left\{\alpha d\left(u\right) + \left(1-\alpha\right) d\left(v\right)\right\}.$$

   Caution: *If the right side of (22) is equal to M, and is attained for $\{u,v\} \in E\left(G\right)$, then*

$$M = \max\left\{\alpha d\left(u\right) + \left(1-\alpha\right) d\left(v\right), \alpha d\left(v\right) + \left(1-\alpha\right) d\left(u\right)\right\}.$$

A similar remark is valid for (23) with appropriate changes.

   It seems that equality in (20) and (21) holds only if G is regular, except in the cases $\alpha = 0$ and $\alpha = 1/2$. If true, this fact would need new proof technique, so we raise the following problem:

Problem 22. *Find all cases of equality in (20), (21), (22), and (23).*

   The last bounds in this sections are in the spirit of (20) and (21):

Proposition 23. *Let $\alpha \in [0,1]$. If G be a graph of order n, then*

$$\lambda^{2}\left(A_{\alpha}\left(G\right)\right) \le \max_{k\in V(G)} \alpha d_{G}^{2}\left(k\right) + \left(1-\alpha\right) w_{G}\left(k\right)$$

and

$$\lambda^{2}\left(A_{\alpha}\left(G\right)\right) \ge \min_{k\in V(G)} \alpha d_{G}^{2}\left(k\right) + \left(1-\alpha\right) w_{G}\left(k\right).$$

Proof. Let $A_{\alpha} := A_{\alpha}\left(G\right)$, $A := A\left(G\right)$, $D := D\left(G\right)$. First, we show that for any $k \in [n]$, the kth rowsum of $A_{\alpha}^{2}\left(G\right)$ is equal to

$$\alpha d_{G}^{2}\left(k\right) + \left(1-\alpha\right) w_{G}\left(k\right).$$

Indeed, for the square of A_{α}, we see that

$$A_{\alpha}^{2} = \alpha^{2}D^{2} + \left(1-\alpha\right)^{2} A^{2} + \alpha\left(1-\alpha\right) DA + \alpha\left(1-\alpha\right) AD.$$

Thus, for the kth rowsum $r_{k}\left(A_{\alpha}^{2}\right)$ we find that

$$
\begin{aligned}
r_{k}(A_{\alpha}^{2}) &= \alpha^{2} r_{k}(D^{2}) + \left(1-\alpha\right)^{2} r_{k}(A^{2}) + \alpha\left(1-\alpha\right) r_{k}\left(DA\right) + \alpha\left(1-\alpha\right) r_{k}\left(AD\right) \\
&= \alpha^{2} d_{G}^{2}\left(k\right) + \left(1-\alpha\right)^{2} w_{G}\left(k\right) + \alpha\left(1-\alpha\right) d_{G}^{2}\left(k\right) + \alpha\left(1-\alpha\right) w_{G}\left(k\right) \\
&= \alpha d_{G}^{2}\left(k\right) + \left(1-\alpha\right) w_{G}\left(k\right).
\end{aligned}
$$

Since $\lambda^{2}\left(A_{\alpha}\right) = \lambda\left(A_{\alpha}^{2}\right)$, the assertions follow, because $\lambda\left(A_{\alpha}^{2}\right)$ is between the smallest and the largest rowsums of A_{α}^{2}. $\qquad\square$

5. SOME SPECTRAL EXTREMAL PROBLEMS

Recall that one of the central problem of classical extremal graph theory is of the following type:

Problem A *Given a graph F, what is the maximum number of edges of a graph of order n, with no subgraph isomorphic to F?*

Such problems are fairly well understood nowadays; see, e.g., [**3**] for comprehensive discussion and [**26**] for some newer results. During the past two decades, some subtler versions of Problem A have been investigated, namely for $\lambda(A(G))$ and $\lambda(Q(G))$. In these problems, the principal questions are the following ones:

Problem B *Given a graph F, what is the maximum $\lambda(A(G))$ of a graph G of order n, with no subgraph isomorphic to F?*

Problem C *Given a graph F, what is the maximum $\lambda(Q(G))$ of a graph G of order n, with no subgraph isomorphic to F?*

Many instances of Problem B have been solved, see, e.g., the second part of the survey paper [**26**]. There is also considerable progress with Problem C: see, e.g., the papers [**1**], [**2**], [**15**], [**16**], [**18**], [**27**], [**29**], [**30**], and [**32**].

Now, having the family $A_\alpha(G)$, we can merge Problems B and C into one, namely:

Problem D *Given a graph F, what is the maximum $\lambda(A_\alpha(G))$ of a graph G of order n, with no subgraph isomorphic to F?*

In this survey we shall solve Problem D when F is a complete graph. Several other cases seem particularly interesting:

Problem 24. *Solve problem D if F is a path or a cycle of given order.*

5.1 Chromatic number and $\lambda(A_\alpha(G))$

A graph is called *r-chromatic* (or *r-partite*) if its vertices can be partitioned into r edgeless sets. An interesting topic in spectral graph theory is to find relations between eigenvalues and the chromatic number of graphs. In particular, here we are interested in the maximum $\lambda(A_\alpha(G))$ if G is an r-partite graph of order n.

