

FIXED POINT THEOREMS IN FRÉCHET ALGEBRAS AND FRÉCHET SPACES AND APPLICATIONS TO NONLINEAR INTEGRAL EQUATIONS

Szymon Dudek

In this paper, we present two new fixed point theorems in Fréchet algebras and Fréchet spaces. Our fixed point results are expressed with the help of family of measures of noncompactness and generalizes Darbo theorem. As an application, we establish some existence results for various types of nonlinear integral equations.

1. INTRODUCTION

In the last years there appeared many papers devoted to the nonlinear integral equations. A lot of these equations were considered in Banach algebra (cf. [1, 5, 6, 9, 10, 14, 15]) but only few were investigated in Fréchet algebra [8]. However, it seems that convenient environment for integral equations on unbounded interval \mathbb{R}_+ are various Fréchet function spaces, which in the case of some types of the product integral equations, naturally lead to Fréchet algebras. Throughout this paper we try to fill this gap in the theory of the nonlinear integral equations in the Fréchet algebras.

In first section we establish a fixed point theorem for the Fréchet space which generalizes the classical Darbo fixed point theorem. In further parts of this paper we provide an example of applications of this theorem to some nonlinear integral equation of Volterra type, which does not satisfy the assumptions of classical Darbo fixed point theorem, but it satisfies the assumptions of our new theorem. In the sequel we prove another fixed point theorem for operators in Fréchet algebras and

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we give the examples of its applications. One of the main assumptions of this theorem is so-called (m) condition. In the last section we investigate this condition in the Fréchet algebra $L^1_{loc}(\mathbb{R}_+)$ and in the Banach algebra $L^1(\mathbb{R}_+)$.

2. NOTATION AND AUXILIARY FACTS

In this section we collect some definitions and results which will be needed later.

Assume that F is a real Fréchet space with the sequence of seminorms $\|\cdot\|_n$, $n \in \mathbb{N}$.

A set $X \subset F$ is said to be *bounded* if the set $\{\|x\|_n : x \in X\}$ is bounded for all $n \in \mathbb{N}$ and for such set X we will denote

$$\|X\|_n := \sup\{\|x\|_n : x \in X\}.$$

If X is a subset of the Fréchet space F then the symbols \overline{X} and $\text{Conv}X$ stand for the closure and the convex closure of X , respectively. Further, let \mathfrak{M}_F denote the family of all nonempty and bounded subsets of F and \mathfrak{N}_F the family of all relatively compact subsets of F . Obviously $\mathfrak{N}_F \subset \mathfrak{M}_F$.

We will proceed our further considerations also in Banach space E with the norm $\|\cdot\|$. Moreover, we will use analogous notions for this space.

Throughout this paper, we accept the following definition of the notion of a measure of noncompactness in Banach space E (see [3]).

Definition 2.1. A function $\mu : \mathfrak{M}_E \rightarrow [0, \infty]$ is said to be a *measure of noncompactness* in the Banach space E if it satisfies the following conditions:

- 1° the family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$,
- 2° $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$,
- 3° $\mu(\text{Conv}X) = \mu(X)$,
- 4° if (X_n) is a sequence of closed sets from \mathfrak{M}_E such that $X_{n+1} \subset X_n$ ($n = 1, 2, \dots$) and if $\lim_{n \rightarrow \infty} \mu(X_n) = 0$, then the intersection $X_\infty = \bigcap_{n=1}^\infty X_n$ is nonempty.

Let α denotes the Kuratowski measure of noncompactness in E . The properties of α may be found in [3].

For the Fréchet space we accept the following definition of the family of measures of noncompactness.

Definition 2.2. A family of functions $\{\mu_n\}_{n \in \mathbb{N}}$, where $\mu_n : \mathfrak{M}_F \rightarrow [0, \infty)$, is said to be a *family of measures of noncompactness* in the real Fréchet space F if it satisfies the following conditions:

- 1° The family $\ker\{\mu_n\} := \{X \in \mathfrak{M}_F : \mu_n(X) = 0 \text{ for } n \in \mathbb{N}\}$ is nonempty and $\ker\{\mu_n\} \subset \mathfrak{N}_F$.
- 2° $\mu_n(X) \leq \mu_n(Y)$ for $X \subset Y$, $n \in \mathbb{N}$.
- 3° $\mu_n(\text{Conv}X) = \mu_n(X)$ for $n \in \mathbb{N}$.
- 4° If (X_i) is a sequence of closed sets from \mathfrak{M}_F such that $X_{i+1} \subset X_i$ ($i = 1, 2, \dots$) and if $\lim_{i \rightarrow \infty} \mu_n(X_i) = 0$ for each $n \in \mathbb{N}$, then the intersection set $X_\infty := \bigcap_{i=1}^{\infty} X_i$ is nonempty.

We call the family of measures of noncompactness $\{\mu_n\}_{n \in \mathbb{N}}$ to be *homogeneous* if

$$5^\circ \mu_n(\lambda X) = |\lambda| \mu_n(X)$$

for $\lambda \in \mathbb{R}$, $n \in \mathbb{N}$. If the family $\{\mu_n\}_{n \in \mathbb{N}}$ satisfied the condition

$$6^\circ \mu_n(X + Y) \leq \mu_n(X) + \mu_n(Y)$$

for $n \in \mathbb{N}$, it is called *subadditive*. It is *sublinear* if both conditions (5°, 6°) hold. We say that the family of measures $\{\mu_n\}_{n \in \mathbb{N}}$ has *the maximum property* if

$$7^\circ \mu_n(X \cup Y) = \max\{\mu_n(X), \mu_n(Y)\}$$

for $n \in \mathbb{N}$.

Definition 2.3. The family of measures of noncompactness $\{\mu_n\}_{n \in \mathbb{N}}$ is said to be *regular* if it is full ($\ker\{\mu_n\} = \mathfrak{N}_F$), sublinear and has maximum property.

REMARK 2.4. In the Fréchet space F we can also consider families of measures $\{\mu_T\}_{T \geq 0}$ indexed by nonnegative numbers instead of families of measures $\{\mu_n\}_{n \in \mathbb{N}}$ indexed by natural numbers.

In the sequel we will formulate two fixed point theorems. Let us recall some useful term.

