Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **11** (2017), 244–257.
https://doi.org/10.2298/AADM1702244H

GRAPHS WITH TWO MAIN AND TWO PLAIN EIGENVALUES

*Sakander Hayat, Muhammad Javaid, Jack H. Koolen**

In honor of Dragoš Cvetković on the occasion of his 75th birthday.

In this note, we study graphs with two main and two plain eigenvalues. Besides some other characterization results, we characterize the disconnected graphs with two main and two plain eigenvalues. Moreover, we provide several families of examples of such graphs.

1. INTRODUCTION

For undefined notations, see the next section.

In 1970, Doob [**14**] suggested the study of graphs with a few eigenvalues and he proposed at most five. A connected regular graph with at most three distinct eigenvalues is known to be strongly regular; see, for example [**4**] for a survey on strongly regular graphs. The connected nonregular graphs with three distinct eigenvalues have been studied by, for example, De Caen, Van Dam & Spence [**9**], Bridges & Mena [**2**], Muzychuk & Klin [**19**], Van Dam [**10**] and Cheng, Gavrilyuk, Greaves & Koolen [**7**]. The connected regular graphs with four distinct eigenvalues were studied by Van Dam [**11**], Van Dam & Spence [**12**] and Huang & Huang [**18**], among others. Cioabă, Haemers, Vermette & Wong [**5**] (resp. Cioabă, Haemers & Vermette [**6**]) studied connected graphs with at most two eigenvalues not equal to 1 and -1 (resp. 0 and -2). In this note, we look at a certain class of nonregular graphs with at most four distinct eigenvalues.

*Corresponding author. Jack H. Koolen
2010 Mathematics Subject Classification. 05C50, 05E30.
Keywords and Phrases. Main eigenvalue, Plain eigenvalue, Refined spectrum, Strong graphs, Strongly regular graphs.

To be more precise, we first define the main and plain eigenvalues of a real symmetric matrix T. An eigenvalue σ of T is said to be a *main eigenvalue* (resp. *plain eigenvalue*) if there exists an eigenvector $\mathbf{x}$ corresponding to σ, such that $\mathbf{x}^\top \mathbf{j} \neq 0$ (resp. $\mathbf{x}^\top \mathbf{j} = 0$). Moreover, σ is called *non-main* if the whole eigenspace is orthogonal to $\mathbf{j}$. Note that the notion of plain eigenvalue is also known in the literature as a restricted eigenvalue. Also note that an eigenvalue can be main and plain simultaneously and a non-main eigenvalue is always plain.

In this note, we will look at graphs whose adjacency matrix has a small number of main and plain eigenvalues. Our main focus is to look at graphs with two main and two plain eigenvalues. Such a graph has at least two and at most four distinct eigenvalues. Among the examples of graphs with two main and two plain eigenvalues are the connected biregular graphs with three distinct eigenvalues and the nonregular strong graphs derived from non-trivial regular two-graphs. In [**17**], it was shown that the number of valencies for the class of strong graphs is unbounded. Our main result classifies the disconnected graphs with two main and two plain eigenvalues (Theorem 2). We conjecture that if a graph has two main and two plain eigenvalues and has at least four distinct valencies then the graph must be strong. We will show that a disconnected graph with two main and two plain eigenvalues is biregular, and we do not have any example with three distinct valencies that is not strong.

In 1999, De Caen [**9**] posed the following question: Does a connected graph with three distinct eigenvalues have at most three distinct valencies? In a similar fashion to de Caen's question, we formulate the following conjecture.

Conjecture 1. *Let Γ be a connected t-valenced graph with two main and two plain eigenvalues. There exists a positive integer C such that if $t \geq C$, then Γ is a strong graph.*

We wonder whether the Conjecture 1 is true for $C = 4$.

This note is organized as follows. In Section 2, we give some preliminaries and in Section 3, besides discussing the main-plain index and refined spectrum, we use switching to construct many graphs with a few main and plain eigenvalues. In Section 4, we use known characterizations to classify graphs with r main and s plain eigenvalues such that $r + s \leq 3$. Moreover, we determine the disconnected graphs with two main and two plain eigenvalues. In Section 5, we give several families of examples of graphs with two main and two plain eigenvalues. Although for many of these families, it was known that they have two main eigenvalues, we show that they also have two plain eigenvalues. We also show that for every even integer n at least 6, there exist at least two connected co-connected non-strong equitable biregular graphs with two main and two plain eigenvalues with n vertices. So this shows that there are many graphs of order n with two main and two plain eigenvalues.

2. PRELIMINARIES

Let Γ be a simple graph with vertex set $V(\Gamma)$ and adjacency matrix A. The *complement* of Γ, is the graph $\overline{\Gamma}$, with the same vertex set such that two distinct vertices are adjacent in $\overline{\Gamma}$ if and only if they are not adjacent in Γ. We call Γ *co-connected*, if its complement is connected. Unless stated otherwise, by the *eigenvalues*, *eigenvectors* and *spectrum* of Γ, we mean the eigenvalues, eigenvectors and spectrum of its adjacency matrix A. We say that a graph Ω is *cospectral* with Γ if they are non-isomorphic and their spectra are same. We will denote the matrix of all-ones, the identity matrix and the vector of all-ones by J, I and $\mathbf{j}$ respectively.