Let us write $T_r(n)$ for the r-partite Turán graph of order n, and recall that $T_r(n)$ is a complete r-partite graph of order n, whose partition sets are of size $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$. Note that if $r = 2$, then $T_2(n) = K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$. It is known that $T_r(n)$ has maximum number of edges among all r-partite graphs of order n. The corresponding problem for $\lambda(A_\alpha(G))$ is not so straightforward, so for reader's sake we shall consider the case $r = 2$ first.

Theorem 25. *Let G be a bipartite graph of order n.*

(i) If $\alpha < 1/2$, then

$$\lambda\left(A_\alpha\left(G\right)\right) < \lambda\left(A_\alpha\left(T_2\left(n\right)\right)\right),$$

unless $G = T_2\left(n\right).$

(ii) If $\alpha > 1/2$, then

$$\lambda\left(A_\alpha\left(G\right)\right) < \lambda\left(A_\alpha\left(K_{1,n-1}\right)\right),$$

unless $G = K_{1,n-1}.$

(iii) If $\alpha = 1/2$, then

$$\lambda\left(A_\alpha\left(G\right)\right) \le n/2,$$

with equality if and only if G is a complete bipartite graph.

Proof. Suppose that G is a bipartite graph of order n with maximum $\lambda\left(A_\alpha\left(G\right)\right)$ among all bipartite graphs of order n. Proposition 14,(d) implies that G is a complete bipartite graph. Suppose that the partition sets V_1 and V_2 of G are of size n_1 and n_2, where $n_1 + n_2 = n$. Set $\lambda := \lambda\left(A_\alpha\left(G\right)\right)$ and let $(x_1, \ldots, x_n)$ be a positive eigenvector to λ. Proposition 16 implies that the entries of $(x_1, \ldots, x_n)$ corresponding to vertices of the same partition set have the same value, say, z_i for V_i, $i = 1, 2$. Thus, the equations (6) give

$$\lambda z_1 = \alpha n_2 z_1 + (1 - \alpha)\, n_2 z_2,$$
$$\lambda z_2 = \alpha n_1 z_2 + (1 - \alpha)\, n_1 z_1.$$

Excluding z_1 and z_2, we find that

$$(\lambda - \alpha n_2)\,(\lambda - \alpha n_1) = (1 - \alpha)^2\, n_1 n_2$$

and therefore,

$$\lambda = \frac{\alpha n + \sqrt{\alpha^2 n^2 + 4 n_1 n_2\,(1 - 2\alpha)}}{2}.$$

Clearly if $\alpha < 1/2$, then λ is maximum whenever $n_1 n_2$ is maximum; hence $G = T_2\left(n\right)$. Likewise if $\alpha > 1/2$, then λ is maximum whenever $n_1 n_2$ is minimum, and so $G = K_{1,n-1}$. Finally if $\alpha = 1/2$, then $\lambda = n/2$ for every complete bipartite graph. $\qquad\square$

For general r the statement reads as:

Theorem 26. *Let $r \ge 2$ and G be an r-partite graph of order n.*

(i) If $\alpha < 1 - 1/r$, then

$$\lambda\left(A_\alpha\left(G\right)\right) < \lambda\left(A_\alpha\left(T_r\left(n\right)\right)\right),$$

unless $G = T_r\left(n\right).$

(ii) If $1 > \alpha > 1 - 1/r$, then

$$\lambda\left(A_\alpha\left(G\right)\right) < \lambda\left(A_\alpha\left(S_{n,r-1}\right)\right),$$

unless $G = S_{n,r-1}$.

(iii) If $\alpha = 1 - 1/r$, then

$$\lambda\left(A_\alpha\left(G\right)\right) \le \left(1 - 1/r\right)n,$$

with equality if and only if G is a complete r-partite graph.

Proof. Suppose that G is an r-partite graph of order n with maximum $\lambda\left(A_\alpha\left(G\right)\right)$ among all r-partite graphs of order n. Proposition 14, (d) implies that G is a complete r-partite graph. Suppose that $V_1, \ldots, V_r$ are the partition sets of G, with sizes $n_1, \ldots, n_r$; obviously $n_1 + \cdots + n_r = n$. Set $\lambda := \lambda\left(A_\alpha\left(G\right)\right)$ and let $(x_1, \ldots, x_n)$ be a positive eigenvector to λ. Proposition 16 implies that the entries of $\mathbf{x}$ corresponding to vertices of the same partition set have the same value, say z_i for V_i, $i = 1, \ldots, r$. Hence, equations (6) reduce to the following r equations

$$(24) \qquad \lambda z_k = \alpha\left(n - n_k\right)z_k + \left(1 - \alpha\right)\sum_{i\in[r]\setminus\{k\}} n_i z_i, \quad 1 \le k \le r.$$

If $\alpha = 1 - 1/r$, we see that the value $\lambda := \left(1 - 1/r\right)n$ always is an eigenvalue with an eigenvector defined by $z_i = 1/\left(rn_i\right)$, $i = 1, \ldots, r$. This observation proves *(iii)*.

