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HIGHER ORDER BELL POLYNOMIALS AND THE RELEVANT INTEGER SEQUENCES

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Dedicated to the Memory of Dr. Massimo Marchetti
an unforgettable dear friend

The recurrence relation for the coefficients of higher order Bell polynomials, i.e. of the Bell polynomials relevant to *nth* derivative of a multiple composite function, is proved. Therefore, starting from this recurrence relation and by using the computer algebra program Mathematica©, some tables for complete higher order Bell polynomials and the relevant numbers are derived.

1. INTRODUCTION

The Bell polynomials [**3**] are a mathematical tool for representing the nth derivative of a composite function. They are strictly related to partitions [**1**], [**2**], [**21**].

Several applications of the classical Bell polynomials have been considered in [**5**], [**7**], [**9**] (in connection with [**22**]), [**13**], [**14**].

Some generalized forms of Bell polynomials appeared in literature, see e.g. [**11**], [**20**]. Further generalizations can be found in [**15**], [**16**], and for the multidimensional case in [**6**], [**19**].

In particular, in [**15**], the higher order Bell polynomials and their main properties were introduced and recently, in [**18**], a recursion formula for the polynomial coefficients $A_{n,k}$ of the classical Bell polynomials was derived. This last result allows to compute the complete Bell polynomials B_n and the relevant Bell numbers b_n, for every integer n.

In this article, after recalling this theory, and by using a more compact notation borrowed from [**6**], we prove the recurrence relation formula for the polynomial

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coefficients $A_{n,k}^{[r]}$ of the rth order Bell polynomials, generalizing in this way the result obtained in [**18**]. Therefore, by using this formula and by means of the computer algebra program Mathematica©, we can obtain, for all integer n, tables of every order complete Bell polynomials $B_n^{[r]}$ and the relevant Bell numbers $b_n^{[r]}$. Here, as examples, we consider the cases $r = 2, 3, 4, 5$.

It is worth to note that the higher order Bell numbers appeared in literature as the McLaurin coefficients of a particular nested exponential function, while in our approach they assume a more general meaning. To our knowledge, tables of higher order Bell polynomials were never considered at all.

2. RECALLING THE BELL POLYNOMIALS

We recall that the Bell polynomials are a classical mathematical tool for representing the n^{th} derivative of a composite function. In fact by considering the composite function $\Phi(t) := f(g(t))$ of functions $x = g(t)$ and $y = f(x)$ defined in suitable intervals of the real axis and n times differentiable with respect to the relevant independent variables and by using the following notations:

$$\Phi_h := D_t^h \Phi(t), \qquad f_h := D_x^h f(x)|_{x=g(t)}, \qquad g_h := D_t^h g(t)\,, \tag{1}$$

and

$$([f,g]_n) := (f_1, g_1; f_2, g_2; \ldots; f_n, g_n)\,, \tag{2}$$

they are defined as follows

$$Y_n\left([f,g]_n\right) := \Phi_n. \tag{3}$$

For example one has:

$$\begin{aligned} &Y_1([f,g]_1) = f_1 g_1, \\ &Y_2([f,g]_2) = f_1 g_2 + f_2 g_1^2, \\ &Y_3([f,g]_3) = f_1 g_3 + f_2(3g_2 g_1) + f_3 g_1^3. \end{aligned}$$

Further examples can be found in [**21**, p. 49].

Inductively, using the notation

$$[g]_n := (g_1, g_2, \ldots, g_n)\,,$$

we can write:

$$Y_n\left([f,g]_n\right) = \sum_{k=1}^{n} A_{n,k}([g]_n) f_k, \tag{4}$$

where the coefficient $A_{n,k}$, for any $k = 1, \ldots, n$, is a polynomial in $g_1, g_2, \ldots, g_n$, homogeneous of degree k and *isobaric* of weight n (i.e. it is a linear combination of monomials $g_1^{k_1} g_2^{k_2} \cdots g_n^{k_n}$ whose weight is constantly given by $k_1 + 2k_2 + \ldots + nk_n = n$).

For them the following result holds true:

Proposition 1. *The Bell polynomials satisfy the recurrence relation:*

$$
\left\{
\begin{array}{l}
Y_0\left([f,g]_0\right) := f_1 \\
Y_{n+1}\left([f,g]_{n+1}\right) = \displaystyle\sum_{k=0}^{n} \binom{n}{k} Y_{n-k}\left([f_1,g]_{n-k}\right)\; g_{k+1},
\end{array}
\right. \tag{5}
$$

where

$$\left([f_1,g]_{n-k}\right) := (f_2, g_1; f_3, g_2; \ldots; f_{n-k+1}, g_{n-k}).$$

An explicit expression for the Bell polynomials is also given by the Faà di Bruno formula [**10**]:

$$\Phi_n = Y_n\left([f,g]_n\right) = \sum_{\pi(n)} \frac{n!}{j_1! j_2! \ldots j_n!}\, f_j \left[\frac{g_1}{1!}\right]^{j_1} \left[\frac{g_2}{2!}\right]^{j_2} \cdots \left[\frac{g_n}{n!}\right]^{j_n}, \tag{6}$$

where the sum runs over all partitions $\pi(n)$ of the integer n (i.e. $n = j_1 + 2j_2 + \cdots + nj_n$), j_h denotes the number of parts of size h and $j = j_1 + j_2 + \cdots + j_n$ denotes the number of parts of the considered partition. A proof of the Faà di Bruno formula can be found in [**21**]. In [**23**] the proof is based on the *umbral calculus* (see [**24**] and the references therein).