We call a regular graph *strongly regular* if there are constants a and c such that every pair of distinct vertices has a or c common neighbors if they are adjacent or non-adjacent respectively. A strongly regular graph Γ with at least two vertices is called *primitive* if both Γ and its complement $\overline{\Gamma}$ are connected, otherwise *imprimitive*. Note that the only imprimitive strongly regular graphs are the regular complete multipartite graphs. We refer to the book of Godsil & Royle [**15**, Chapter 10] for more details on strongly regular graphs.

We assume that the graph Γ has s distinct valencies $k_1, \ldots, k_s$. We write $V_i := \{v \in V(\Gamma) \mid d_v = k_i\}$ and $n_i :=\mid V_i \mid$ for $i \in \{1, \ldots, s\}$. Clearly the subsets V_i partition the vertex set of Γ and this partition is called the *valency partition* of Γ. A graph Γ is called *t-valenced* if the number of distinct valencies in Γ is t. A 2-valenced graph is also called a *biregular* graph. Let $\pi = \{\pi_1, \ldots, \pi_s\}$ be a partition of the vertex set of Γ. For each vertex x in π_i, write $d_x^{(j)}$ for the number of neighbors of x in π_j. Then we write $b_{ij} = \frac{1}{|\pi_i|} \sum_{x \in \pi_i} d_x^{(j)}$ for the average number of neighbors in π_j of the vertices in π_i. The matrix $B_\pi := (b_{ij})$ is called the *quotient matrix* of π and π is called *equitable* if for all i and j, we have $d_x^{(j)} = b_{ij}$ for each $x \in \pi_i$. A graph Γ is called an *equitable* graph if its valency partition is equitable. Following Muzychuk & Klin [**19**], we say that a biregular graph is *strongly biregular* if it has exactly three distinct eigenvalues.

The *multicone over a graph* Γ is the graph with vertex set $S \cup V(\Gamma)$, where S is a set of isolated vertices with $|S| \geq 1$, such that every vertex of S is adjacent to all vertices of Γ. We just call it a *multicone* when Γ is not specified. When $|S| = 1$, we call it a *cone*. If Γ, Ω are graphs, with $V(\Gamma) = \{x_1, \ldots, x_n\}$, then the *corona* $\Gamma \circ \Omega$ is obtained from Γ by adding n disjoint copies of Ω and joining x_i by an edge to each vertex in the i-th copy of Ω $(i = 1, \ldots, n)$.

3. THE MAIN-PLAIN INDEX AND THE REFINED SPECTRUM

In this section, we will introduce some terminology which will be used in subsequent sections.

For a real symmetric matrix T and its eigenvalue θ, we denote by $\xi_T(\theta)$ the eigenspace of θ for T. For an eigenvalue θ of T with multiplicity m, the *plain multiplicity* p of θ is defined by $\dim(\xi_T(\theta) \cap \mathbf{j}^\top)$. The set $\xi_T(\theta) \cap \mathbf{j}^\top$ is called the *plain eigenspace* of θ and a non-zero vector $\mathbf{x} \in \xi_T(\theta) \cap \mathbf{j}^\top$ is said to be a *plain eigenvector* of T corresponding to θ. Note that $p \leq m \leq p+1$ and θ is main if and only if $m \neq p$ and plain if and only if $p \neq 0$. We define the *refined spectrum* of T by $\big(r, s; \mu_1, \dots, \mu_r; [\pi_1]^{p_1}, \dots, [\pi_s]^{p_s}\big)$, where μ_i's for $i = 1, \dots, r$ are the main eigenvalues of T and π_i's for $i = 1, \dots, s$ are the plain eigenvalues of T with corresponding plain multiplicities p_i's. Note that by the above discussion, it is easy to obtain the spectrum of T from its refined spectrum. The pair (r, s) is said to be the *main-plain index* of T. When T is assumed to be the adjacency matrix of a graph Γ we usually leave out the index T from $\xi_T(\theta)$ and write $\xi(\theta)$, if no confusion can occur.

Now we introduce the generalized adjacency matrix of a graph. Let Γ be a graph. For $h \in \mathbb{R}$, the *generalized adjacency matrix* of Γ is defined as $B_h := A + h(J - I)$, where A is the adjacency matrix of Γ. Note that our definition of the generalized adjacency matrix is just an affine transformation of the usual definition as we prefer 0's on the diagonal. The adjacency matrix $\overline{A} = J - I - A$ of the complement of Γ and the Seidel matrix $S = J - I - 2A$ can be written as $\overline{A} := -B_{-1}$ and $S := -2B_{-\frac{1}{2}}$ respectively. The following lemma presents the fact that the matrix B_h has the same main-plain index for every $h \in \mathbb{R}$.