Further, letting $S := n_1 z_1 + \cdots + n_r z_r$, equations (24) imply that

$$\left(\lambda - \alpha\left(n - n_k\right) + \left(1 - \alpha\right)n_k\right)n_k z_k = \left(1 - \alpha\right)n_k S, \quad 1 \le k \le r.$$

After some algebra, we see that λ satisfies the equation

$$(25) \qquad \sum_{k\in[r]} \frac{n_k}{\lambda - \alpha n + n_k} = \frac{1}{1 - \alpha}.$$

Now, if $\alpha < 1 - 1/r$, then $1/\left(1 - \alpha\right) < r$. Hence some of the summands in the left side of (25) is less than 1 and so $\lambda - \alpha n > 0$. Letting

$$f\left(z\right) := \frac{z}{\lambda - \alpha n + z} = 1 - \frac{\lambda - \alpha n}{\lambda - \alpha n + z},$$

it is easy to see that

$$f''\left(z\right) = \frac{-2\left(\lambda - \alpha n\right)}{\left(\lambda - \alpha n + z\right)^3} < 0$$

for $z \ge 0$; thus, $f\left(z\right)$ is strictly concave for $z \ge 0$. It is not hard to see that the maximum

$$(26) \quad \max\left\{\sum_{k\in[r]} f\left(t_k\right) : \ t_1 + \cdots + t_r = n, t_k \ge 0 \text{ is an integer for each } k \in [r]\right\}$$

is attained if and only if $t_k = \lfloor n/r \rfloor$ or $t_k = \lceil n/r \rceil$ for any $k \in [r]$. Indeed, it is clear that the maximum (26) is attained, as there are finitely many vectors $(t_1, \ldots, t_r)$ satisfying the constraints. Suppose that the maximum (26) is attained for some $t_1, \ldots, t_r$ and assume by symmetry that $t_1 \leq \cdots \leq t_r$. If $t_r - t_1 \leq 1$, we are done, so assume for a contradiction that $t_r - t_1 \geq 2$. Set

$$t'_1 = t_1 + 1, \ t'_2 = t_2, \ \ldots \ t'_{r-1} = t_{r-1}, \ t'_r = t_r - 1.$$

The Mean Value Theorem implies that there exist $\theta_1 \in (t_1, t_1 + 1)$ and $\theta_r \in (t_r - 1, t_r)$ such that

$$\sum_{k \in [r]} f(t'_k) - \sum_{k \in [r]} f(t_k) = f(t_1 + 1) - f(t_1) + f(t_r - 1) - f(t_r) = f'(\theta_1) - f'(\theta_r).$$

However, since $f'(z)$ is decreasing in z and $\theta_r > \theta_1$, we see that $f'(\theta_1) - f'(\theta_r) > 0$, and therefore

$$\sum_{k \in [r]} f(t'_k) > \sum_{k \in [r]} f(t_k),$$

contrary to the assumption that $\sum_{k \in [r]} f(t_k)$ is maximal. Therefore, $t_k = \lfloor n/r \rfloor$ or $t_k = \lceil n/r \rceil$ for any $k \in [r]$.

Let $\lambda_T := \lambda(A_\alpha(T_r(n)))$ and let $t_1, \ldots, t_r$ be the sizes of the partition sets of $T_r(n)$, that is to say, $t_i = \lfloor n/r \rfloor$ or $t_i = \lceil n/r \rceil$ and $t_1 + \cdots + t_r = n$. In view of (25), we have

$$\sum_{k \in [r]} \frac{t_k}{\lambda_T - \alpha n + t_k} = \frac{1}{1 - \alpha}.$$

Now, we see that

$$\sum_{k \in [r]} \frac{t_k}{\lambda_T - \alpha n + t_k} = \frac{1}{1 - \alpha} = \sum_{k \in [r]} \frac{n_k}{\lambda - \alpha n + n_k} \leq \sum_{k \in [r]} \frac{t_k}{\lambda - \alpha n + t_k}.$$

Hence $\lambda_T \geq \lambda$, with equality if and only if $n_i = \lfloor n/r \rfloor$ or $n_i = \lceil n/r \rceil$ for all $i \in [r]$. This argument proves *(i)*.

The proof of *(ii)* goes along the same lines. Let G be an r-partite graph of order n with

$$\lambda \geq \lambda(A_\alpha(S_{n,r-1})).$$

We shall prove that $G = S_{n,r-1}$. Since $\Delta(S_{n,r-1}) = n - 1$, Corollary 13 implies that

(27) $\qquad \lambda \geq \lambda(A_\alpha(S_{n,r-1})) \geq \alpha(n-1) + (1-\alpha)^2/\alpha = \alpha n + 1/\alpha - 2.$