Definition 2.5. Let Ω be a nonempty subset of a Fréchet space F , and let $\mathcal{A} : \Omega \rightarrow F$ be a continuous operator which transforms bounded subsets of Ω onto bounded ones. One says that \mathcal{A} satisfies *the Darbo condition* with constants $(k_n)_{n \in \mathbb{N}}$ with respect to a family of measures of noncompactness $\{\mu_n\}_{n \in \mathbb{N}}$ if

$$\mu_n(\mathcal{A}(X)) \leq k_n \mu_n(X)$$

for each $n \in \mathbb{N}$ and $X \in \mathfrak{M}_F$ such that $X \subset \Omega$. If $k_n < 1$, $n \in \mathbb{N}$, then \mathcal{A} is called a *contraction* with respect to $\{\mu_n\}_{n \in \mathbb{N}}$.

Assuming additionally that Ω is a nonempty, bounded, closed and convex subset of F and $\mathcal{A} : \Omega \rightarrow \Omega$ is a contraction with respect to $\{\mu_n\}_{n \in \mathbb{N}}$, we obtain

a generalization of the classical Darbo fixed point theorem for the Fréchet spaces (see [11]).

In the sequel we will usually assume that the space F has the structure of Fréchet algebra. In such a case we write xy in order to denote the product of elements $x, y \in F$. Similarly, we will write XY to denote the product of subsets X, Y of F , that is, $XY = \{xy : x \in X, y \in Y\}$.

Definition 2.6. We say that the family of measures of noncompactness $\{\mu_n\}_{n \in \mathbb{N}}$ defined on the Fréchet algebra F satisfies the *condition (m)* if for arbitrary sets $X, Y \in \mathfrak{M}_F$ the following inequality is satisfied:

$$\mu_n(XY) \leq \|X\|_n \mu_n(Y) + \|Y\|_n \mu_n(X) \quad \text{for } n \in \mathbb{N}.$$

Assume that $(E, \|\cdot\|)$ is a Banach algebra. Based on it, we form the Fréchet algebra $F := C(\mathbb{R}_+, E)$ consisting of all functions defined and continuous on \mathbb{R}_+ and with values in the Banach space E . The space $C(\mathbb{R}_+, E)$ is equipped with the family of seminorms $(\|\cdot\|_n)_{n \in \mathbb{N}}$ as follows

$$\|x\|_n := \sup\{\|x(t)\| : t \in [0, n]\}, \quad x \in C(\mathbb{R}_+, E), n \in \mathbb{N}.$$

Let us recall two facts:

- (A) A sequence (x_n) is convergent to x in $C(\mathbb{R}_+, E)$ if and only if (x_n) is uniformly convergent to x on compact subsets of \mathbb{R}_+ .
- (B) A family $X \subset C(\mathbb{R}_+, E)$ is relatively compact if and only if for each $T > 0$, the restrictions to $[0, T]$ of all functions from X form an equicontinuous set and $X(t)$ is relatively compact in E for each $t \in \mathbb{R}_+$.

In the Fréchet space $C(\mathbb{R}_+, E)$ we define a product as follows

$$(xy)(t) := x(t)y(t), \quad x, y \in C(\mathbb{R}_+, E), t \in \mathbb{R}_+.$$

Now we introduce the family $\{\mu_n\}_{n \in \mathbb{N}}$ in the Fréchet algebra $C(\mathbb{R}_+, E)$ defined by the formula

$$(1) \quad \mu_n(X) := \omega_0^n(X) + \bar{\alpha}_n(X), \quad X \in \mathfrak{M}_{C(\mathbb{R}_+, E)}, n \in \mathbb{N},$$

where

$$\bar{\alpha}_t(X) := \sup\{\alpha(X(\tau)) : \tau \in [0, t]\},$$

and

$$\omega_0^n(X) := \lim_{\varepsilon \rightarrow 0^+} \omega_0^n(X, \varepsilon),$$

$$\omega_0^n(X, \varepsilon) := \sup\{\|x(t) - x(s)\| : x \in X, t, s \in [0, n], |t - s| \leq \varepsilon\}.$$

The family of maps $\{\mu_n\}_{n \in \mathbb{N}}$ defined above is a family of measures of noncompactness in the Fréchet space $F = C(\mathbb{R}_+, E)$ (see [12]).

REMARK 2.7. The kernel $\ker\{\mu_n\}$ of the family $\{\mu_n\}_{n \in \mathbb{N}}$ consists of all sets $X \in \mathfrak{M}_F$ such that functions belonging to X are equicontinuous on compact subsets of \mathbb{R}_+ and $X(t)$ is relatively compact in E for each $t \in \mathbb{R}_+$.

In our further considerations we will need the following lemmas.

Lemma 2.8. *The following conditions are satisfied:*

1) *The Kuratowski measure of noncompactness α satisfies condition (m) in Banach algebra E , that is,*

$$(2) \quad \alpha(XY) \leq \|X\|\alpha(Y) + \|Y\|\alpha(X), \quad X, Y \in \mathfrak{M}_E.$$

2) *The family of measures of noncompactness $\{\mu_n\}_{n \in \mathbb{N}}$ in the Fréchet space $C(\mathbb{R}_+, E)$, defined by the formula (1), satisfies condition (m) in following form*

$$(3) \quad \mu_n(XY) \leq \|X\|_n \mu_n(Y) + \|Y\|_n \mu_n(X), \quad n \in \mathbb{N}, \quad X, Y \in \mathfrak{M}_{C(\mathbb{R}_+, E)}.$$

Proof.

1) Let us fix $X, Y \in \mathfrak{M}_E$. For an arbitrary $\varepsilon > 0$ there exist $A_1, \dots, A_n \subset E$ and $B_1, \dots, B_m \subset E$ such that

$$\text{diam}A_i < \alpha(X) + \varepsilon, \quad \text{diam}B_j < \alpha(Y) + \varepsilon, \quad i = 1, \dots, n, \quad j = 1, \dots, m$$

and $X \subset \cup_{i=1}^n A_i$, $Y \subset \cup_{j=1}^m B_j$. Let us put

$$C_{ij} := A_i B_j, \quad i = 1, \dots, n, \quad j = 1, \dots, m.$$

Obviously

$$XY \subset \cup_{i=1}^n \cup_{j=1}^m C_{ij}.$$

Let us fix C_{ij} and take arbitrary $x_1 y_1, x_2 y_2 \in C_{ij}$ where $x_1, x_2 \in A_i$, $y_1, y_2 \in B_j$. Then

$$\begin{aligned} \|x_1 y_1 - x_2 y_2\| &\leq \|x_1\| \cdot \|y_1 - y_2\| + \|x_1 - x_2\| \cdot \|y_2\| \\ &\leq \|X\|(\text{diam}(B_j) + \text{diam}(A_i))\|Y\| \\ &\leq \|X\|(\alpha(Y) + \varepsilon) + (\alpha(X) + \varepsilon)\|Y\| \end{aligned}$$

and therefore

$$\alpha(XY) \leq \|X\|(\alpha(Y) + \varepsilon) + (\alpha(X) + \varepsilon)\|Y\|.$$

In view of arbitrariness of $\varepsilon > 0$ we obtain (2).