Lemma 1. *Let $h \in \mathbb{R}$. If the adjacency matrix of a graph Γ has refined spectrum $\big(r, s; \mu_1, \dots, \mu_r;\ [\pi_1]^{p_1}, \dots, [\pi_s]^{p_s}\big)$, then B_h has refined spectrum $\big(r, s; \mu'_1, \dots, \mu'_r;$ $[\pi_1 - h]^{p_1}, \dots, [\pi_s - h]^{p_s}\big)$ for some real numbers $\mu'_1, \dots, \mu'_r$. In particular, B_h has the same main-plain index for every $h \in \mathbb{R}$.*

As a consequence of Lemma 1, we have the following known result, see [**16, 21**].

Remark 1. *Let the adjacency matrix of a graph Γ have refined spectrum $\big(r, s; \mu_1,$ $\dots, \mu_r; [\pi_1]^{p_1}, \dots, [\pi_s]^{p_s}\big)$. Then the following holds:*

(i) *The adjacency matrix $\overline{A}$ of the complement of Γ has refined spectrum $\big(r, s; \mu'_1,$ $\dots, \mu'_r; [-\pi_1 - 1]^{p_1}, \dots, [-\pi_s - 1]^{p_s}\big)$ for some real numbers $\mu'_1, \dots, \mu'_r$.*

(ii) *The Sediel matrix S of Γ has refined spectrum $\big(r, s; \mu''_1, \dots, \mu''_r; [-\pi_1 - 2]^{p_1}, \dots,$ $[-\pi_s - 2]^{p_s}\big)$ for some real numbers $\mu''_1, \dots, \mu''_r$.*

In particular, A, $\overline{A}$ and S have the same main-plain index (r, s).

In the following we discuss the refined spectrum of a disjoint union of cliques.

Example 1. *Let $\Gamma = K_{l_1} \dot\cup K_{l_2} \dot\cup \dots \dot\cup K_{l_t}$ with $l_1 \geq l_2 \geq \dots, \geq l_t \geq 1$. We denote the multiset of the l_i's by $\{[m_1]^{b_1}, [m_2]^{b_2}, \dots, [m_d]^{b_d}\}$ where $\{l_1, l_2, \dots, l_t\} = \{m_1, m_2, \dots, m_d\}$, $b_i = |\{j \mid l_j = m_i\}|$ for $i = 1, 2, \dots, d$ and $m_1 > m_2 > \dots > m_d$. Let $U = \{i \mid b_i \geq 2\} = \{u_1, u_2, \dots, u_k\}$. Then the refined spectrum of Γ is $(d, k+1; m_1 - 1, \dots, m_d - 1; [-1]^{(l_1 + \dots + l_t) - t}, [m_{u_1} - 1]^{b_1 - 1}, \dots, [m_{u_k} - 1]^{b_k - 1})$.*

Without proof, we give the following trivial bound on the main-plain index of a graph.

Lemma 2. *Let Γ be a connected graph with distinct eigenvalues $\theta_1, \theta_2, \ldots, \theta_d$ and respective multiplicities $m_1, m_2, \ldots, m_d$. Let (r, s) be the main-plain index of Γ and $l := |\{i \mid m_i \geq 2\}|$. Then $r \leq d$, $s \geq l$, and $d \leq r + s \leq l + d$ all hold.*

Remark 2. *In Section 5, we will discuss a class of graphs with four distinct eigenvalues of which two are simple and main and the remaining two eigenvalues are plain. This shows that the lower bound $r + s = d$ can be achieved. Also in Section 5, we will see that the upper bound $r + s = l + d$ can also be achieved.*

In this note, by switching we mean the Seidel switching of graphs. Let Γ be a graph and σ be a bipartition of $V(\Gamma)$. The *switched graph* Γ^σ (with respect to σ) is the graph obtained by switching Γ with respect to σ. A well-known fact is that the Seidel matrices of Γ and Γ^σ have the same spectrum. So if the Seidel matrix of Γ has a few distinct eigenvalues, then Lemma 2 shows that Γ^σ has only a few distinct main and plain (adjacency) eigenvalues. For example, if one starts with a connected strongly regular graph Γ, then Γ^σ has at most three main eigenvalues and at most two plain eigenvalues. In Section 5, we will give some families of graphs with two main and two plain eigenvalues using switching.

Note that two cospectral graphs can have a different numbers of main eigenvalues and thus different refined spectra. For example, the cospectral pair $K_{1,4}\dot{\cup}K_1$ and $C_4\dot{\cup}2K_1$ have a different number of main eigenvalues as $K_{1,4}\dot{\cup}K_1$ has refined spectrum $(3, 1; 2, 0, -2; [0]^3)$ and $C_4\dot{\cup}2K_1$ has refined spectrum $(2, 2; 2, 0; [0]^3, [-2]^1)$.

In view of the above discussion, we would like to ask the following question.

Question 1. *Suppose that the graphs Γ_1 and Γ_2 have the same refined spectra. Are their complements cospectral?*

In general, the answer is probably no.

The following result is an immediate consequence of the Perron-Frobenius Theorem [**3**, Theorem 2.2.1].

Lemma 3. [**21**] *The spectral radius of a nonnegative irreducible real symmetric matrix is always simple and main. In particular, the spectral radius of a connected graph is always simple and main.*

The following result was shown by Hagos.