If $\alpha > 1 - 1/r$, then $1/(1 - \alpha) < r$. Hence some of the summands in the left side of (25) is less than 1 and so $\lambda - \alpha n < 0$. Letting

$$f(z) := \frac{z}{\lambda - \alpha n + z} = 1 - \frac{\lambda - \alpha n}{\lambda - \alpha n + z},$$

it is easy to see that

$$f''(z) = \frac{-2(\lambda - \alpha n)}{(\lambda - \alpha n + z)^3} > 0$$

whenever $z + \lambda - \alpha n > 0$. In view of (27), we find that

$$\lambda - \alpha n + 1 \geq 1/\alpha - 1 > 0,$$

thus $f(z)$ is strictly convex for $z \geq 1$. It is not hard to see that the maximum

$$(28) \quad \max\left\{\sum_{k \in [r]} f(s_k) : \ s_1 + \cdots + s_r = n, s_k > 0 \text{ is an integer for each } k \in [r]\right\}$$

is attained if and only if all but one of the s_ks are equal to 1. Indeed, it is clear that the maximum (28) is attained, as there are finitely many vectors $(s_1, \ldots, s_r)$ satisfying the constraints. Let the maximum (28) be attained for some $s_1, \ldots, s_r$ and assume by symmetry that $s_1 \leq \cdots \leq s_r$. If $s_{r-1} = 1$, we are done, so assume for a contradiction that $s_{r-1} \geq 2$. Set

$$s_1' = s_1, \ldots s_{r-2}' = s_{r-2}, \ s_{r-1}' = s_{r-1} - 1, \ s_r' = s_r + 1.$$

The Mean Value Theorem implies that there exist $\theta_{r-1} \in (s_{r-1} - 1, s_{r-1})$ and $\theta_r \in (s_r, s_r + 1)$ such that

$$\sum_{k \in [r]} f(s_k') - \sum_{k \in [r]} f(s_k) = f(s_r + 1) - f(s_r) + f(s_{r-1} - 1) - f(s_r) = f'(\theta_r) - f'(\theta_{r-1}).$$

However, since $f'(z)$ is increasing in z for $z \geq 1$ and $\theta_r > \theta_{r-1}$, we see that $f'(\theta_r) - f'(\theta_{r-1}) > 0$, and therefore

$$\sum_{k \in [r]} f(s_k') > \sum_{k \in [r]} f(s_k),$$

contrary to the assumption that $\sum_{k \in [r]} f(s_k)$ is maximal. Therefore, $s_1 = \cdots = s_{r-1} = 1$.

Let $\lambda_S := \lambda(A_\alpha(S_{n,r-1}))$ and let $s_1, \ldots, s_r$ be the sizes of the partition sets of $S_{n,r-1}$, that is to say, $s_1 = \cdots = s_{r-1} = 1$ and $s_r = n - r + 1$. In view of (25), we have

$$\sum_{k \in [r]} \frac{s_k}{\lambda_S - \alpha n + s_k} = \frac{1}{1 - \alpha}.$$

Now, we see that

$$\sum_{k \in [r]} \frac{s_k}{\lambda_S - \alpha n + s_k} = \frac{1}{1 - \alpha} = \sum_{k \in [r]} \frac{n_k}{\lambda - \alpha n + n_k} \leq \sum_{k \in [r]} \frac{s_k}{\lambda - \alpha n + s_k}.$$

Hence, $\lambda_S = \lambda$, and all but one of the partition sets of G are of cardinality 1, that is to say, $G = S_{n,r-1}$. The proof of Theorem 26 is completed. $\quad\square$

5.2 Clique number and $\lambda\left(A_\alpha\left(G\right)\right)$

A graph is called K_r-*free* if it does not contain a complete graph on r vertices. It is known (see, e.g., [**26**] and [**18**]) that if G is a K_{r+1}-free graph of order n, then

$$\lambda\left(A\left(G\right)\right) \leq \lambda\left(A\left(T_r\left(n\right)\right)\right),$$
$$\lambda\left(Q\left(G\right)\right) \leq \lambda\left(Q\left(T_r\left(n\right)\right)\right).$$

The generalization of these results to $\lambda\left(A_\alpha\left(G\right)\right)$ is summarized in the following encompassing, but somewhat unexpected theorem:

Theorem 27. *Let $r \geq 2$ and G be an K_{r+1}-free graph of order n.*
  (i) If $0 \leq \alpha < 1 - 1/r$, then

$$\lambda\left(A_\alpha\left(G\right)\right) < \lambda\left(A_\alpha\left(T_r\left(n\right)\right)\right),$$

unless $G = T_r\left(n\right)$.
  (ii) If $1 > \alpha > 1 - 1/r$, then

$$\lambda\left(A_\alpha\left(G\right)\right) < \lambda\left(A_\alpha\left(S_{n,r-1}\right)\right),$$

unless $G = S_{n,r-1}$.
  (iii) If $\alpha = 1 - 1/r$, then

$$\lambda\left(A_\alpha\left(G\right)\right) \leq \left(1 - 1/r\right)n,$$

with equality if and only if G is a complete r-partite graph.

We shall show that Theorem 27 can be reduced to Theorem 26 via the following technical lemma.