2) Let us fix $X, Y \in \mathfrak{M}_{C(\mathbb{R}_+, E)}$ and $n \in \mathbb{N}$. Similarly as in [4] we derive

$$(4) \quad \omega_0^n(XY) \leq \|X\|_n \omega_0^n(Y) + \|Y\|_n \omega_0^n(X).$$

Moreover, according to (2) we have

$$\begin{aligned} \bar{\alpha}_n(XY) &= \sup\{\alpha((XY)(\tau)) : \tau \in [0, n]\} \\ &\leq \sup\{\|X(\tau)\|\alpha(Y(\tau)) + \|Y(\tau)\|\alpha(X(\tau)) : \tau \in [0, n]\} \\ &\leq \|X\|_n \bar{\alpha}_n(Y) + \|Y\|_n \bar{\alpha}_n(X). \end{aligned}$$

Adding above inequality to (4) and keeping in mind (1) we obtain (3). \square

Lemma 2.9. [7] *If the set $X \subset C(\mathbb{R}_+, E)$ is equicontinuous on compact subsets of \mathbb{R}_+ , then $\text{Conv}X$ is also equicontinuous on these intervals.*

Lemma 2.10. [7] *Assume $X \subset C(\mathbb{R}_+, E)$ is equicontinuous on compact intervals of \mathbb{R}_+ and $X(t)$ is bounded for all $t \geq 0$. Then*

$$\alpha\left(\int_0^t X(s)ds\right) \leq \int_0^t \alpha(X(s))ds, \quad t \geq 0.$$

3. FIXED POINT THEOREMS

In this section we will provide two fixed point theorems. The first one is inspired by paper [10]. An example considered in the section 4.1 shows that sometimes an operator generated by an integral equation does not satisfy the assumptions of classical Darbo fixed point theorem. Namely, some operators have this property for "n-th composition". Motivated by this fact, for the purposes of Theorem 4.16 we provide following one.

Theorem 3.11. *Assume that Ω is a nonempty, bounded, convex and closed subset of the Fréchet space F and the mapping $\mathcal{A} : \Omega \rightarrow \Omega$ is continuous. For an arbitrary set $X \subset \Omega$ let us put*

$$\tilde{\mathcal{A}}^1(X) := \mathcal{A}(\text{Conv}X), \quad \tilde{\mathcal{A}}^n(X) := \mathcal{A}(\text{Conv}\tilde{\mathcal{A}}^{n-1}(X)), \quad n = 2, 3, \dots .$$

If there exist a sequence $(k_n) \subset [0, 1)$ and sequence of natural numbers (m_n) such that

$$(5) \quad \mu_n(\tilde{\mathcal{A}}^{m_n}(X)) \leq k_n \mu_n(X) \quad \text{for } X \subset \Omega, \quad n \in \mathbb{N},$$

where $\{\mu_n\}_{n \in \mathbb{N}}$ is the family of measures of noncompactness in Fréchet space F , then \mathcal{A} has at least one fixed point in the set Ω .

Proof. Notice that for an arbitrary nonempty set $X \subset \Omega$ we have

$$(6) \quad \tilde{\mathcal{A}}^{a+b}(X) = \tilde{\mathcal{A}}^a(\tilde{\mathcal{A}}^b(X)).$$

Now we denote

$$Y_0 := \Omega, \quad Y_i := \text{Conv}\mathcal{A}(Y_{i-1}), \quad i = 1, 2, 3, \dots .$$

Obviously the sets Y_i are nonempty, convex, closed and

$$(7) \quad Y_i = \text{Conv}\tilde{\mathcal{A}}^i(\Omega), \quad i = 1, 2, \dots .$$

Moreover, we have implication

$$(8) \quad i, j \in \mathbb{N}_0, \quad \text{and } i \leq j \implies Y_j \subset Y_i.$$

Indeed, keeping in mind (6) we get

$$Y_j = \text{Conv}\tilde{\mathcal{A}}^j(\Omega) = \text{Conv}\tilde{\mathcal{A}}^i(\tilde{\mathcal{A}}^{j-i}(\Omega)) \subset \text{Conv}\tilde{\mathcal{A}}^i(\Omega) = Y_i.$$

Let us take additional symbols

$$I_0 := 0, \quad I_i := i \cdot m_1 \cdot m_2 \cdot \dots \cdot m_i, \quad i = 1, 2, \dots,$$

$$Z_i := Y_{I_i}, \quad i = 0, 1, 2, \dots$$

The condition (8) provides the following inclusion

$$Z_0 \supset Z_1 \supset Z_2 \supset \dots$$

Let us fix $n \in \mathbb{N}$ and consider $i \geq n$. In virtue of (7), (6) and assumptions we have

$$\begin{aligned} \mu_n(Z_i) &= \mu_n(\text{Conv}\tilde{\mathcal{A}}^{I_i}(\Omega)) = \mu_n(\tilde{\mathcal{A}}^{I_i}(\Omega)) = \mu_n(\tilde{\mathcal{A}}^{m_n}(\tilde{\mathcal{A}}^{I_i-m_n}(\Omega))) \\ &\leq k_n \mu_n(\tilde{\mathcal{A}}^{I_i-m_n}(\Omega)) \leq k_n \mu_n(\tilde{\mathcal{A}}^{m_n}(\tilde{\mathcal{A}}^{I_i-2m_n}(\Omega))) \\ &\leq k_n^2 \mu_n(\tilde{\mathcal{A}}^{I_i-2m_n}(\Omega)) \leq \dots \leq k_n^{\frac{I_i}{m_n}} \mu_n(\Omega). \end{aligned}$$

Hence, putting $i \rightarrow \infty$ we obtain $\lim_{i \rightarrow \infty} \mu_n(Z_i) = 0$. In view of condition 4° from Definition 2.2 we deduce that the set Z

$$Z := \bigcap_{i=0}^{\infty} Z_i$$

is nonempty. In virtue of the above conclusions we infer that Z is also a convex and closed subset of F . Moreover, since $\mu_n(Z) \leq \mu_n(Z_i) \rightarrow 0$ for $i \rightarrow \infty$, we obtain $\mu_n(Z) = 0$ for $n = 1, 2, \dots$ and from term 1° of Definition 2.2 we deduce that Z is compact set.