Proposition 1. ([**16**, Theorem 2.1]) *The number of main eigenvalues of Γ is the largest integer k such that the vectors $\mathbf{j}, A\mathbf{j}, A^2\mathbf{j}, \ldots, A^{k-1}\mathbf{j}$ are linearly independent.*

As an immediate consequence, we have the following well-known fact.

Lemma 4. [**21**, Proposition 1.4] *A graph Γ has one main eigenvalue if and only if Γ is regular.*

Now we turn our attention to the graphs with two main eigenvalues. We assume that the graph Γ has two main eigenvalues. If we denote $\mathbf{d} := A\mathbf{j}$, then by Proposition 1, the graph Γ satisfies the following relation:

$$A\mathbf{d} = a\mathbf{d} + b\mathbf{j}, \tag{1}$$

where $a,\ b$ are some real numbers. Note that Γ cannot be a regular graph.

The following result was essentially shown by Cvetković [**8**].

Theorem 1. [**17**, Theorem 2.1] *Let Γ be a connected graph and π be an equitable partition of Γ. Let Q be the quotient matrix of π, say with exactly r distinct eigenvalues. Then*

(1) *Γ has at most r main eigenvalues.*

In particular, the following hold:

(a) *If Γ is equitable and t-valenced, then Γ has at most t main eigenvalues;*

(b) *If, moreover Γ is equitable and biregular, then Γ has exactly two main eigenvalues.*

Hayat et al. [**17**] also showed that the number of distinct valencies of a connected graph with two main eigenvalues is unbounded so they are far from equitable and biregular. Now we show that a biregular graph with two main eigenvalues is equitable.

Proposition 2. *Let Γ be a biregular graph. Then Γ has exactly two main eigenvalues if and only if Γ is equitable.*

Proof. By Theorem 1(b), we only need to show the 'only if part' of the statement. Let Γ be a biregular graph with two main eigenvalues and adjacency matrix A. Let $\mathbf{d} := A\mathbf{j}$. Then by Equation 1,

$$A\mathbf{d} = a\mathbf{d} + b\mathbf{j}$$

holds for some real numbers a and b. This means that $\sum_{y:y\sim x} d_y = (A\mathbf{d})_x = ad_x + b$ holds for any vertex x. Let d_1 and d_2 be the two distinct valencies of Γ. This implies that the number of neighbors of x with valency d_1 (resp. d_2) depends only on whether $d_x = d_1$ or $d_x = d_2$. This shows that the valency partition is equitable. $\square$

Note that the cone over a graph Ω has adjacency matrix $A = \begin{pmatrix} 0 & \mathbf{j}^\top \\ \mathbf{j} & A' \end{pmatrix}$, where A' is the adjacency matrix of Ω. Now, for an eigenvalue θ of Ω, if $\mathbf{x}$ is a plain eigenvector of Ω, then $\begin{pmatrix} 0 \\ \mathbf{x} \end{pmatrix}$ is a plain eigenvector of the cone over Ω. Consequently we have the following result.

Proposition 3. *Let Ω be a graph with r main eigenvalues. Then the cone over Ω has at most $r+1$ main eigenvalues.*

Note that if $\Omega = K_n$, then the resulting cone has exactly one main eigenvalue. We wonder whether such an example also exists for larger r.

4. SOME CHARACTERIZATIONS

In this section, we present some characterizations of graphs with a small number of main and plain eigenvalues.

Note that the connected graphs with one main eigenvalue and one plain eigenvalue are the complete graphs with at least two vertices, as there are at most two distinct eigenvalues and thus the diameter is exactly one. As a connected regular graph with three distinct eigenvalues is strongly regular (see for example [**15**, Lemma 10.2.1]), we obtain:

Lemma 5. *A connected graph Γ has one main eigenvalue and at most two plain eigenvalues if and only if Γ is a strongly regular graph.*

Note that the connected graphs with one main eigenvalue and three plain eigenvalues are exactly the connected regular graphs with four distinct eigenvalues. These graphs are studied by Van Dam [**11**], Van Dam & Spence [**12**] and Huang & Huang [**18**], among others.

The following lemma shows that the connected graphs with two main eigenvalues and one plain eigenvalue are the nonregular complete bipartite graphs.

Lemma 6. *A connected graph Γ has two main eigenvalues and one plain eigenvalue if and only if Γ is a nonregular complete bipartite graph.*

Proof. Since Γ is not complete, it has three distinct eigenvalues. Let ρ and τ be the main eigenvalues and θ be the plain eigenvalue of Γ. Note that the two main eigenvalues ρ and τ are simple. The lemma is true for $n = 3$, thus we may assume that $n \geq 4$. Then $\theta \in \mathbb{Z}$, as the other two eigenvalues are simple. Note that the rank of $A - \theta I$ is two. Consider three distinct vertices x, y, z such that $x \sim y \sim z$ and $x \nsim z$. The subgraph induced on $\{x, y, z\}$ has eigenvalues 0, $\sqrt{2}$, and $-\sqrt{2}$. The principal sub matrix P of $A-\theta I$ with respect to $\{x, y, z\}$ has rank at most two, and this implies that $\theta \in \{0, \pm\sqrt{2}\}$. As $\theta \in \mathbb{Z}$ we find $\theta = 0$ and hence $\rho = -\tau$. This implies that Γ is bipartite. Note that Γ has diameter two so it must be complete bipartite. Moreover, Γ is nonregular as it has two main eigenvalues. The converse is obvious, and this completes the proof. $\square$

The following result classifies the disconnected graphs with two main and two plain eigenvalues.