The following result gives us a recursion formula for the coefficients $A_{n,k}$ which appear in the Bell formula (4) and are known as partial Bell polynomials. It was proved in [**18**], but we will observe that it derives as a particular case of Theorem 7 proved here in Section 3.

Theorem 2. *We have,* $\forall\, n$*:*

$$A_{n+1,1} = g_{n+1}\,, \qquad A_{n+1,n+1} = g_1^{n+1}\,. \tag{7}$$

Furthermore, $\forall\, k = 1, 2, \ldots, n-1$*, the* $A_{n,k}$ *coefficients can be computed by the recurrence relation*

$$A_{n+1,k+1}([g]_{n+1}) = \sum_{h=0}^{n-k} \binom{n}{h} A_{n-h,k}([g]_{n-h})\, g_{h+1}. \tag{8}$$

The complete Bell polynomials, considered in literature, are defined by

$$B_n([g]_n) = Y_n(1, g_1; 1, g_2; \ldots 1, g_n) = \sum_{k=1}^{n} A_{n,k}([g]_n)\,, \tag{9}$$

and the Bell numbers by

$$b_n = Y_n(1,1;1,1;\ldots;1,1) = \sum_{k=1}^{n} A_{n,k}(1,1,\ldots,1)\,. \tag{10}$$

3. BELL POLYNOMIALS OF ORDER r

In [**6**] the following extension of the classical Bell polynomials was achieved.

Consider $\Phi(t) := f(\varphi^{(1)}(\varphi^{(2)}(\cdots(\varphi^{(r)}(t)))))$, i.e. the composition of functions $x^{(r)} = \varphi^{(r)}(t), \ldots, x^{(2)} = \varphi^{(2)}\left(x^{(3)}\right), x^{(1)} = \varphi^{(1)}\left(x^{(2)}\right), y = f\left(x^{(1)}\right)$ defined in suitable intervals of the real axis, and suppose that the functions $\varphi^{(r)}, \ldots, \varphi^{(2)}, \varphi^{(1)}, f$ are n times differentiable with respect to the relevant independent variables so that, by using the chain rule, $\Phi(t)$ can be differentiated n times with respect to t.

We use the following notations:

$$
\begin{aligned}
&\Phi_h := D_t^h \Phi(t), \\
&f_h := D^h_{x^{(1)}} f|_{x^{(1)}=\varphi^{(1)}(\cdots(\varphi^{(r)}(t)))}, \\
&\varphi_h^{(1)} := D^h_{x^{(2)}} \varphi^{(1)}|_{x^{(2)}=\varphi^{(2)}(\cdots(\varphi^{(r)}(t)))}, \\
&\ldots\ldots\ldots\ldots\ldots \\
&\varphi_h^{(r)} := D_t^h \varphi^{(r)}(t),
\end{aligned}
\tag{11}
$$

and

$$\left(\left[f, \varphi^{(1)}, \ldots, \varphi^{(r)}\right]_n\right) := (f_1, \varphi_1^{(1)}, \ldots, \varphi_1^{(r)}; \ldots; f_n, \varphi_n^{(1)}, \ldots, \varphi_n^{(r)}).$$

Then the n^{th} derivative of the function Φ allows us to define the (one-dimensional) Bell polynomials of order r, $Y_n^{[r]}$, as follows:

$$Y_n^{[r]}\left(\left[f, \varphi^{(1)}, \ldots, \varphi^{(r)}\right]_n\right) := \Phi_n. \tag{12}$$

For $r = 1$ we obtain the ordinary Bell polynomials $Y_n^{[1]}\left(\left[f, \varphi^{(1)}\right]_n\right) = Y_n\left(\left[f, \varphi^{(1)}\right]_n\right)$. Note that we are considering here the one-dimensional case, while in [**6**] even the multi-dimensional Bell polynomials were introduced.

The first polynomials have the following explicit expressions:

$$
\begin{aligned}
&Y_1^{[r]}\left(\left[f, \varphi^{(1)}, \ldots, \varphi^{(r)}\right]_1\right) = f_1 \varphi_1^{(1)} \cdots \varphi_1^{(r)}, \\
&Y_2^{[r]}\left(\left[f, \varphi^{(1)}, \ldots, \varphi^{(r)}\right]_2\right) = f_2\left(\varphi_1^{(1)} \cdots \varphi_1^{(r)}\right)^2 + f_1 \varphi_2^{(1)}\left(\varphi_1^{(2)} \cdots \varphi_1^{(r)}\right)^2 \\
&\qquad + f_1 \varphi_1^{(1)} \varphi_2^{(2)}\left(\varphi_1^{(3)} \cdots \varphi_1^{(r)}\right)^2 + f_1 \varphi_1^{(1)} \varphi_1^{(2)} \cdots \varphi_1^{(r-1)} \varphi_2^{(r)}.
\end{aligned}
\tag{13}
$$

In general, we have

$$Y_n^{[r]}([f, \varphi^{(1)}, \ldots, \varphi^{(r)}]_n) = \sum_{k=1}^{n} A_{n,k}^{[r]}([\varphi^{(1)}, \ldots, \varphi^{(r)}]_n) f_k. \tag{14}$$