Now we will prove the following inclusion

$$(9) \quad \mathcal{A}(Z_i) \subset Z_i, \quad i = 0, 1, 2, \dots$$

Let us fix $i \in \mathbb{N}$. In view of (7) and (6) we infer that

$$\begin{aligned} \mathcal{A}(Z_i) &= \mathcal{A}(Y_{I_i}) = \mathcal{A}(\text{Conv}\tilde{\mathcal{A}}^{I_i}(\Omega)) = \mathcal{A}(\text{Conv}\tilde{\mathcal{A}}^{I_i-1}(\tilde{\mathcal{A}}^1(\Omega))) \\ &\subset \mathcal{A}(\text{Conv}\tilde{\mathcal{A}}^{I_i-1}(\Omega)) = \tilde{\mathcal{A}}^{I_i}(\Omega) \subset \text{Conv}\tilde{\mathcal{A}}^{I_i}(\Omega) = Y_{I_i} = Z_i. \end{aligned}$$

Moreover, (9) yields that

$$\mathcal{A}(Z) = \bigcap_{i=0}^{\infty} \mathcal{A}(Z_i) \subset \bigcap_{i=0}^{\infty} Z_i = Z.$$

Applying Tikhonov theorem for the mapping $\mathcal{A} : Z \rightarrow Z$ we infer that \mathcal{A} has at least one fixed point x . \square

REMARK 3.12. Notice that if $m_n \equiv 1$ for $n = 1, 2, \dots$ then we get Corollary 3.3 obtained in [11] saying that if \mathcal{A} is a contraction with respect to $\{\mu_n\}_{n \in \mathbb{N}}$, then \mathcal{A} has at least one fixed point x .

For our purposes we will need the following fixed point theorem.

Theorem 3.13. *Assume that Ω is a nonempty, bounded, closed and convex subset of the Fréchet algebra F and the operators $\mathcal{G}, \mathcal{F}, \mathcal{V}$ transform continuously the set Ω into algebra F in such way that \mathcal{G} is compact and $\mathcal{F}(\Omega), \mathcal{V}(\Omega)$ are bounded and the operator $\mathcal{A} = \mathcal{G} + \mathcal{F} \cdot \mathcal{V}$ transforms Ω into Ω . Let the operators \mathcal{F} and \mathcal{V} satisfy on the set Ω the Darbo condition with the constants (k_n) and (l_n) with respect to the regular family of measures of noncompactness $\{\mu_n\}_{n \in \mathbb{N}}$ which satisfy condition (m). Then the operator \mathcal{A} fulfills Darbo condition with the constants $(\|\mathcal{F}(\Omega)\|_n l_n + \|\mathcal{V}(\Omega)\|_n k_n)_{n \in \mathbb{N}}$ with respect to the family of measures of noncompactness $\{\mu_n\}_{n \in \mathbb{N}}$. Particularly, if*

$$(10) \quad \|\mathcal{F}(\Omega)\|_n l_n + \|\mathcal{V}(\Omega)\|_n k_n < 1, \quad n \in \mathbb{N},$$

then \mathcal{A} has at least one fixed point in the set Ω .

Proof. Let us fix $\phi \neq X \subset \Omega$ and $n \in \mathbb{N}$. Since the family $\{\mu_n\}_{n \in \mathbb{N}}$ is regular and the operator \mathcal{G} is compact, we get

$$\mu_n(\mathcal{A}(X)) \leq \mu_n(\mathcal{G}(X)) + \mu_n(\mathcal{F}(X) \cdot \mathcal{V}(X)) = \mu_n(\mathcal{F}(X) \cdot \mathcal{V}(X)).$$

Moreover, the family $\{\mu_n\}_{n \in \mathbb{N}}$ satisfies the condition (m). This fact yields that

$$\begin{aligned} \mu_n(\mathcal{A}(X)) &\leq \mu_n(\mathcal{F}(X) \cdot \mathcal{V}(X)) \leq \|\mathcal{F}(X)\|_n \mu_n(\mathcal{V}(X)) + \|\mathcal{V}(X)\|_n \mu_n(\mathcal{F}(X)) \\ &\leq \|\mathcal{F}(\Omega)\|_n \mu_n(\mathcal{V}(X)) + \|\mathcal{V}(\Omega)\|_n \mu_n(\mathcal{F}(X)) \\ &\leq (\|\mathcal{F}(\Omega)\|_n l_n + \|\mathcal{V}(\Omega)\|_n k_n) \mu_n(X). \end{aligned}$$

In view of (10) and the generalization of the classical Darbo fixed point theorem in the Fréchet space ([11]) we infer that \mathcal{A} has at least one fixed point in the set Ω . \square

REMARK 3.14. Notice that every Banach algebra $(E, \|\cdot\|)$ can be regarded as a Fréchet algebra $(E, \|\cdot\|_n)$, where $\|\cdot\|_n = \|\cdot\|$, $n \in \mathbb{N}$. This yields that in the special case $k_n = k$ and $l_n = l, n \in \mathbb{N}$ Theorem 3.13 becomes Theorem 4 from [4].

4. INTEGRAL EQUATIONS IN THE FRÉCHET SPACE OF CONTINUOUS FUNCTIONS

In this section we consider two types of nonlinear integral Volterra equation in the Fréchet algebra $F = C(\mathbb{R}_+, E)$, where we assume that E is Banach algebra.

4.1 A product of Volterra operators

First, we will investigate the following integral equation:

$$(11) \quad x(t) = \int_0^t v_1(t, s, x(s)) ds \cdot \int_0^t v_2(t, s, x(s)) ds, \quad t \in \mathbb{R}_+,$$

where we will look for solutions $x = x(t)$ in the space $C(\mathbb{R}_+, E)$. We start with the following assumptions.

(A₁) The mappings $v_i : \{(t, s, x) : t \in \mathbb{R}_+, s \in [0, t], x \in E\} \rightarrow E$ are uniformly continuous on bounded subsets of their domains for $i = 1, 2$.