Theorem 2. *Let Γ be a disconnected graph with two main and two plain eigenvalues. Then Γ is a member of one of the following families.*

(i) *The disjoint union of cliques $K_{l_1} \dot\cup K_{l_2} \dot\cup \ldots \dot\cup K_{l_t}$, where $t \geq 3$, $l_1 \neq l_2$ and $l_2 = l_3 = \ldots = l_t$.*

(ii) *The disjoint union of isolated vertices and a regular complete multipartite graph.*

(iii) *The disjoint union of an isolated vertex and a strongly regular graph.*

(iv) *The disjoint union of two strongly regular graphs with different valencies and the same non-trivial eigenvalues.*

Proof. The case when each connected component is a clique follows immediately from Example 1. And this shows (i).

Next we may assume that there exists a connected component, say, Γ_1 of Γ with at least three distinct eigenvalues. Let Γ_1 have vertex set V_1 with n_1 vertices and let Γ_2 be the induced subgraph on $V_2 = V - V_1$ with $n_2 = n - n_1$ vertices. Let ρ_1 and ρ_2 be the spectral radii of Γ_1 and Γ_2, respectively. Note that Γ has at most four distinct eigenvalues. Assume that Γ has exactly four distinct eigenvalues. As the main eigenvalues are simple, we see that ρ_1 and ρ_2 are the main eigenvalues. This shows that both Γ_1 and Γ_2 are regular and Γ_2 is connected. Also for $i = 1, 2$, Γ_i has at most three distinct eigenvalues, so Γ_i is either a clique or a non-complete strongly regular graph. As Γ_1 is non-complete, we find that Γ_2 is strongly regular. If both Γ_1 and Γ_2 are non-complete, we are in case (iv) of the theorem. So we may assume that Γ_1 is non-complete and Γ_2 is a clique. The graph Γ_2 does not have -1 as an eigenvalue, hence Γ_2 is an isolated vertex. This shows that we are in case (iii) of the theorem.

Now we assume that Γ has at most three distinct eigenvalues, and hence exactly three distinct eigenvalues. Then Γ_1 has three distinct eigenvalues, say, $\rho_1 > \theta_1 > \theta_2$ where $\rho_2 \in \{\rho_1, \theta_1\}$, as $\rho_2 \geq 0$. This implies that Γ_1 is strongly regular, and θ_1 and θ_2 are the plain eigenvalues, which gives $\rho_2 = \theta_1$. Then Γ_2 has at most two distinct eigenvalues and hence Γ_2 is the disjoint union of cliques. Now $\rho_2 > 0$, implies $\theta_2 = -1$, a contradiction. So $\rho_2 = \theta_1 = 0$ and hence $\theta_1 = 0$ implies that Γ_1 is complete multipartite. This shows that we are in the case (ii). This completes the proof.

□

Using Remark 1, as a corollary we have:

Corollary 1. *Let Γ be a connected graph with two main and two plain eigenvalues. Assume that the complement of Γ is disconnected. Then Γ is a member of one of the following families.*

(i) *The complete t-partite graph $K_{l_1,l_2,\ldots,l_t}$ where $t \geq 3$, $l_1 \neq l_2$ and $l_2 = l_3 = \ldots = l_t$.*

(ii) *The multicone over a regular complete multipartite graph.*

(iii) *The cone over a strongly regular graph.*

(iv) *The join of two strongly regular graphs with different valencies and the same non-trivial eigenvalues.*

5. EXAMPLES AND DISCUSSION

In this section, we give some families of graphs with two main and two plain eigenvalues. For several of these families, it was already known that they have two main eigenvalues.

Strong graphs. Recall that the Seidel matrix S of a graph, with adjacency matrix A, is defined by $S = J - I - 2A$. A *strong graph* is a graph such that its Seidel matrix S satisfies $S^2 \in \langle S, I, J \rangle$, where $\langle \dots \rangle$ denotes the $\mathbb{R}$-span. We call a graph *non-strong*, if it is not strong. Seidel [**22**] showed that:

Proposition 4. [**22**] *Let Γ be a graph with Seidel matrix S. Then Γ is strong if and only if at least one of the following holds:*

(i) *Γ is strongly regular.*

(ii) *S has exactly two distinct eigenvalues.*

Using a result of Van Dam et al. [**13**], Hayat et al. showed the following result.