Some useful properties, proved in [**15**], satisfied by the polynomials $Y_n^{[r]}$ are the following:

Theorem 3. *For every integer n, the polynomials $Y_n^{[r]}$ are expressed in terms of the Bell polynomials of lower order, by means of the following equation:*

$$Y_n^{[r]}\left(\left[f,\varphi^{(1)},\ldots,\varphi^{(r)}\right]_n\right)=Y_n\left(\left[f,Y^{[r-1]}\left(\left[\varphi^{(1)},\ldots,\varphi^{(r)}\right]\right)\right]_n\right), \tag{15}$$

where

$$\left(\left[f,Y^{[r-1]}\left(\left[\varphi^{(1)},\ldots,\varphi^{(r)}\right]\right)\right]_n\right)$$

$$:=\left(f_1,Y_1^{[r-1]}\left(\left[\varphi^{(1)},\ldots,\varphi^{(r)}\right]_1\right);\ldots;f_n,Y_n^{[r-1]}\left(\left[\varphi^{(1)},\ldots,\varphi^{(r)}\right]_n\right)\right).$$

Theorem 4. *The following recurrence relation for the Bell polynomials $Y_n^{[r]}$ holds true:*

$$\begin{cases} Y_0^{[r]}\left(\left[f,\varphi^{(1)},\ldots,\varphi^{(r)}\right]_0\right)=f_1 \\ Y_{n+1}^{[r]}\left(\left[f,\varphi^{(1)},\ldots,\varphi^{(r)}\right]_{n+1}\right)=\sum_{k=0}^{n}\binom{n}{k} \\ \qquad\times\, Y_{n-k}^{[r]}\left(\left[f_1,\varphi^{(1)},\ldots,\varphi^{(r)}\right]_{n-k}\right)Y_{k+1}^{[r-1]}\left(\left[\varphi^{(1)},\ldots,\varphi^{(r)}\right]_{k+1}\right), \end{cases} \tag{16}$$

where

$$\left(\left[f_1,\varphi^{(1)},\ldots,\varphi^{(r)}\right]_{n-k}\right):=\left(f_2,\varphi_1^{(1)},\ldots,\varphi_1^{(r)};\ldots;f_{n-k+1},\varphi_{n-k}^{(1)},\ldots,\varphi_{n-k}^{(r)}\right).$$

Theorem 5. *The generalized Faà di Bruno formula holds true:*

$$Y_n^{[r]}\left(\left[f,\varphi^{(1)},\ldots,\varphi^{(r)}\right]_n\right)=\sum_{\pi(n)}\frac{n!}{j_1!j_2!\ldots j_n!}\,f_j\left[\frac{Y_1^{[j-1]}\left(\left[\varphi^{(1)},\ldots,\varphi^{(r)}\right]_1\right)}{1!}\right]^{j_1}$$

$$\times\left[\frac{Y_2^{[j-1]}\left(\left[\varphi^{(1)},\ldots,\varphi^{(r)}\right]_2\right)}{2!}\right]^{j_2}\cdots\left[\frac{Y_n^{[j-1]}\left(\left[\varphi^{(1)},\ldots,\varphi^{(r)}\right]_n\right)}{n!}\right]^{j_n}. \tag{17}$$

By putting, for every integer s $(1\le s\le r-1)$,

$$\varphi^{(s+1)}(\varphi^{(s+2)}\cdots(\varphi^{(r)}(t)))=:g(t),\qquad f(\varphi^{(1)}(\cdots(\varphi^{(s)}(x))))=:f(x)$$

where $x=g(t)$, the composite function $\Phi(t):=f(\varphi^{(1)}(\cdots(\varphi^{(r)}(t))))$, can be written as follows

$$\Phi(t)=f(g(t)).$$

Therefore the following result holds true:

Theorem 6. *For every integer n, the polynomials $Y_n^{[r]}$ are expressed in terms of the Bell polynomials of lower order, by means of the following equation:*

$$
\begin{aligned}
&Y_n^{[r]}\left(\left[f,\varphi^{(1)},\ldots,\varphi^{(r)}\right]_n\right)\\
&=Y_n\left(\left[Y^{[s]}\left(\left[f,\varphi^{(1)},\ldots,\varphi^{(s)}\right]\right),Y^{[r-s-1]}\left(\left[\varphi^{(s+1)},\ldots,\varphi^{(r)}\right]\right)\right]_n\right).
\end{aligned}
\tag{18}
$$

The complete Bell polynomials of order r, $B_n^{[r]}$, are defined by the equation:

$$
\begin{aligned}
B_n^{[r]}\left(\left[\varphi^{(1)},\ldots,\varphi^{(r)}\right]_n\right)&=Y_n^{[r]}(1,\varphi_1^{(1)},\ldots,\varphi_1^{(r)};\ldots;1,\varphi_n^{(1)},\ldots,\varphi_n^{(r)})\\
&=\sum_{k=1}^{n}A_{n,k}^{[r]}\left(\left[\varphi^{(1)},\ldots,\varphi^{(r)}\right]_n\right),
\end{aligned}
$$

and the rth order Bell numbers by

$$
b_n^{[r]}=Y_n^{[r]}(1,1,1;\ldots;1,1,1)=\sum_{k=1}^{n}A_{n,k}^{[r]}(1,1;\ldots;1,1).
$$