(A₂) There exist the integrable and locally bounded functions $p_i : \{(t, s) : t \in \mathbb{R}_+, s \in [0, t]\} \rightarrow \mathbb{R}_+$, $i = 1, 2$ such that

$$\alpha(v_i(t, s, X)) \leq p_i(t, s)\alpha(X), \quad 0 \leq s \leq t, X \in \mathfrak{M}_E.$$

(A₃) There exist the integrable functions $a_i : \{(t, s) : t \in \mathbb{R}_+, s \in [0, t]\} \rightarrow \mathbb{R}_+$, $i = 1, 2$, nondecreasing functions $b_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $i = 1, 2$ and the function $r : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$\|v_i(t, s, x)\| \leq a_i(t, s)b_i(\|x\|), \quad i = 1, 2, 0 \leq s \leq t, x \in E$$

and

$$\int_0^t a_1(t, s)b_1(r(s))ds \cdot \int_0^t a_2(t, s)b_2(r(s))ds \leq r(t), \quad t \in \mathbb{R}_+.$$

REMARK 4.15. Notice that assumption (A₁) yields that the mappings v_i are bounded on bounded subsets of the set $\{(t, s, x) : t \in \mathbb{R}_+, s \in [0, t], x \in E\}$.

Now we can formulate one of two main results of this section.

Theorem 4.16. *Under the assumptions (A₁) – (A₃) the integral equation (11) has at least one solution $x \in C(\mathbb{R}_+, E)$.*

Proof. Let us define the operators $\mathcal{V}_1, \mathcal{V}_2, \mathcal{A} : C(\mathbb{R}_+, E) \rightarrow C(\mathbb{R}_+, E)$ by the formulas

$$\mathcal{V}_i x(t) := \int_0^t v_i(t, s, x(s))ds, \quad i = 1, 2, x \in C(\mathbb{R}_+, E),$$

$$\mathcal{A}x(t) := (\mathcal{V}_1 x \cdot \mathcal{V}_2 x)(t), \quad x \in C(\mathbb{R}_+, E).$$

Theorem 3.11 applied to the operator \mathcal{A} provides thesis of this Theorem. Moreover, using assumptions (A₁) – (A₃) we can show that the above operators are well defined. Further, let us put

$$\widehat{\Omega} := \{x \in C(\mathbb{R}_+, E) : \|x(t)\| \leq r(t), t \in \mathbb{R}_+\},$$

where r is specified in (A₃). Obviously, the set $\widehat{\Omega}$ is nonempty, bounded, closed and convex in the Fréchet algebra $C(\mathbb{R}_+, E)$. In view of assumption (A₃) we have additionally that $\mathcal{A} : \widehat{\Omega} \rightarrow \widehat{\Omega}$. Let us consider the set

$$\Omega := \text{Conv}\mathcal{A}(\widehat{\Omega}).$$

Then we obtain

$$\mathcal{A}(\Omega) = \mathcal{A}(\text{Conv}\mathcal{A}(\widehat{\Omega})) \subset \mathcal{A}(\text{Conv}\widehat{\Omega}) = \mathcal{A}(\widehat{\Omega}) \subset \text{Conv}\mathcal{A}(\widehat{\Omega}) = \Omega$$

and this yields that $\mathcal{A} : \Omega \rightarrow \Omega$. Obviously, Ω is bounded, closed and convex. Now, we prove that Ω is a locally equicontinuous set. Let us fix $T > 0$, $\varepsilon > 0$ and $t_1, t_2 \in [0, T]$, $t_1 \leq t_2$. Then, for fixed $x \in \widehat{\Omega}$ we get

$$(12) \quad \|\mathcal{V}_i x(t_2) - \mathcal{V}_i x(t_1)\| \leq \int_{t_1}^{t_2} \|v_i(t_2, s, x(s))\| ds + \int_0^{t_1} \|v_i(t_2, s, x(s)) - v_i(t_1, s, x(s))\| ds.$$

Applying Remark 4.15 we derive

$$M_i := \sup\{\|v_i(t, s, x)\| : 0 \leq s \leq t \leq T, \|x\| \leq \sup_{\tau \leq T} r(\tau)\} < \infty.$$

Based on local equicontinuity of v_i we have that there exists a sufficiently small $\delta > 0$ such that

$$\|v_i(t_2, s, x) - v_i(t_1, s, x)\| \leq \frac{\varepsilon}{2T}$$

and

$$\delta M_i \leq \frac{\varepsilon}{2}.$$

Keeping in mind (12) we obtain

$$\|\mathcal{V}_i x(t_2) - \mathcal{V}_i x(t_1)\| \leq \varepsilon \text{ for } |t_2 - t_1| \leq \delta$$

which proves that $\mathcal{V}_i(\widehat{\Omega})$ is equicontinuous on $[0, T]$. Further, using the following estimate

$$\|\mathcal{A}x(t_2) - \mathcal{A}x(t_1)\| \leq \|\mathcal{V}_2 x(t_2)\| \cdot \|\mathcal{V}_1 x(t_2) - \mathcal{V}_1 x(t_1)\| + \|\mathcal{V}_1 x(t_2)\| \cdot \|\mathcal{V}_2 x(t_2) - \mathcal{V}_2 x(t_1)\|,$$

boundedness of $\|\mathcal{V}_i x(t_j)\|$ on appropriate sets and from equicontinuity of $\mathcal{V}_i(\widehat{\Omega})$ on $[0, T]$ proved above, we have that $\mathcal{A}(\widehat{\Omega})$ is equicontinuous on $[0, T]$. For proving the local equicontinuity of $\Omega = \text{Conv} \mathcal{A}(\widehat{\Omega})$ it is enough to utilize Lemma 2.9. This implies that, for arbitrary $X \subset \Omega$, we have

$$(13) \quad \omega_0^n(X) = 0, \quad n \in \mathbb{N}.$$

Reasoning similarly as above we infer that

$$(14) \quad \mathcal{V}_i(\Omega) \text{ is equicontinuous on } [0, T], \quad i = 1, 2.$$

Now, we prove that \mathcal{A} satisfies the condition (5). Let us fix $n \in \mathbb{N}$, $t \in [0, n]$, $X \subset \Omega$. In view of Remark 4.15 we obtain the boundedness of the following constants

$$V_i := \sup\{\|\mathcal{V}_i(\Omega)(\tau)\| : \tau \in [0, n]\}, \quad i = 1, 2.$$