Theorem 3. [**17**] *Let Γ be an n-vertex nonregular strong graph. If Γ has Seidel eigenvalues $[-1-2\theta_0]^{m_0}, [-1-2\theta_1]^{m_1}$, then it has four distinct (adjacency) eigenvalues $\mu_0, \mu_1, [\theta_0]^{m_0-1}, [\theta_1]^{m_1-1}$ of which μ_0 and μ_1 are main eigenvalues and θ_0, θ_1 are plain eigenvalues. The main eigenvalues μ_0, μ_1 are uniquely determined by θ_0, θ_1, m_0, m_1, n and the number of edges.*

Thus, the nonregular strong graphs are the graphs with two main and two plain eigenvalues, and Van Dam et al. [**13**] showed that they are connected unless they have an isolated vertex.

Strongly biregular graphs. Let Γ be a connected strongly biregular graph. By a result of Van Dam [**10**], Γ is equitable and thus has two main eigenvalues of which one is the spectral radius. Consequently, Γ has two main and at most two plain eigenvalues. By Lemma 6, a strongly biregular graph Γ has exactly two main and two plain eigenvalues if and only if Γ is nonbipartite.

Rowlinson [**20**] characterized the strongly biregular graphs among the graphs with three distinct eigenvalues. For the convenience of the reader, we include a proof of it.

Lemma 7. *Let Γ be a connected graph with three distinct eigenvalues. Then exactly two of them are main if and only if Γ is strongly biregular.*

Proof. Let Γ have three distinct eigenvalues, say $\theta_0 > \theta_1 > \theta_2$. Then there exists a positive vector α such that

$$(A - \theta_1 I)(A - \theta_2 I) = \alpha\alpha^\top, \quad \text{and} \quad A\alpha = \theta_0\alpha \tag{2}$$

both hold. From this equation we obtain $d_i = \alpha_i^2 - \theta_1\theta_2$, where d_i is the valency of vertex i. Let Γ have two main eigenvalues. Then by Equation (1) we obtain $A^2\mathbf{j} \in \langle \mathbf{d}, \mathbf{j}\rangle$. Now by Equation (2), $\alpha(\alpha^\top \mathbf{j}) \in \langle \mathbf{d}, \mathbf{j}\rangle$ and $\alpha^\top \mathbf{j} \neq 0$. Accordingly there exist real numbers a and b such that $\alpha = a\mathbf{d} + b\mathbf{j}$. It follows that

$$a\alpha_i^2 - \alpha_i - a\theta_1\theta_2 + b = 0 \quad (i = 1, \ldots, n),$$

and thus α_i's take just two values. Consequently, by Equation (2), the graph Γ has exactly two valencies. The other direction immediately follows from Proposition 2. $\square$

Bridges & Mena [**2**] and De Caen et al. [**9**] gave examples of connected graphs with three eigenvalues and three distinct valencies. They have three main eigenvalues by Lemma 7. Thus the upper bound $r + s = l + d$ in Lemma 2 can be achieved.

Vertex-deleted subgraph of a strongly regular graph. Let Γ be a connected strongly regular graph with parameters (n, k, a, c) and distinct eigenvalues $k > \theta_1 > \theta_2$. Fix a vertex u of Γ. Let N_u be the set of neighbors of vertex u. Then the partition $\sigma = \{u, N_u, V - (\{u\} \cup N_u)\}$ is equitable with quotient matrix $Q = \begin{pmatrix} 0 & k & 0 \\ 1 & a & k-a-1 \\ 0 & c & k-c \end{pmatrix}$. As Q has constant row sum, the all-ones vector $\mathbf{j}$ is an eigenvector of Q. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be two further eigenvectors of Q. Let P be the characteristic matrix of σ. Then $\mathbf{w}_1 = P\mathbf{x}_1$ and $\mathbf{w}_2 = P\mathbf{x}_2$ are two eigenvectors of Γ with eigenvalues θ_1 and θ_2, respectively. The graph $\Gamma - u$ is equitable biregular and thus has two main eigenvalues. Note that the plain eigenvectors $\mathbf{w}_1$ and $\mathbf{w}_2$ of Γ both give eigenvectors of $\Gamma - u$ by deleting $(\mathbf{w}_1)_u$ and $(\mathbf{w}_2)_u$. Therefore, $\Gamma - u$ also has exactly two plain eigenvalues. Note that Rowlinson [**21**] already showed that these graphs have just two main eigenvalues.

Switching in graphs. The following result shows that a switched graph of a graph with two plain and at most two main eigenvalues is again a graph with two plain and at most two main eigenvalues.

Proposition 5. *Let Γ be an n-vertex graph with two plain and at most two main eigenvalues. Let $U \subseteq V(\Gamma)$ be such that the bipartition $\sigma = \{U, V - U\}$ is equitable. Then Γ^σ is a graph with two plain and at most two main eigenvalues.*

Proof. Assume Γ is a graph with two plain and at most two main eigenvalues with equitable bipartition σ. Let $Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$ be the quotient matrix of Γ corresponding to σ with $n_1 = |U|$ and $n_2 = |V - U| = n - n_1$. Then Γ^σ is an equitable graph with quotient matrix

$$Q' = \begin{pmatrix} q_{11} & n_2 - q_{12} \\ n_1 - q_{21} & q_{22} \end{pmatrix}.$$