Now, in order to derive tables for complete higher order Bell polynomials $B_n^{[r]}$ and the relevant higher order Bell numbers $b_n^{[r]}$, we generalize the result given in Theorem 2, by means of the following theorem

Theorem 7. *We have,* $\forall\, n$

$$
\begin{aligned}
A_{n+1,1}^{[r]}&=Y_{n+1}^{[r-1]}\left(\left[\varphi^{(1)},\ldots,\varphi^{(r)}\right]_{n+1}\right),\\
A_{n+1,n+1}^{[r]}&=\left(Y_1^{[r-1]}\left(\left[\varphi^{(1)},\ldots,\varphi^{(r)}\right]_1\right)\right)^{n+1}=\left(\varphi_1^{(1)}\cdots\varphi_1^{(r)}\right)^{n+1}.
\end{aligned}
\tag{19}
$$

Furthermore, $\forall\, k=1,2,\ldots,n-1$, the $r-th$ order partial Bell polynomials $A_{n,k}^{[r]}$ satisfy the recursion:

$$
\begin{aligned}
A_{n+1,k+1}^{[r]}\left(\left[\varphi^{(1)},\ldots,\varphi^{(r)}\right]_{n+1}\right)&=\sum_{h=0}^{n-k}\binom{n}{h}A_{n-h,k}^{[r]}\left(\left[\varphi^{(1)},\ldots,\varphi^{(r)}\right]_{n-h}\right)\\
&\quad\times Y_{h+1}^{[r-1]}\left(\left[\varphi^{(1)},\ldots,\varphi^{(r)}\right]_{h+1}\right).
\end{aligned}
\tag{20}
$$

Proof. - According to equations (14) and (4), using Theorem 3, we can write

$$Y_{n+1}^{[r]}\left(\left[f,\varphi^{(1)},\ldots,\varphi^{(r)}\right]_{n+1}\right)=\sum_{k=1}^{n+1}A_{n+1,k}^{[r]}\left(\left[\varphi^{(1)},\ldots,\varphi^{(r)}\right]_{n+1}\right)f_k$$

$$=Y_{n+1}\left(f_1,Y_1^{[r-1]}\left(\left[\varphi^{(1)},\ldots,\varphi^{(r)}\right]_1\right);\ldots;f_{n+1},Y_{n+1}^{[r-1]}\left(\left[\varphi^{(1)},\ldots,\varphi^{(r)}\right]_{n+1}\right)\right)$$

$$=\sum_{k=1}^{n+1}A_{n+1,k}\left(Y_1^{[r-1]}\left(\left[\varphi^{(1)},\ldots,\varphi^{(r)}\right]_1\right);\ldots;Y_{n+1}^{[r-1]}\left(\left[\varphi^{(1)},\ldots,\varphi^{(r)}\right]_{n+1}\right)\right)f_k\,,$$

so that we find the following relations between the classical polynomial coefficients and the r-th order ones:

$$\begin{aligned}&A_{n+1,k}^{[r]}\left(\left[\varphi^{(1)},\ldots,\varphi^{(r)}\right]_{n+1}\right)\\ &=A_{n+1,k}\left(Y_1^{[r-1]}\left(\left[\varphi^{(1)},\ldots,\varphi^{(r)}\right]_1\right);\ldots;Y_{n+1}^{[r-1]}\left(\left[\varphi^{(1)},\ldots,\varphi^{(r)}\right]_{n+1}\right)\right).\end{aligned}\tag{21}$$

Equations (19) can be obtained from relations (21), for $k=1$ and $k=n+1$, as a direct consequence of the definition of the ordinary coefficient $A_{n+1,k}$ given in (4). In order to prove equation (20), note that, taking into account the first relation in (19), we can write the equation (14) in the form

$$Y_{n+1}^{[r]}\left(\left[f,\varphi^{(1)},\ldots,\varphi^{(r)}\right]_{n+1}\right)=\sum_{k=0}^{n}A_{n+1,k+1}^{[r]}\left(\left[\varphi^{(1)},\ldots,\varphi^{(r)}\right]_{n+1}\right)f_{k+1}$$

$$=A_{n+1,1}^{[r]}\,f_1+\sum_{k=1}^{n}A_{n+1,k+1}^{[r]}\left(\left[\varphi^{(1)},\ldots,\varphi^{(r)}\right]_{n+1}\right)f_{k+1}$$

$$=Y_{n+1}^{[r-1]}\left(\left[\varphi^{(1)},\ldots,\varphi^{(r)}\right]_{n+1}\right)f_1+\sum_{k=1}^{n}A_{n+1,k+1}^{[r]}\left(\left[\varphi^{(1)},\ldots,\varphi^{(r)}\right]_{n+1}\right)f_{k+1}.$$