Applying the (m) property of Kuratowski measure α , (14), Lemma 2.10 and assumption (A_2) we derive

$$\begin{aligned}
\alpha(\tilde{\mathcal{A}}^1(X)(t)) &= \alpha(\mathcal{A}(\text{Conv}X)(t)) = \alpha((\mathcal{V}_1 \cdot \mathcal{V}_2)(\text{Conv}X)(t)) \\
&\leq \alpha(\mathcal{V}_1(\text{Conv}X)(t) \cdot \mathcal{V}_2(\text{Conv}X)(t)) \\
&\leq \|\mathcal{V}_2(\text{Conv}X)(t)\| \alpha(\mathcal{V}_1(\text{Conv}X)(t)) \\
&\quad + \|\mathcal{V}_1(\text{Conv}X)(t)\| \alpha(\mathcal{V}_2(\text{Conv}X)(t)) \\
&\leq \|\mathcal{V}_2(\Omega)(t)\| \alpha\left(\int_0^t v_1(t, s, \text{Conv}X(s)) ds\right) \\
&\quad + \|\mathcal{V}_1(\Omega)(t)\| \alpha\left(\int_0^t v_2(t, s, \text{Conv}X(s)) ds\right) \\
&\leq V_2 \int_0^t \alpha(v_1(t, s, \text{Conv}X(s))) ds + V_1 \int_0^t \alpha(v_2(t, s, \text{Conv}X(s))) ds \\
&\leq V_2 \int_0^t p_1(t, s) \alpha(\text{Conv}X(s)) ds + V_1 \int_0^t p_2(t, s) \alpha(\text{Conv}X(s)) ds \\
&\leq V_2 \int_0^t p_1(t, s) \alpha(X(s)) ds + V_1 \int_0^t p_2(t, s) \alpha(X(s)) ds \\
&\leq \int_0^t (V_2 p_1(t, s) + V_1 p_2(t, s)) \alpha(X(s)) ds.
\end{aligned}$$

Hence

$$\alpha(\tilde{\mathcal{A}}^1(X)(t)) \leq \int_0^t p(t, s) \alpha(X(s)) ds,$$

where

$$p(t, s) := V_2 p_1(t, s) + V_1 p_2(t, s) \quad \text{for } 0 \leq s \leq t.$$

Further, we have

$$\begin{aligned}
\alpha(\tilde{\mathcal{A}}^2(X)(t)) &= \alpha(\mathcal{A}(\text{Conv}\tilde{\mathcal{A}}^1(X))(t)) \leq \int_0^t p(t, s) \alpha(\tilde{\mathcal{A}}^1(X)(s)) ds \\
&\leq \int_0^t p(t, s) \int_0^s p(s, s_1) \alpha(X(s_1)) ds_1 ds.
\end{aligned}$$

In the sequel we get

$$\begin{aligned}
\alpha(\tilde{\mathcal{A}}^m(X)(t)) &\leq \int_0^t p(t, s) \int_0^s p(s, s_1) \dots \int_0^{s_{m-2}} p(s_{m-2}, s_{m-1}) \alpha(X(s_{m-1})) ds_{m-1} \dots ds_1 ds \\
(15) \quad &\leq \int_0^t p(t, s) \int_0^s p(s, s_1) \dots \int_0^{s_{m-2}} p(s_{m-2}, s_{m-1}) ds_{m-1} \dots ds_1 ds \bar{\alpha}_n(X).
\end{aligned}$$

In virtue of assumption (A_2) we have that $p(t, s)$ is bounded on $\{(t, s) : 0 \leq s \leq t \leq n\}$. Hence,

$$P := \text{ess sup}\{p(t, s) : 0 \leq s \leq t \leq n\} < \infty.$$

From above information and inequality (15) we have

$$\alpha(\tilde{\mathcal{A}}^m(X)(t)) \leq \frac{(Pn)^m}{m!} \bar{\alpha}_n(X),$$

hence,

$$\bar{\alpha}_n(\tilde{\mathcal{A}}^m(X)) \leq \frac{(Pn)^m}{m!} \bar{\alpha}_n(X).$$

Keeping in mind (1) and (13) we obtain

$$\mu_n(\tilde{\mathcal{A}}^m(X)) \leq \frac{(Pn)^m}{m!} \mu_n(X).$$

Since $\lim_{m \rightarrow \infty} \frac{(Pn)^m}{m!} = 0$ we infer that there exists sufficiently big m_n such that $\frac{(Pn)^{m_n}}{m_n!} < 1$ which means that assumptions of Theorem 3.11 are satisfied and the operator \mathcal{A} has a fixed point. \square

4.2 Nonlinear Integral Quadratic Volterra Equation in Fréchet algebra $C(\mathbb{R}_+, E)$

In this subsection we will investigate the following quadratic Volterra integral equation

$$(16) \quad x(t) = g(t) + f(t, x(t)) \cdot \int_0^t v(t, s, x(s)) ds, \quad t \in \mathbb{R}_+.$$

We will seek the solutions $x = x(t)$ also in the space $C(\mathbb{R}_+, E)$, where E is Banach algebra. We consider Eq. (16) under the following assumptions:

- (A₁) The function $g : \mathbb{R}_+ \rightarrow E$ is continuous.
- (A₂) The function $f : \mathbb{R}_+ \times E \rightarrow E$ is continuous and there exists $k \in \mathbb{R}_+$ such that

$$\|f(t, x_1) - f(t, x_2)\| \leq k \|x_1 - x_2\|, \quad t \in \mathbb{R}_+, \quad x, y \in E.$$

- (A₃) The function $v : \{(t, s, x) : t \in \mathbb{R}_+, s \in [0, t], x \in E\} \rightarrow E$ is uniformly continuous on the bounded subsets of the domain of the mapping v .
- (A₄) There is an integrable function $p : \{(t, s) : t \in \mathbb{R}_+, s \in [0, t]\} \rightarrow \mathbb{R}_+$, such that

$$\alpha(v(t, s, X)) \leq p(t, s) \alpha(X), \quad 0 \leq s \leq t, \quad X \in \mathfrak{M}_E.$$

- (A₅) There is an integrable function $a : \{(t, s) : t \in \mathbb{R}_+, s \in [0, t]\} \rightarrow \mathbb{R}_+$, nondecreasing function $b : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and a function $r : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$\|v(t, s, x)\| \leq a(t, s) b(\|x\|), \quad 0 \leq s \leq t, \quad x \in E$$

and

$$\|g(t)\| + (\|f(t, \bar{0})\| + kr(t)) \int_0^t a(t, s) b(r(s)) ds \leq r(t), \quad t \in \mathbb{R}_+.$$

(A₆) The following inequality holds

$$k \left(\sup_{t \in [0, n]} \int_0^t a(t, s) b(r(s)) ds \right) + \left(\sup_{t \in [0, n]} \|f(t, \bar{0})\|_n + k\bar{r}(n) \right) \sup_{t \in [0, n]} \int_0^t p(t, s) ds < 1,$$

for $n \in \mathbb{N}$, where $\bar{r}(t) := \sup_{s \in [0, t]} r(s)$.