Lemma 28. *Let $\alpha \in [0, 1)$ and $n \geq r \geq 2$. If G is a graph with maximum $\lambda\left(A_\alpha\left(G\right)\right)$ among all K_{r+1}-free graphs of order n, then G is complete r-partite.*

For the proof of the lemma, we introduce some notation: Let $\alpha \in [0, 1)$. Given a graph G of order n and a vector $\mathbf{x} := \left(x_1, \ldots, x_n\right)$, set

$$S_G\left(\mathbf{x}\right) := \left\langle A_\alpha\left(G\right)\mathbf{x}, \mathbf{x}\right\rangle,$$

and for any $v \in V\left(G\right)$, set

$$S_G(v, \mathbf{x}) := \alpha d_G\left(v\right) x_v + \left(1 - \alpha\right) \sum_{\{v,i\} \in E(G)} x_i.$$

Proof. Let G be a graph with maximum $\lambda\left(A_\alpha\left(G\right)\right)$ among all K_{r+1}-free graphs of order n. For short, let $\lambda := \lambda\left(A_\alpha\left(G\right)\right)$. Clearly, G is connected; thus, there is a positive unit eigenvector $\mathbf{x} := \left(x_1, \ldots, x_n\right)$ to $\lambda\left(A_\alpha\left(G\right)\right)$, and therefore,

$$\lambda = S_G\left(\mathbf{x}\right) = \sum_{v \in V(G)} x_v S_G(v, \mathbf{x}).$$

Note that the eigenequation (6) for any vertex $v \in V(G)$ can be written as

$$(29) \qquad\qquad \lambda x_v = S_G(v, \mathbf{x}).$$

To prove the lemma, we need two claims.

Claim A *There exists a coclique $W \subset G$ such that*

$$G = W \vee G',$$

where $G' = G - V(W)$.

Proof Select a vertex u with

$$S_G(u, \mathbf{x}) := \max \{ S_G(v, \mathbf{x}) : v \in V(G) \},$$

and set $U := \Gamma_G(u)$ and $W := G - U$. Remove all edges within W and join each vertex in U to each vertex in W. Write H for the resulting graph, which is obviously of order n and is K_{r+1}-free. We shall show that $S_H(v, \mathbf{x}) \geq S_G(v, \mathbf{x})$ for each $v \in V(G)$. This is obvious if $v \in U$, since then $\Gamma_G(v) \subset \Gamma_H(v)$, so $S_H(v, \mathbf{x}) \geq S_G(v, \mathbf{x})$. Now, let $v \in V(W)$. Note that

$$S_H(v, \mathbf{x}) = \alpha d_G(u) x_v + (1 - \alpha) \sum_{\{u,i\} \in E(G)} x_i$$
$$= \alpha d_G(u) x_v + S_G(u, \mathbf{x}) - \alpha d_G(u) x_u.$$

Hence,

$$S_H(v, \mathbf{x}) - S_G(v, \mathbf{x}) = S_G(u, \mathbf{x}) - S_G(v, \mathbf{x}) - \alpha d_G(u)(x_u - x_v).$$

Now, equation (29) implies that $S_G(u, \mathbf{x}) = \lambda x_u$ and $S_G(v, \mathbf{x}) = \lambda x_v$. Hence

$$S_H(v, \mathbf{x}) - S_G(v, \mathbf{x}) = \lambda(x_u - x_v) - \alpha d_G(u)(x_u - x_v) = (\lambda - \alpha d_G(u))(x_u - x_v).$$

But Corollary 13 implies that $\lambda - \alpha d_G(u) > 0$, and equation (29) implies that $x_u \geq x_v$. Hence $S_H(v, \mathbf{x}) \geq S_G(v, \mathbf{x})$ for any $v \in V(G)$, and so

$$\lambda(A_\alpha(H)) \geq S_H(\mathbf{x}) \geq S_G(\mathbf{x}) = \lambda \geq \lambda(A_\alpha(H)).$$

Therefore, $\lambda(A_\alpha(H)) = \lambda$, implying, in particular, that $S_H(v, \mathbf{x}) = S_G(v, \mathbf{x})$ for each $v \in U$; thus each $v \in U$ is joined in G to each $w \in W$, so $G = H = W \vee G[U]$, completing the proof of Claim A.

To finish the proof of the lemma we need another technical assertion:

Claim B *Let $1 \leq k < r$. If F is an induced subgraph of G and $W_1, \ldots, W_k$ are disjoint cocliques of G such that*

$$G = W_1 \vee \cdots \vee W_k \vee F$$

then there is a coclique $W_{k+1} \subset F$ such that

$$G = W_1 \vee \cdots \vee W_{k+1} \vee F',$$

where $F' = F - V(W_{k+1})$.