Furthermore, recalling equation $(16)_1$, the equation $(16)_2$ becomes

$$Y_{n+1}^{[r]}\left(\left[f,\varphi^{(1)},\dots,\varphi^{(r)}\right]_{n+1}\right)$$

$$=\sum_{h=0}^{n}\binom{n}{h}Y_{n-h}^{[r]}\left(\left[f_1,\varphi^{(1)},\dots,\varphi^{(r)}\right]_{n-h}\right)Y_{h+1}^{[r-1]}\left(\left[\varphi^{(1)},\dots,\varphi^{(r)}\right]_{h+1}\right)$$

$$=f_1Y_{n+1}^{[r-1]}\left(\left[\varphi^{(1)},\dots,\varphi^{(r)}\right]_{n+1}\right)$$

$$+\sum_{h=0}^{n-1}\binom{n}{h}Y_{n-h}^{[r]}\left(\left[f_1,\varphi^{(1)},\dots,\varphi^{(r)}\right]_{n-h}\right)Y_{h+1}^{[r-1]}\left(\left[\varphi^{(1)},\dots,\varphi^{(r)}\right]_{h+1}\right)$$

so that, neglecting the first term in both the above sums, we find:

$$\sum_{k=1}^{n}A_{n+1,k+1}^{[r]}\left(\left[\varphi^{(1)},\dots,\varphi^{(r)}\right]_{n+1}\right)f_{k+1}$$

$$=\sum_{h=0}^{n-1}\binom{n}{h}\left(\sum_{\ell=1}^{n-h}A_{n-h,\ell}^{[r]}\left(\left[\varphi^{(1)},\dots,\varphi^{(r)}\right]_{n-h}\right)f_{\ell+1}\right)Y_{h+1}^{[r-1]}\left(\left[\varphi^{(1)},\dots,\varphi^{(r)}\right]_{h+1}\right)$$

and inverting summations by the Dirichlet formula:

$$\sum_{k=1}^{n}A_{n+1,k+1}^{[r]}\left(\left[\varphi^{(1)},\dots,\varphi^{(r)}\right]_{n+1}\right)f_{k+1}$$

$$=\sum_{\ell=1}^{n}\left(\sum_{h=0}^{n-\ell}\binom{n}{h}A_{n-h,\ell}^{[r]}\left(\left[\varphi^{(1)},\dots,\varphi^{(r)}\right]_{n-h}\right)Y_{h+1}^{[r-1]}\left(\left[\varphi^{(1)},\dots,\varphi^{(r)}\right]_{h+1}\right)\right)\ f_{\ell+1}.$$

Therefore, changing ℓ into k in the last formula and equating the coefficients of f_{k+1}, the equation (20) follows.

4. TABLES OF COMPLETE HIGHER ORDER BELL POLYNOMIALS, FOR $r=2,3,4,5$

By using the recurrence relation (19)-(20) and by means of the computer algebra program Mathematica© we can construct the complete Bell polynomials of every order. We will limit ourselves to present here the very first of them. Putting, for shortness: $x=\varphi^{(1)},\ y=\varphi^{(2)},\ z=\varphi^{(3)},\ u=\varphi^{(4)},\ v=\varphi^{(5)}$, and denoting by indices the order of derivatives, we have found:

Second order Bell polynomials:

$$
\begin{aligned}
&B_1^{[2]}\left([x,y]_1\right) = x_1y_1;\\
&B_2^{[2]}\left([x,y]_2\right) = x_1^2y_1^2 + x_2y_1^2 + x_1y_2;\\
&B_3^{[2]}\left([x,y]_3\right) = x_1^3y_1^3 + 3x_1x_2y_1^3 + x_3y_1^3 + 3x_1^2y_1y_2 + 3x_2y_1y_2 + x_1y_3;\\
&B_4^{[2]}\left([x,y]_4\right) = x_1^4y_1^4 + 6x_1^2x_2y_1^4 + 3x_2^2y_1^4 + 4x_1x_3y_1^4 + x_4y_1^4 + 6x_1^3y_1^2y_2\\
&\quad +18x_1x_2y_1^2y_2 + 6x_3y_1^2y_2 + 3x_1^2y_2^2 + 3x_2y_2^2 + 4x_1^2y_1y_3 + 4x_2y_1y_3 + x_1y_4;\\
&B_5^{[2]}\left([x,y]_5\right) = x_1^5y_1^5 + 10x_1^3x_2y_1^5 + 15x_1x_2^2y_1^5 + 10x_1^2x_3y_1^5 + 10x_2x_3y_1^5 + 5x_1x_4y_1^5\\
&\quad +x_5y_1^5 + 10x_1^4y_1^3y_2 + 60x_1^2x_2y_1^3y_2 + 30x_2^2y_1^3y_2 + 40x_1x_3y_1^3y_2 + 10x_4y_1^3y_2\\
&\quad +15x_1^3y_1y_2^2 + 45x_1x_2y_1y_2^2 + 15x_3y_1y_2^2 + 10x_1^3y_1^2y_3 + 30x_1x_2y_1^2y_3 + 10x_3y_1^2y_3\\
&\quad +10x_1^2y_2y_3 + 10x_2y_2y_3 + 5x_1^2y_1y_4 + 5x_2y_1y_4 + x_1y_5\,.
\end{aligned}
$$