Theorem 4.17. *Under the assumptions (A₁) – (A₆) the integral equation (16) has at least one solution $x \in C(\mathbb{R}_+, E)$.*

Proof. Let us consider a set

$$\Omega := \{x \in C(\mathbb{R}_+, E) : \|x(t)\| \leq r(t), t \in \mathbb{R}_+\}.$$

The set Ω is bounded, convex and closed. Next, we define the operators $\mathcal{G}, \mathcal{F}, \mathcal{V}, \mathcal{A} : C(\mathbb{R}_+, E) \rightarrow C(\mathbb{R}_+, E)$ by the formulas

$$\mathcal{G}x(t) := g(t), \quad x \in C(\mathbb{R}_+, E),$$

$$\mathcal{F}x(t) := f(t, x(t)), \quad x \in C(\mathbb{R}_+, E),$$

$$\mathcal{V}x(t) := \int_0^t v(t, s, x(s)) ds, \quad x \in C(\mathbb{R}_+, E),$$

$$\mathcal{A}x(t) := \mathcal{G}x(t) + (\mathcal{F}x \cdot \mathcal{V}x)(t), \quad x \in C(\mathbb{R}_+, E).$$

The last part of assumption (A₅) implies that

$$\mathcal{A} : \Omega \rightarrow \Omega.$$

The assumptions guarantee that the above operators are well defined and continuous. To this end, we apply Theorem 3.13 to the operator \mathcal{A} . Let us fix a nonempty set $X \subset \Omega$ and $n \in \mathbb{N}$. Standard calculations show that

$$\omega_0^n(\mathcal{F}(X)) \leq k\omega_0^n(X),$$

$$\omega_0^n(\mathcal{V}(X)) = 0,$$

$$\bar{\alpha}_n(\mathcal{F}(X)) = \sup_{t \in [0, n]} \alpha(f(t, X(t))) \leq k\bar{\alpha}_n(X),$$

$$\bar{\alpha}_n(\mathcal{V}(X)) \leq \sup_{t \in [0, n]} \int_0^t p(t, s) ds \bar{\alpha}_n(X).$$

Taking into account (1) and the above estimate we obtain

$$\mu_n(\mathcal{F}(X)) \leq k\mu_n(X),$$

$$\mu_n(\mathcal{V}(X)) \leq \sup_{t \in [0, n]} \int_0^t p(t, s) ds \mu_n(X).$$

Moreover, we have

$$\begin{aligned} \|\mathcal{F}(\Omega)\|_n &= \sup_{t \in [0, n]} \|f(t, \bar{0})\|_n + k\bar{r}(n), \\ \|\mathcal{V}(\Omega)\|_n &= \sup_{t \in [0, n]} \int_0^t a(t, s)b(r(s))ds. \end{aligned}$$

Referring to Theorem 3.13 let us put

$$k_n := k, \quad l_n := \sup_{t \in [0, n]} \int_0^t p(t, s)ds.$$

From the above estimates and assumption (A₆) we infer that

$$\begin{aligned} \|\mathcal{V}(\Omega)\|_n k_n + \|\mathcal{F}(\Omega)\|_n l_n &= \left(\sup_{t \in [0, n]} \int_0^t a(t, s)b(r(s))ds \right) k \\ &+ \left(\sup_{t \in [0, n]} \|f(t, \bar{0})\|_n + k\bar{r}(n) \right) \sup_{t \in [0, n]} \int_0^t p(t, s)ds < 1. \end{aligned}$$

Thus, according to Theorem 3.13 we conclude that the operator \mathcal{A} has at least one fixed point. In view of Remark 2.7 we have that the set X of all solutions of Eq. (16) is equicontinuous on compact subsets of \mathbb{R}_+ and $X(t)$ is relatively compact in E for each $t \in \mathbb{R}_+$. □

REMARK 4.18. In the similar way we can show that equation

$$x(t) = (\mathcal{U}_1 x \cdot \mathcal{U}_2 x)(t),$$

where

$$(\mathcal{U}_i x)(t) = g_i(t) + f_i(t, x(t)) \cdot \int_0^t v_i(t, s, x(s))ds, \quad t \in \mathbb{R}_+, \quad i = 1, 2$$

with analogous assumptions has at least fixed point in the Fréchet space $C(\mathbb{R}_+, E)$. The proof will be omitted.

5. CONDITION (m) IN FRÉCHET ALGEBRA $L^1_{loc}(\mathbb{R}_+)$ AND IN BANACH ALGEBRA $L^1(\mathbb{R}_+)$

Theorem 3.13 and Remark 3.14 show us an importance of condition (m). We will offer some variants of condition (m) in Fréchet algebra $L^1_{loc}(\mathbb{R}_+)$ and in Banach algebra $L^1(\mathbb{R}_+)$ with suitable topologies and the families of measures of noncompactness.

Let us recall ([13]) that $L^1_{loc}(\mathbb{R}_+)$ means the space of all real functions locally integrable on \mathbb{R}_+ with a sequence of seminorms $(\|\cdot\|_n)_{n \in \mathbb{N}}$, where

$$\|x\|_n := \int_0^n |x(t)| dt, \quad n \in \mathbb{N}, \quad x \in L^1_{loc}(\mathbb{R}_+).$$

We treat the product of vectors x and y in $L^1_{loc}(\mathbb{R}_+)$ as a convolution $x * y$ i.e.

$$(17) \quad (x * y)(t) := \int_0^\infty x(t-s)y(s)ds, \quad t \in \mathbb{R}_+, \quad x, y \in L^1_{loc}(\mathbb{R}_+)$$

where we assume that $x(t) = y(t) = 0$ for $t < 0$. Standard calculations lead us to the inequality $\|x * y\|_T \leq \|x\|_T \|y\|_T$ for $T \geq 0$.

In the algebra $L^1_{loc}(\mathbb{R}_+)$ we consider weak topology τ^* and the family of measures of noncompactness $(a_n)_{n \in \mathbb{N}}$ defined by the formula (see [13])

$$a_n(X) := \lim_{\varepsilon \rightarrow 0^+} \sup_{x \in X} \sup \left\{ \int_D |x(t)| dt : D \subset [0, n], \quad m(D) \leq \varepsilon \right\}$$

for $n \in \mathbb{N}$, $X \in \mathfrak{M}_{L^1_{loc}(\mathbb{R}_+)}$. Under the above notions we have following result.

Theorem 5.19. *In the Fréchet algebra $L^1_{loc}(\mathbb{R}_+)$ condition (m) has the following form*

$$(18) \quad a_n(X * Y) \leq \frac{1}{2} \|X\|_n a_n(Y) + \frac{1}{2} \|Y\|_n a_n(X),$$

for $n \in \mathbb{N}$, $X, Y \in \mathfrak{M}_{L^1_{loc}(\mathbb{R}_+)}$.