Proof Select a vertex $u \in V(F)$ with

$$S_G(u, \mathbf{x}) = \max \{S_G(v, \mathbf{x}) : v \in V(F)\},$$

and set $U := \Gamma_F(u)$ and $W := F - U$. Remove all edges within W and join each vertex in U to each vertex in W. Write H for the resulting graph, which is obviously of order n and is K_{r+1}-free. We shall show that $S_H(v, \mathbf{x}) \geq S_G(v, \mathbf{x})$ for each $v \in V(G)$. This is obvious if $v \in V \backslash V(W)$, since then either $\Gamma_G(v) = \Gamma_H(v)$ or $\Gamma_G(v) \subset \Gamma_H(v)$, so $S_H(v, \mathbf{x}) \geq S_G(v, \mathbf{x})$. Now, let $v \in V(W)$. Exactly as in the proof of Claim A we see that

$$S_H(v, \mathbf{x}) - S_G(v, \mathbf{x}) = (\lambda - \alpha d_G(u))(x_u - x_v).$$

Hence, $S_H(v, \mathbf{x}) \geq S_G(v, \mathbf{x})$ and

$$\lambda(A_\alpha(H)) \geq S_H(\mathbf{x}) \geq S_G(\mathbf{x}) = \lambda \geq \lambda(A_\alpha(H)).$$

Therefore, $\lambda(A_\alpha(H)) = \lambda$, implying, in particular, that $S_H(v, \mathbf{x}) = S_G(v, \mathbf{x})$ for each $v \in U$; thus each $v \in U$ is joined in F to each $w \in W$, and so $F = W \vee G[U]$. Letting $W_{k+1} := W$, the proof of Claim B is completed.

To complete the proof of the lemma, we first apply Claim A and then repeatedly apply Claim B until $k = r - 1$. In this way we find that

$$G = W_1 \vee \cdots \vee W_{r-1} \vee F,$$

where $W_1 \vee \cdots \vee W_{r-1}$ are cocliques of G and F is an induced subgraph of G. Because G is K_{r+1}-free, F must be a coclique too and so, G is a complete r-partite graph. $\qquad \square$

6. MISCELANEOUS

In this section, we briefly touch a few rather different topics, some of which deserve a much more thorough investigation.

6.1 The smallest eigenvalue $\lambda_{\min}(A_\alpha(G))$

The smallest eigenvalue of the adjacency matrix, which is second in importance after the spectral radius, has numerous relations with the structure of the graph. To a great extent this is also true for $\lambda_{\min}(Q(G))$; see, e.g., [12], [22], and [23]. In particular, the smallest eigenvalues of $A(G)$ and $Q(G)$ have close relations

to bipartite subgraphs of G. A simple relation of this type can be obtained also for $\lambda_{\min}(A_\alpha(G))$.

Let G be a graph of order n with m edges. Let $V(G) = V_1 \cup V_2$ be a bipartition and let the n-vector $\mathbf{x} := (x_1, \ldots, x_n)$ be -1 on V_1 and 1 on V_2. We see that

$$
\begin{aligned}
\langle A_\alpha(G)\mathbf{x}, \mathbf{x}\rangle &= \alpha\langle D(G)\mathbf{x}, \mathbf{x}\rangle + (1-\alpha)\langle A(G)\mathbf{x}, \mathbf{x}\rangle \\
&= 2\alpha m + 2(1-\alpha)(e(V_1) + e(V_2)) - 2(1-\alpha)e(V_1, V_2) \\
&= 2\alpha m + 2(1-\alpha)m - 4(1-\alpha)e(V_1, V_2) \\
&= 2m - 4(1-\alpha)e(V_1, V_2)
\end{aligned}
$$

Hence, scaling $(x_1, \ldots, x_n)$ to unit length, we get:

Proposition 29. *If G is a graph of order n with m edges, then*

$$
\lambda_{\min}(A_\alpha(G)) \leq \frac{2m}{n} - \frac{4(1-\alpha)}{n}\ maxcut(G).
$$

It is interesting to determine how small $\lambda_{\min}(A_\alpha(G))$ can be if G is a graph of order n. For $\alpha \geq 1/2$ the question is easy. Indeed, if $\alpha \geq 1/2$, the matrix $A_\alpha(G)$ is positive semidefinite, so $\lambda_{\min}(A_\alpha(G)) \geq 0$. On the other hand, if G has an isolated vertex, then $\lambda_{\min}(A_\alpha(G)) = 0$, so if $\alpha \in [1/2, 1]$, then

$$
\min\{\lambda_{\min}(A_\alpha(G)) : v(G) = n\} = 0.
$$

By contrast,

$$
\min\{\lambda_{\min}(A(G)) : v(G) = n\} = -\sqrt{\lfloor n/2 \rfloor \lceil n/2 \rceil};
$$

hence it is worth to study the following problem:

Problem 30. *For any $\alpha \in (0, 1/2)$ determine*

$$
\min\{\lambda_{\min}(A_\alpha(G)) : v(G) = n\}.
$$

6.2 The second largest eigenvalue $\lambda_2(A_\alpha(G))$

In this subsection, we discuss how large $\lambda_2(A_\alpha(G))$ can be if G is a graph of order n.

Proposition 31. *Let G be a graph of order n with $A_\alpha(G) = A_\alpha$.*
  (a) If $1/2 \leq \alpha \leq 1$, then

$$
\lambda_2(A_\alpha) \leq \alpha n - 1.
$$

If $\alpha > 1/2$, equality is attained if and only if $G = K_n$.
  (b) If $0 \leq \alpha < 1/2$, then

$$
\lambda_2(A_\alpha) \leq \frac{n}{2} - 1.
$$

If n is even equality holds for the graph $G = 2K_{n/2}$.