Third order Bell polynomials:

$$
\begin{aligned}
&B_1^{[3]}\left([x,y,z]_1\right) = z_1y_1x_1;\\
&B_2^{[3]}\left([x,y,z]_2\right) = z_2y_1x_1 + z_1^2y_2x_1 + z_1^2y_1^2x_1^2 + z_1^2y_1^2x_2;\\
&B_3^{[3]}\left([x,y,z]_3\right) = z_3y_1x_1 + 3z_1z_2y_2x_1 + z_1^3y_3x_1 + 3z_1z_2y_1^2x_1^2 + 3z_1^3y_1y_2x_1^2\\
&\quad +z_1^3y_1^3x_1^3 + 3z_1z_2y_1^2x_2 + 3z_1^3y_1y_2x_2 + 3z_1^3y_1^3x_1x_2 + z_1^3y_1^3x_3;\\
&B_4^{[3]}\left([x,y,z]_4\right) = z_4y_1x_1 + 3z_2^2y_2x_1 + 4z_1z_3y_2x_1 + 6z_1^2z_2y_3x_1 + z_1^4y_4x_1\\
&\quad +3z_2^2y_1^2x_1^2 + 4z_1z_3y_1^2x_1^2 + 18z_1^2z_2y_1y_2x_1^2 + 3z_1^4y_2^2x_1^2 + 4z_1^4y_1y_3x_1^2\\
&\quad +6z_1^2z_2y_1^3x_1^3 + 6z_1^4y_1^2y_2x_1^3 + z_1^4y_1^4x_1^4 + 3z_2^2y_1^2x_2 + 4z_1z_3y_1^2x_2 + 18z_1^2z_2y_1y_2x_2\\
&\quad +3z_1^4y_2^2x_2 + 4z_1^4y_1y_3x_2 + 18z_1^2z_2y_1^3x_1x_2 + 18z_1^4y_1^2y_2x_1x_2 + 6z_1^4y_1^4x_1^2x_2\\
&\quad +3z_1^4y_1^4x_2^2 + 6z_1^2z_2y_1^3x_3 + 6z_1^4y_1^2y_2x_3 + 4z_1^4y_1^4x_1x_3 + z_1^4y_1^4x_4\,.
\end{aligned}
$$

Fourth order Bell polynomials:

$$
\begin{aligned}
&B_1^{[4]}\left([x,y,z,u]_1\right) = u_1z_1y_1x_1;\\
&B_2^{[4]}\left([x,y,z,u]_2\right) = u_2z_1y_1x_1 + u_1^2z_2y_1x_1 + u_1^2z_1^2y_2x_1 + u_1^2z_1^2y_1^2x_1^2 + u_1^2z_1^2y_1^2x_2;\\
&B_3^{[4]}\left([x,y,z,u]_3\right) = u_3z_1y_1x_1 + 3u_1u_2z_2y_1x_1 + u_1^3z_3y_1x_1 + 3u_1u_2z_1^2y_2x_1\\
&\quad +3u_1^3z_1z_2y_2x_1 + u_1^3z_1^3y_3x_1 + 3u_1u_2z_1^2y_1^2x_1^2 + 3u_1^3z_1z_2y_1^2x_1^2 + 3u_1^3z_1^3y_1y_2x_1^2\\
&\quad +u_1^3z_1^3y_1^3x_1^3 + 3u_1u_2z_1^2y_1^2x_2 + 3u_1^3z_1z_2y_1^2x_2 + 3u_1^3z_1^3y_1y_2x_2 + 3u_1^3z_1^3y_1^3x_1x_2\\
&\quad +u_1^3z_1^3y_1^3x_3;\\
&B_4^{[4]}\left([x,y,z,u]_4\right) = u_4z_1y_1x_1 + 3u_2^2z_2y_1x_1 + 4u_1u_3z_2y_1x_1 + 6u_1^2u_2z_3y_1x_1\\
&\quad +u_1^4z_4y_1x_1 + 3u_2^2z_1^2y_2x_1 + 4u_1u_3z_1^2y_2x_1 + 18u_1^2u_2z_1z_2y_2x_1 + 3u_1^4z_2^2y_2x_1\\
&\quad +4u_1^4z_1z_3y_2x_1 + 6u_1^2u_2z_1^3y_3x_1 + 6u_1^4z_1^2z_2y_3x_1 + +u_1^4z_1^4y_4x_1 + 3u_2^2z_1^2y_1^2x_1^2\\
&\quad +4u_1u_3z_1^2y_1^2x_1^2 + 18u_1^2u_2z_1z_2y_1^2x_1^2 + 3u_1^4z_2^2y_1^2x_1^2 + 4u_1^4z_1z_3y_1^2x_1^2\\
&\quad +18u_1^2u_2z_1^3y_1y_2x_1^2 + 18u_1^4z_1^2z_2y_1y_2x_1^2 + 3u_1^4z_1^4y_2^2x_1^2 + 4u_1^4z_1^4y_1y_3x_1^2\\
&\quad +6u_1^2u_2z_1^3y_1^3x_1^3 + 6u_1^4z_1^2z_2y_1^3x_1^3 + 6u_1^4z_1^4y_1^2y_2x_1^3 + u_1^4z_1^4y_1^4x_1^4 + 3u_2^2z_1^2y_1^2x_2\\
&\quad +4u_1u_3z_1^2y_1^2x_2 + 18u_1^2u_2z_1z_2y_1^2x_2 + 3u_1^4z_2^2y_1^2x_2 + 4u_1^4z_1z_3y_1^2x_2\\
&\quad +18u_1^2u_2z_1^3y_1y_2x_2 + 18u_1^4z_1^2z_2y_1y_2x_2 + 3u_1^4z_1^4y_2^2x_2 + 4u_1^4z_1^4y_1y_3x_2\\
&\quad +18u_1^2u_2z_1^3y_1^3x_1x_2 + 18u_1^4z_1^2z_2y_1^3x_1x_2 + 18u_1^4z_1^4y_1^2y_2x_1x_2 + 6u_1^4z_1^4y_1^4x_1^2x_2\\
&\quad +3u_1^4z_1^4y_1^4x_2^2 + 6u_1^2u_2z_1^3y_1^3x_3 + 6u_1^4z_1^2z_2y_1^3x_3 + 6u_1^4z_1^4y_1^2y_2x_3 + 4u_1^4z_1^4y_1^4x_1x_3\\
&\quad +u_1^4z_1^4y_1^4x_4\,.
\end{aligned}
$$