Thus by Theorem 1, the graph Γ^σ has at most two main eigenvalues. Let

$$A = \begin{array}{cc} & \begin{array}{cc} U & \overline{U} \end{array} \\ \begin{array}{c} U \\ \overline{U} \end{array} & \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \end{array}.$$

be the adjacency matrix of Γ, where $\overline{U} = V - U$. Then the adjacency matrix of Γ^σ can be partitioned accordingly as

$$A(\Gamma^\sigma) = \begin{pmatrix} A_{11} & J_1 - A_{12} \\ J_2 - A_{21} & A_{22} \end{pmatrix},$$

where J_1 (resp. J_2) is the $n_1 \times n_2$ (resp. $n_2 \times n_1$) all-ones matrix. Let $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_p$ be eigenvectors corresponding to plain eigenvalues $\pi_1, \pi_2, \dots, \pi_p$ respectively, where p is the number of plain eigenvalues of Γ. Then any eigenvector $\mathbf{x}_i$ for $i = 1, \dots, p$ can be partitioned according to σ as $\mathbf{x}_i = \begin{pmatrix} \mathbf{x}_i' \\ \mathbf{x}_i'' \end{pmatrix}$, where $\mathbf{x}_i'$ (resp. $\mathbf{x}_i''$) is the block corresponding to U (resp. $\overline{U}$). We define $\widehat{\mathbf{x}}_i := \begin{pmatrix} \mathbf{x}_i' \\ -\mathbf{x}_i'' \end{pmatrix}$. Note that $\widehat{\mathbf{x}}_i$ is also an eigenvector of $A(\Gamma^\sigma)$ with eigenvalue π_i for $i = 1, \dots, p$. Thus Γ^σ has the same number of plain eigenvalues as Γ. This completes the proof. □

Note that Proposition 5 helps us to construct graphs with two main and two plain eigenvalues as long as the switched graph Γ^σ is not regular. For example, let Γ be a strongly regular graph with $\sigma = \{C, V - C\}$, where the induced subgraph C is a Delsarte clique, so that σ is an equitable partition. Then by Proposition 5, Γ^σ is a graph with two main and two plain eigenvalues if Γ^σ is nonregular. We give the following example as an instance of this fact.

Example 2. *Let Γ be $L(K_{6,6})$, which is a strongly regular graph with parameters $(36, 10, 4, 2)$. Let σ be a Delsarte clique in Γ which is a 6-clique. Then Γ^σ is an equitable biregular graph with spectrum $\{[7+\sqrt{129}]^1, [4]^9, [-2]^{25}, [7-\sqrt{129}]^1\}$ of which 4 and -2 are the two plain eigenvalues and $7 \pm \sqrt{129}$ are the two main eigenvalues.*

Note that the matrix B_{-2} of the switched graph Γ^σ of the above example has spectrum $\{[6]^9, [0]^{26}, [-54]^1\}$. Moreover, B_{-2} is a scalar multiple of the distance matrix of Γ^σ. Thus we obtain a first example of a nonregular nonbipartite graph with three distinct distance eigenvalues.

In a follow-up paper, we will show that, for graphs with two main and two plain eigenvalues there always exists a real number h such that the generalized adjacency matrix B_h has at most three distinct eigenvalues.

Corona of a clique and an isolated vertex. If K_n is a complete graph on n vertices i.e. an n-clique with $n > 1$, then the corona $\Gamma = K_n \circ K_1$ is an equitable biregular graph with $2n$ vertices, see for example with $n = 3$ in Figure 1.

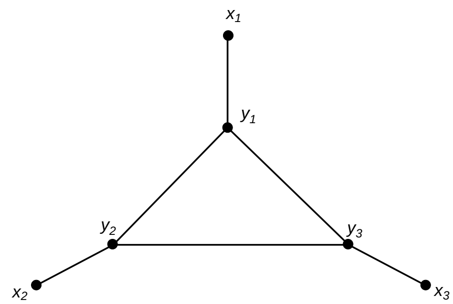

Figure 1: The graph $K_3 \circ K_1$.

By Theorem 1, Γ has exactly two main eigenvalues. Let Q be the quotient matrix of Γ, where

$$Q = \begin{pmatrix} 0 & 1 \\ 1 & n-1 \end{pmatrix}.$$

Since the eigenvalues of Q are the two main eigenvalues of Γ, both of them are non-integral and thus simple. To show that Γ has two plain eigenvalues, it is enough to show that it has at most four distinct eigenvalues. Let $x_1, \ldots, x_n$ (resp. $y_1, \ldots, y_n$) are the vertices of valency one (resp. valency n) in Γ such that $x_i \sim y_i$ for all i (see Figure 1). Note that for any i, the stabilizer of x_i coincides with the stabilizer of its neighbor y_i i.e. if you stabilize x_i, then you have to stabilize its corresponding y_i. Since any such point stabilizer has four orbits in the automorphism group, the graph Γ has at most four distinct eigenvalues by [**1**, Theorem 4.2]. Thus Γ has exactly two main and two plain eigenvalues. Note that the complement of Γ is also a connected graph with two main and two plain eigenvalues and is not isomorphic to Γ if $n \geq 6$. We conclude our discussion of this subsection with the following observation.