Proof. For $X \in \mathfrak{M}_{L^1_{loc}(\mathbb{R}_+)}$, $n \in \mathbb{N}$ and $\varepsilon > 0$ let us denote

$$a_n(X, \varepsilon) := \sup_{x \in X} \sup \left\{ \int_D |x(t)| dt : D \subset [0, n], \quad m(D) \leq \varepsilon \right\}.$$

Further, let us fix $X, Y \in \mathfrak{M}_{L^1_{loc}(\mathbb{R}_+)}$, $x \in X$, $y \in Y$, $\varepsilon > 0$ and $D \subset [0, n]$, $m(D) \leq \varepsilon$. Then we obtain

$$(19) \quad \begin{aligned} \int_D |(x * y)(t)| dt &\leq \int_D \int_{\mathbb{R}_+} |x(t-s)| \cdot |y(s)| ds dt \leq \int_{\mathbb{R}_+} \int_D |x(t-s)| \cdot |y(s)| dt ds \\ &= \int_{\mathbb{R}_+} |y(s)| \int_D |x(t-s)| dt ds = \int_{\mathbb{R}_+} |y(s)| \int_{D-s} |x(\tau)| d\tau ds. \end{aligned}$$

Since $D-s \subset [-s, n-s]$, $m((D-s) \cap [0, n]) \leq \varepsilon$ and $x(\tau) = 0$ for $\tau < 0$ we deduce that

$$\int_{\mathbb{R}_+} |y(s)| \int_{D-s} |x(\tau)| d\tau ds \leq \int_0^n |y(s)| \int_{D-s} |x(\tau)| d\tau ds \leq \|Y\|_n a_n(X, \varepsilon).$$

Applying evaluation (19) we get

$$\int_D |(x * y)(t)| dt \leq \|Y\|_n a_n(X, \varepsilon)$$

and similarly

$$\int_D |(y * x)(t)| dt \leq \|X\|_n a_n(Y, \varepsilon).$$

Further

$$\begin{aligned} \int_D |(x * y)(t)| dt &= \frac{1}{2} \int_D |(x * y)(t)| dt + \frac{1}{2} \int_D |(x * y)(t)| dt \\ &\leq \frac{1}{2} \|Y\|_n a_n(X, \varepsilon) + \frac{1}{2} \|X\|_n a_n(Y, \varepsilon). \end{aligned}$$

Taking supremum on $x \in X, y \in Y$ we infer that

$$a_n(X * Y, \varepsilon) \leq \frac{1}{2} \|Y\|_n a_n(X, \varepsilon) + \frac{1}{2} \|X\|_n a_n(Y, \varepsilon)$$

and for $\varepsilon \rightarrow 0$ this implies (18). □

Now, we are concerned with the condition (m) in Banach algebra $L^1(\mathbb{R}_+)$ furnished with weak topology and the product of vectors defined also by formula (17). Let us recall that ([2]) one of the examples of the weak measure of noncompactness in $L^1(\mathbb{R}_+)$ is the measure μ defined by formula

$$(20) \quad \mu(X) := a(X) + b(X), \quad X \in \mathfrak{M}_{L^1(\mathbb{R}_+)}$$

where

$$\begin{aligned} a(X) &:= \lim_{\varepsilon \rightarrow 0^+} \sup_{x \in X} \sup \left\{ \int_D |x(t)| dt : D \subset \mathbb{R}, m(D) \leq \varepsilon \right\}, \\ b(X) &:= \lim_{T \rightarrow \infty} \sup_{x \in X} \int_T^\infty |x(t)| dt. \end{aligned}$$

Theorem 5.20. *In the Banach algebra $L^1(\mathbb{R}_+)$ condition (m) has the following form*

$$\mu(X * Y) \leq \|X\| \mu(Y) + \|Y\| \mu(X)$$

for $X, Y \in \mathfrak{M}_{L^1(\mathbb{R}_+)}$.

Proof. Reasoning similarly to the proof of Theorem 5.19 we infer that

$$a(X * Y) \leq \frac{1}{2} \|X\| a(Y) + \frac{1}{2} \|Y\| a(X)$$

for $X, Y \in \mathfrak{M}_{L^1(\mathbb{R}_+)}$.

For proving the thesis of the above Theorem it is enough to prove following inequality

$$(21) \quad b(X * Y) \leq \|X\| b(Y) + \|Y\| b(X)$$

and apply the formula (20). Thus, let us fix $X, Y \in \mathfrak{M}_{L^1(\mathbb{R}_+)}$, $x \in X, y \in Y$ and two numbers $0 < U < T$. Further, we get

$$\begin{aligned}
 \int_T^\infty |(x * y)(t)| dt &= \int_T^\infty \int_0^\infty |x(t-s)y(s)| ds dt = \int_0^\infty |y(s)| \int_T^\infty |x(t-s)| dt ds \\
 &= \int_0^U |y(s)| \int_T^\infty |x(t-s)| dt ds + \int_U^\infty |y(s)| \int_T^\infty |x(t-s)| dt ds \\
 &= \int_0^U |y(s)| \int_{T-s}^\infty |x(\tau)| d\tau ds + \int_U^\infty |y(s)| \int_{T-s}^\infty |x(\tau)| d\tau ds \\
 &\leq \int_0^U |y(s)| \int_{T-U}^\infty |x(\tau)| d\tau ds + \int_U^\infty |y(s)| \int_0^\infty |x(\tau)| d\tau ds \\
 &\leq \|y\| \int_{T-U}^\infty |x(\tau)| d\tau + \|x\| \int_U^\infty |y(s)| ds \\
 &\leq \|Y\| \sup_{x \in X} \int_{T-U}^\infty |x(\tau)| d\tau + \|X\| \sup_{y \in Y} \int_U^\infty |y(s)| ds.
 \end{aligned}$$

Overall,

$$\sup_{x \in X, y \in Y} \int_T^\infty |(x * y)(t)| dt \leq \|Y\| \sup_{x \in X} \int_{T-U}^\infty |x(\tau)| d\tau + \|X\| \sup_{y \in Y} \int_U^\infty |y(s)| ds.$$

Putting $T = 2U$ where $U \rightarrow \infty$ we obtain the estimate (21). \square

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Szymon Dudek

Faculty of Mathematics and Applied Physics,
Rzeszów University of Technology,
al.Powstańców Warszawy 8,
35-959 Rzeszów,
Poland
E-mail: sdudek@prz.edu.pl

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