Fifth order Bell polynomials:

$$\begin{aligned}
&B_1^{[5]}\left([x,y,z,u,v]_1\right) = v_1u_1z_1y_1x_1;\\
&B_2^{[5]}\left([x,y,z,u,v]_2\right) = v_2u_1z_1y_1x_1 + v_1^2u_2z_1y_1x_1 + v_1^2u_1^2z_2y_1x_1 + v_1^2u_1^2z_1^2y_2x_1\\
&\quad +v_1^2u_1^2z_1^2y_1^2x_1^2 + v_1^2u_1^2z_1^2y_1^2x_2;\\
&B_3^{[5]}\left([x,y,z,u,v]_3\right) = v_3u_1z_1y_1x_1 + 3v_1v_2u_2z_1y_1x_1 + v_1^3u_3z_1y_1x_1\\
&\quad +3v_1v_2u_1^2z_2y_1x_1 + 3v_1^3u_1u_2z_2y_1x_1 + v_1^3u_1^3z_3y_1x_1 + 3v_1v_2u_1^2z_1^2y_2x_1\\
&\quad +3v_1^3u_1u_2z_1^2y_2x_1 + 3v_1^3u_1^3z_1z_2y_2x_1 + v_1^3u_1^3z_1^3y_3x_1 + 3v_1v_2u_1^2z_1^2y_1^2x_1^2\\
&\quad +3v_1^3u_1u_2z_1^2y_1^2x_1^2 + 3v_1^3u_1^3z_1z_2y_1^2x_1^2 + 3v_1^3u_1^3z_1^3y_1y_2x_1^2 + v_1^3u_1^3z_1^3y_1^3x_1^3\\
&\quad +3v_1v_2u_1^2z_1^2y_1^2x_2 + 3v_1^3u_1u_2z_1^2y_1^2x_2 + 3v_1^3u_1^3z_1z_2y_1^2x_2 + 3v_1^3u_1^3z_1^3y_1y_2x_2\\
&\quad +3v_1^3u_1^3z_1^3y_1^3x_1x_2 + v_1^3u_1^3z_1^3y_1^3x_3;\\
&B_4^{[5]}\left([x,y,z,u,v]_4\right) = v_4u_1z_1y_1x_1 + 3v_2^2u_2z_1y_1x_1 + 4v_1v_3u_2z_1y_1x_1\\
&\quad +6v_1^2v_2u_3z_1y_1x_1 + v_1^4u_4z_1y_1x_1 + 3v_2^2u_1^2z_2y_1x_1 + 4v_1v_3u_1^2z_2y_1x_1\\
&\quad +18v_1^2v_2u_1u_2z_2y_1x_1 + 3v_1^4u_2^2z_2y_1x_1 + 4v_1^4u_1u_3z_2y_1x_1 + 6v_1^2v_2u_1^3z_3y_1x_1\\
&\quad +6v_1^4u_1^2u_2z_3y_1x_1 + v_1^4u_1^4z_4y_1x_1 + 3v_2^2u_1^2z_1^2y_2x_1 + 4v_1v_3u_1^2z_1^2y_2x_1\\
&\quad +18v_1^2v_2u_1u_2z_1^2y_2x_1 + 3v_1^4u_2^2z_1^2y_2x_1 + 4v_1^4u_1u_3z_1^2y_2x_1 + 18v_1^2v_2u_1^3z_1z_2y_2x_1\\
&\quad +18v_1^4u_1^2u_2z_1z_2y_2x_1 + 3v_1^4u_1^4z_2^2y_2x_1 + 4v_1^4u_1^4z_1z_3y_2x_1 + 6v_1^2v_2u_1^3z_1^3y_3x_1\\
&\quad +6v_1^4u_1^2u_2z_1^3y_3x_1 + 6v_1^4u_1^4z_1^2z_2y_3x_1 + v_1^4u_1^4z_1^4y_4x_1 + 3v_2^2u_1^2z_1^2y_1^2x_1^2\\
&\quad +4v_1v_3u_1^2z_1^2y_1^2x_1^2 + 18v_1^2v_2u_1u_2z_1^2y_1^2x_1^2 + 3v_1^4u_2^2z_1^2y_1^2x_1^2 + 4v_1^4u_1u_3z_1^2y_1^2x_1^2\\
&\quad +18v_1^2v_2u_1^3z_1z_2y_1^2x_1^2 + 18v_1^4u_1^2u_2z_1z_2y_1^2x_1^2 + 3v_1^4u_1^4z_2^2y_1^2x_1^2 + 4v_1^4u_1^4z_1z_3y_1^2x_1^2\\
&\quad +18v_1^2v_2u_1^3z_1^3y_1y_2x_1^2 + 18v_1^4u_1^2u_2z_1^3y_1y_2x_1^2 + 18v_1^4u_1^4z_1^2z_2y_1y_2x_1^2\\
&\quad +3v_1^4u_1^4z_1^4y_2^2x_1^2 + 4v_1^4u_1^4z_1^4y_1y_3x_1^2 + 6v_1^2v_2u_1^3z_1^3y_1^3x_1^3 + 6v_1^4u_1^2u_2z_1^3y_1^3x_1^3\\
&\quad +6v_1^4u_1^4z_1^2z_2y_1^3x_1^3 + 6v_1^4u_1^4z_1^4y_1^2y_2x_1^3 + v_1^4u_1^4z_1^4y_1^4x_1^4 + 3v_2^2u_1^2z_1^2y_1^2x_2\\
&\quad +4v_1v_3u_1^2z_1^2y_1^2x_2 + 18v_1^2v_2u_1u_2z_1^2y_1^2x_2 + 3v_1^4u_2^2z_1^2y_1^2x_2 + 4v_1^4u_1u_3z_1^2y_1^2x_2\\
&\quad +18v_1^2v_2u_1^3z_1z_2y_1^2x_2 + 18v_1^4u_1^2u_2z_1z_2y_1^2x_2 + 3v_1^4u_1^4z_2^2y_1^2x_2 + 4v_1^4u_1^4z_1z_3y_1^2x_2\\
&\quad +18v_1^2v_2u_1^3z_1^3y_1y_2x_2 + 18v_1^4u_1^2u_2z_1^3y_1y_2x_2 + 18v_1^4u_1^4z_1^2z_2y_1y_2x_2 + 3v_1^4u_1^4z_1^4y_2^2x_2\\
&\quad +4v_1^4u_1^4z_1^4y_1y_3x_2 + 18v_1^2v_2u_1^3z_1^3y_1^3x_1x_2 + 18v_1^4u_1^2u_2z_1^3y_1^3x_1x_2\\
&\quad +18v_1^4u_1^4z_1^2z_2y_1^3x_1x_2 + 18v_1^4u_1^4z_1^4y_1^2y_2x_1x_2 + 6v_1^4u_1^4z_1^4y_1^4x_1^2x_2 + 3v_1^4u_1^4z_1^4y_1^4x_2^2\\
&\quad +6v_1^2v_2u_1^3z_1^3y_1^3x_3 + 6v_1^4u_1^2u_2z_1^3y_1^3x_3 + 6v_1^4u_1^4z_1^2z_2y_1^3x_3 + 6v_1^4u_1^4z_1^4y_1^2y_2x_3\\
&\quad +4v_1^4u_1^4z_1^4y_1^4x_1x_3 + v_1^4u_1^4z_1^4y_1^4x_4.
\end{aligned}$$