Proposition 6. *For every even integer n at least 4, there exists a graph with two main and two plain eigenvalues.*

Acknowledgments. We would like to thank the referees for their comments and careful reading which improved the note significantly. Sakander Hayat is supported by a CAS-TWAS president's fellowship at USTC, China. Muhammad Javaid is supported by Chinese Academy of Science, President's International Fellowship Initiative (CAS-PIFI), Beijing, China under grant No. 2015 PM 035. Jack Koolen is partially supported by the National Natural Science Foundation of China (Nos. 11471009 and 11671376).

REFERENCES

1. S. Bang, T. Fujisaki, J. H. Koolen: *The spectra of the local graphs of the twisted Grassmann graphs.* European J. Combin., **30** (2009), 638–654.
2. W. G. Bridges, R. A. Mena: *Multiplicative cones–a family of three eigenvalue graphs.* Aequationes Math., **22** (1981), 208–214.
3. A. E. Brouwer, W. H. Haemers: *Spectra of Graphs*, Springer, Heidelberg, 2012.
4. A. E. Brouwer, J. H. van Lint:, *Strongly regular graphs and partial geometries*, In Enumeration and Design: Papers from the conference on combinatorics held at the University of Waterloo, Waterloo, Ont., June 14-July 2, 1982, (Ed. D. M. Jackson and S. A. Vanstone). Toronto, Canada: Academic Press, 85–122, 1984.
5. S. M. Cioabă, W. H. Haemers, J. R. Vermette, W. Wong: *The graphs with all but two eigenvalues equal to* ± 1. J. Algebr. Comb., **41** (2015), 887–897.
6. S. M. Cioabă, W. H. Haemers, J. R. Vermette: *The graphs with all but two eigenvalues equal to* -2 *or* 0. Des. Codes Cryptogr., **84** (2017), 153–163.
7. X.-M. Cheng, A. L. Gavrilyuk, G. R. W. Greaves, J. H. Koolen: *Biregular graphs with three eigenvalues.* European J. Combin., **56** (2016), 57–80.
8. D. M. Cvetković: *The main part of the spectrum, divisors and switching of graphs.* Publ. Inst. Math. (Beograd), **23(37)** (1978), 31–38.
9. D. de Caen, E. R. van Dam, E. Spence:, *A non-regular analogue of conference graphs.* J. Combin. Theory Ser. A, **88** (1999), 194–204.
10. E. R. van Dam: *Nonregular graphs with three eigenvalues.* J. Combin. Theory Ser. B, **73** (1998), 101–118.
11. E. R. van Dam: *Regular graphs with four eigenvalues.* Linear Algebra Appl., **226–228** (1995), 139–162.
12. E. R. van Dam, E. Spence: *Small regular graphs with four eigenvalues.* Discrete Math., **189** (1998), 233–257.
13. E. R. van Dam, J. H. Koolen, Z.-J. Xia: *Graphs with many valencies and few eigenvalues.* Electron. J. Linear Algebra, **28** (2015), 12–24.
14. M. Doob:, *Graphs with a small number of distinct eigenvalues.* Ann. New York Acad. Sci., **175** (1970), 104–110.
15. C. D. Godsil, G. Royle:, *Algebraic Graph Theory*, Springer-Verlag, Berlin, 2001.
16. E. M. Hagos:, *Some results on graph spectra.* Linear Algebra Appl., **356** (2002), 103–111.
17. S. Hayat, J. H. Koolen, F. Liu, Z. Qiao: *A note on graphs with exactly two main eigenvalues.* Linear Algebra Appl., **511** (2016), 318–327.
18. X. Huang, Q. Huang:, *On regular graphs with four distinct eigenvalues.* Linear Algebra Appl., **512** (2017), 219–233.
19. M. Muzychuk, M. Klin: *On graphs with three eigenvalues.* Discrete Math., **189** (1998), 191–207.
20. P. Rowlinson: *On graphs with just three distinct eigenvalues.* Linear Algebra Appl., **507** (2016), 462–473.

21. P. Rowlinson: *The main eigenvalues of a graph: A survey.* Appl. Anal. Discr. Math., **1** (2007), 445–471.

22. J. J. Seidel: *Strongly regular graphs with* $(-1, 1, 0)$*-adjacency matrix having eigenvalue* 3. Linear Algebra Appl., **1** (1968), 281–298.

Sakander Hayat
School of Mathematical Sciences,
University of Science and Technology of China,
Hefei, Anhui, 230026, P.R. China
E-mail: *sakander@mail.ustc.edu.cn*

(Received 25.11.2016.)
(Revised 17.09.2017.)

Muhammad Javaid
School of Mathematical Sciences,
University of Science and Technology of China,
Hefei, Anhui, 230026, P.R. China
E-mail: *javaidmath@gmail.com*

Jack H. Koolen
Wen-Tsun Wu Key Laboratory of CAS,
School of Mathematical Sciences,
University of Science and Technology of China,
Hefei, Anhui, 230026, P.R. China
E-mail: *koolen@ustc.edu.cn*