The proof of (a) follows by

$$\lambda_2 \left(A_\alpha\right) = \lambda_2 \left(A_{1/2}\right) + (\alpha - 1/2) L\left(G\right) \le (n - 2)/2 + (\alpha - 1/2) n.$$

The proof of (b) follows by

$$\lambda_2 \left(A_\alpha\right) \le \lambda_2 \left(A_{1/2}\right) \le \frac{n}{2} - 1.$$

Note that we have not determined precisely how large $\lambda_2 \left(A_\alpha \left(G\right)\right)$ can be if G is a graph of odd order n. Taking $G = K_{\lceil n/2 \rceil} \cup K_{\lceil n/2 \rceil}$, we see that

$$\lambda_2 \left(A_\alpha \left(G\right)\right) = \frac{n - 1}{2} - 1,$$

but this bound still leaves a margin of $1/2$ to close.

6.3 Eigenvalues of $A_\alpha \left(G\right)$ and the diameter of G

The following theorem can be proved using the generic idea of [**6**].

Proposition 32. *Let $a \in [0, 1)$, let G be a graph with $A_\alpha \left(G\right) = A_\alpha$, and let u and v be two vertices of G at distance $k \ge 1$. Let $l \in [k]$ and set $B := A_\alpha^l$.*
(a) If $l = k$, then $b_{u,v} > 0$;
(b) If $l < k$, then $b_{u,v} = 0$.

Proof. Set $A = A\left(G\right)$. If X and Y are matrices of the same size, write $X \succ Y$, if $x_{i,j} \ge y_{i,j}$ for all admissible i, j.

Proof of (a) Note that $A_\alpha \succ (1 - \alpha) A$, and so $A_\alpha^k \succ (1 - \alpha)^k A^k$. However, the (u, v) entry of A^k is positive, since there is a path of length k between u and v. Hence, $b_{u,v} > 0$, proving *(a)*.

Proof of (b) Now suppose that $l < k$, and note that $A + nI \succ A_\alpha$. Hence, $\left(A + nI\right)^l \succ A_\alpha^l$. Since

$$\left(A + nI\right)^l = A^l + a_{l-1} A^{l-1} + \cdots + a_0 I^l$$

for some real $a_0, \ldots, a_{l-1}$, we see that the (u, v) entry of $\left(A + nI\right)^l$ is zero, because there is no path shorter than k between u and v, so the (u, v) entry of each of the matrices $A^l, \ldots, A, I$ is zero. Hence, $b_{u,v} = 0$. $\square$

Corollary 33. *If G is a connected graph of diameter D, then $A_\alpha \left(G\right)$ has at least $D + 1$ distinct eigenvalues.*

6.4 Eigenvalues of $A_\alpha \left(G\right)$ and traces

In this subsection we give two explicit expressions for the sum of the eigenvalues of $A_\alpha \left(G\right)$ and for the sum of their squares.

Proposition 34. *If G is a graph of order n and has m edges, then*

$$\sum_{i=1}^{n} \lambda_i \left(A_\alpha \left(G \right) \right) = \operatorname{tr} A_\alpha \left(G \right) = \alpha \sum_{i \in V(G)} d_G \left(u \right) = 2\alpha m.$$

Here is a similar formula for the sum of the squares of the eigenvalues of A_α.

Proposition 35. *If G is a graph of order n and has m edges, then.*

$$\sum_{i=1}^{n} \lambda_i^2 \left(A_\alpha \left(G \right) \right) = \operatorname{tr} A_\alpha^2 \left(G \right) = 2 \left(1 - \alpha \right)^2 m + \alpha^2 \sum_{i \in V} d_G^2 \left(u \right).$$

Proof. Let $A_\alpha := A_\alpha \left(G \right)$, $A := A \left(G \right)$, and $D := D \left(G \right)$. Calculating the square A_α^2 and taking its trace, we find that

$$
\begin{aligned}
\operatorname{tr} A_\alpha^2 &= \operatorname{tr} \left(\alpha^2 D^2 + \left(1 - \alpha \right)^2 A^2 + \alpha \left(1 - \alpha \right) DA + \alpha \left(1 - \alpha \right) AD \right) \\
&= \alpha^2 \operatorname{tr} D^2 + \left(1 - \alpha \right)^2 \operatorname{tr} A^2 + \alpha \left(1 - \alpha \right) \operatorname{tr} DA + \alpha \left(1 - \alpha \right) \operatorname{tr} AD \\
&= 2 \left(1 - \alpha \right)^2 m + \alpha^2 \sum_{i \in V} d_G^2 \left(u \right),
\end{aligned}
$$

completing the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

7. THE A_α-SPECTRA OF SOME GRAPHS

Equalities (11) and the fact that the eigenvalues of $A \left(K_n \right)$ are $n-1, -1, ..., -1$ give the spectrum of $A_\alpha \left(K_n \right)$ as follows:

Proposition 36. *The eigenvalues of $A_\alpha \left(K_n \right)$ are*

$$\lambda_1 \left(A_\alpha \left(K_n \right) \right) = n - 1 \quad and \quad \lambda_k \left(A_\alpha \left(K_n \right) \right) = \alpha n - 1 \ for\ 2 \leq k \leq n.$$