5. HIGHER ORDER BELL NUMBERS, FOR $r = 2, 3, 4, 5$

It is worth to note that the sequences of higher order Bell numbers which will be presented here appear in the Encyclopedia of Integer Sequences [**25**] under the # A144150, arising from a problem of Combinatorial Analysis and even as the McLaurin coefficients of the functions [**4**], [**12**]

$$\begin{aligned}
&\exp(\exp(\exp(x)-1)-1)\,,\\
&\exp(\exp(\exp(\exp(x)-1)-1)-1)\,,\\
&\exp(\exp(\exp(\exp(\exp(x)-1)-1)-1)-1)\,,\\
&\exp(\exp(\exp(\exp(\exp(\exp(x)-1)-1)-1)-1)-1)\,,
\end{aligned}$$

for the cases $r = 2$, $r = 3$, $r = 4$, $r = 5$, respectively, and so on for the subsequent values of r. Whereas in our approach they assume a more general meaning, as they are independent of the functions $f, \varphi^{(1)}, \dots, \varphi^{(r)}$.

According to the above reference we have found, using the recurrence relation (19)-(20) and by means of the computer algebra program Mathematica©, the following sequences for the higher order Bell numbers $b_n^{[2]}$, $b_n^{[3]}$, $b_n^{[4]}$, $b_n^{[5]}$, $(n = 1, 2, \ldots, 21)$:

n	$b_n^{[2]}$	$b_n^{[3]}$
1	1	1
2	3	4
3	12	22
4	60	154
5	358	1304
6	2471	12915
7	19302	146115
8	167894	1855570
9	1606137	26097835
10	16733779	402215465
11	188378402	6734414075
12	2276423485	121629173423
13	29367807524	2355470737637
14	402577243425	48664218965021
15	5840190914957	1067895971109199
16	89345001017415	24795678053493443
17	1436904211547895	607144847919796830
18	24227076487779802	15630954703539323090
19	427187837301557598	421990078975569031642
20	7859930038606521508	11918095123121138408128
21	150601795280158255827	351369494911150177020241

n	$b_n^{[4]}$	$b_n^{[5]}$
1	1	1
2	5	6
3	35	51
4	315	561
5	3455	7556
6	44590	120196
7	660665	2201856
8	11035095	45592666
9	204904830	1051951026
10	4183174520	26740775306
11	93055783320	742069051906
12	2238954627848	22310563733864
13	57903797748386	722108667742546
14	1601122732128779	25024187820786357
15	47120734323344439	924161461265888370
16	1470076408565099152	36223781285638309482
17	48449426629560437576	1501552062016443881514
18	1681560512531504058350	65615806216647567704054
19	61293054886119796799892	3014172185130478508636024
20	2340495383889150212466948	145182909448358007294714590
21	93417856359082371487510529	7315683185452671138995784017

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