Next we present the spectrum of a join of two regular graphs.[2]

Proposition 37. *Let G_1 be a r_1-regular graph of order n_1, and G_2 be a r_2-regular graph of order n_2. The spectrum of $A_\alpha \left(G_1 \vee G_2 \right)$ is determined as follows:*
(a) The largest and smallest eigenvalues of $G_1 \vee G_2$ are given by

$$\lambda \left(A_\alpha \left(G_1 \vee G_2 \right) \right) = \lambda \begin{pmatrix} r_1 + \alpha n_2 & \left(1 - \alpha \right)^2 n_1 n_2 \\ 1 & r_2 + \alpha n_1 \end{pmatrix},$$

$$\lambda_{\min} \left(A_\alpha \left(G_1 \vee G_2 \right) \right) = \lambda_{\min} \begin{pmatrix} r_1 + \alpha n_2 & \left(1 - \alpha \right)^2 n_1 n_2 \\ 1 & r_2 + \alpha n_1 \end{pmatrix}.$$

(b) The remaining $n_1 + n_2 - 2$ eigenvalues of $G_1 \vee G_2$ are given by

$$\alpha r_1 + \left(1 - \alpha \right) \lambda_k \left(A \left(G_1 \right) \right), \quad 2 \leq k \leq n_1,$$
$$\alpha r_2 + \left(1 - \alpha \right) \lambda_k \left(A \left(G_2 \right) \right), \quad 2 \leq k \leq n_2.$$

---

[2]The idea for Proposition 37 is due to the referee.

The proof of Proposition 37 is based on familiar ideas dating back to Fink and Grohmann [**14**]: take a set of orthogonal eigenvectors to $\lambda_2\left(A\left(G_1\right)\right), ..., \lambda_{n_1}\left(A\left(G_1\right)\right)$ and another set for $\lambda_2\left(A\left(G_2\right)\right), ..., \lambda_{n_2}\left(A\left(G_2\right)\right)$, and extend those vectors in the obvious way to $n_1 + n_2 - 2$ orthogonal eigenvectors of length $n_1 + n_2$ to the same eigenvalues. This approach proves (b).

To prove (a), note that the matrix

$$B := \left[\begin{array}{cc} r_1 + \alpha n_2 & \left(1-\alpha\right)^2 n_1 n_2 \\ 1 & r_2 + \alpha n_1 \end{array} \right]$$

is the 2×2 quotient matrix of $A_\alpha\left(G_1 \vee G_2\right)$ with respect to the bipartition

$$V\left(G_1 \vee G_2\right) = V\left(G_1\right) \cup V\left(G_2\right).$$

Since, G_1 and G_2 are regular, it turns out that the two eigenvalues of B are the largest and the smallest eiganvalues of G.

As an immediate corollary of Proposition 37, we present the A_α-spectrum of the complete bipartite graph $K_{a,b}$.

Proposition 38. *Let $a \geq b \geq 1$. If $\alpha \in \left[0,1\right]$, the eigenvalues of $A_\alpha\left(K_{a,b}\right)$ are*

$$\lambda\left(A_\alpha\left(K_{a,b}\right)\right) = \frac{1}{2}\left(\alpha\left(a+b\right) + \sqrt{\alpha^2\left(a+b\right)^2 + 4ab\left(1-2\alpha\right)}\right),$$

$$\lambda_{\min}\left(A_\alpha\left(K_{a,b}\right)\right) = \frac{1}{2}\left(\alpha\left(a+b\right) - \sqrt{\alpha^2\left(a+b\right)^2 + 4ab\left(1-2\alpha\right)}\right),$$

$$\lambda_k\left(A_\alpha\left(K_{a,b}\right)\right) = \alpha a \quad \textit{for } 1 < k \leq b,$$
$$\lambda_k\left(A_\alpha\left(K_{a,b}\right)\right) = \alpha b \quad \textit{for } b < k < a+b.$$

In particular, the A_α-spectrum of the star $K_{1,n-1}$ is as follows:

Proposition 39. *The eigenvalues of $A_\alpha\left(K_{1,n-1}\right)$ are*

$$\lambda\left(A_\alpha\left(K_{1,n-1}\right)\right) = \frac{1}{2}\left(\alpha n + \sqrt{\alpha^2 n^2 + 4\left(n-1\right)\left(1-2\alpha\right)}\right)$$

$$\lambda_{\min}\left(A_\alpha\left(K_{1,n-1}\right)\right) = \frac{1}{2}\left(\alpha n - \sqrt{\alpha^2 n^2 + 4\left(n-1\right)\left(1-2\alpha\right)}\right)$$

$$\lambda_k\left(A_\alpha\left(K_{1,n-1}\right)\right) = \alpha \quad \textit{for } 1 < k < n.$$

8. CONCLUDING REMARKS

The present survey covers just a small portion of the hundreds of results about $A\left(G\right)$ and $Q\left(G\right)$ that could be extended to $A_\alpha\left(G\right)$. To extend most of these results would be a challenging endeavor. If nothing else, Theorems 26 and 27 show

that it is worth studying $A_\alpha(G)$, for it is unlikely to discover them in a different context.

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