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FROM A COTANGENT SUM TO A GENERALIZED TOTIENT FUNCTION

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In this paper we investigate a certain category of cotangent sums and more specifically the sum

$$\sum_{m=1}^{b-1} \cot\left(\frac{\pi m}{b}\right) \sin^3\left(2\pi m \frac{a}{b}\right)$$

and associate the distribution of its values to a generalized totient function $\phi(n, A, B)$, where

$$\phi(n, A, B) := \sum_{\substack{A \le k \le B \\ (n,k)=1}} 1 \, .$$

One of the methods used consists in the exploitation of relations between trigonometric sums and the fractional part of a real number.

1. INTRODUCTION

For a, b, $n \in \mathbb{N}$, let

$$x_n := \left\{ \frac{na}{b} \right\} = \frac{na}{b} - \left\lfloor \frac{na}{b} \right\rfloor ,$$

where $\lfloor u \rfloor$ stands for the floor function of the real number u. In other words x_n denotes the fractional part of the rational number na/b (for an extensive study of fractional parts of real numbers see [**2**]).
We know (see [**3**], Proposition 2.1) that

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Proposition 1.1. *For every a, b, $n \in \mathbb{N}$, $b \geq 2$, we have*

$$\sum_{m=1}^{b-1} \cot\left(\frac{\pi m}{b}\right) \cos\left(2\pi m n \frac{a}{b}\right) = 0 \,.$$

If $b \not| na$ then we also have

$$x_n = \frac{1}{2} - \frac{1}{2b} \sum_{m=1}^{b-1} \cot\left(\frac{\pi m}{b}\right) \sin\left(2\pi m n \frac{a}{b}\right) \,.$$

Based on the trigonometric identity

$$\cos(n\theta) = \sum_{k=0}^{n} \cos^k(\theta)\ \sin^{n-k}(\theta)\ \cos\left(\frac{n-k}{2}\pi\right)$$

in combination with the above proposition, we can inductively prove that for every a, b, q, $n \in \mathbb{N}$, $b \geq 2$, it holds

$$\sum_{m=1}^{b-1} \cot\left(\frac{\pi m}{b}\right) \cos^q\left(2\pi m n \frac{a}{b}\right) = 0 \,.$$

Additionally, by Proposition 1.1 we can prove that for every a, b, $n \in \mathbb{N}$, $b \geq 2$, we have

$$\sum_{m=1}^{b-1} \cot\left(\frac{\pi m}{b}\right) \sin^2\left(2\pi m n \frac{a}{b}\right) = 0 \,.$$

Hence, the natural question of calculating cotangent sums of the form

$$\sum_{m=1}^{b-1} \cot\left(\frac{\pi m}{b}\right) \sin^r\left(2\pi m n \frac{a}{b}\right) \,,$$

where $r \in \mathbb{N}$ and $r \geq 3$, arises.
Interestingly, the investigation of the above category of cotangent sums, with $r \geq 3$, turns out to be more complex.
In the subsequent sections, we shall calculate the cotangent sum $S(1, a, b)$, where

$$S(n, a, b) = \sum_{m=1}^{b-1} \cot\left(\frac{\pi m}{b}\right) \sin^3\left(2\pi m n \frac{a}{b}\right)$$

and associate the distribution of its values to a generalized totient function $\phi(n, A, B)$, where

$$\phi(n, A, B) := \sum_{\substack{A \leq k \leq B \\ (n,k)=1}} 1 \,.$$

Moreover, we prove several properties of $\phi(n, A, B)$ including an asymptotic formula. Namely, our main results are the following:

Proposition 1.2. *Let a, $b \in \mathbb{N}$, where $(a,b)=1$, $a \geq b \geq 2$ and $b \neq 3$. Then*

$$S(1,a,b) = 0 \text{ or } \pm b/2\,.$$

Proposition 1.3. *Let a, $b \in \mathbb{N}$, $b \geq 2$, where $(a,b)=1$ and $b \neq 3$. Then*

$$S(1,a,b) = 0$$

if and only if $2b = 3a+k+1$, for some k, $0 \leq k \leq b-2$.

Corollary 1.4. *Let a, $b \in \mathbb{N}$, $b \geq 2$, where $(a,b)=1$ and $b \neq 3$. Then, the number of integers a, such that $1 \leq a \leq b-1$ and $S(1,a,b)=0$, is given by the following formula*

$$\#\left\{a \mid S(1,a,b)=0\right\} = \phi\left(b, \left\lceil \frac{b+1}{3} \right\rceil, \left\lfloor \frac{2b-1}{3} \right\rfloor\right).$$

Proposition 1.5. *Let a, $b \in \mathbb{N}$, $b \geq 2$, where $(a,b)=1$ and $b \neq 3$. Then*

$$S(1,a,b) = \frac{b}{2}$$

if and only if $b = 3a+k+1$, for some k, $0 \leq k \leq b-2$.

Corollary 1.6. *Let a, $b \in \mathbb{N}$, $b \geq 2$, where $(a,b)=1$ and $b \neq 3$. Then, the number of integers a, such that $1 \leq a \leq b-1$ and $S(1,a,b)=b/2$, is given by the following formula*

$$\#\left\{a \mid S(1,a,b)=\frac{b}{2}\right\} = \phi\left(b, 1, \left\lfloor \frac{b-1}{3} \right\rfloor\right).$$

Proposition 1.7. *Let a, $b \in \mathbb{N}$, $b \geq 2$, where $(a,b)=1$ and $b \neq 3$. Then*

$$S(1,a,b) = -\frac{b}{2}$$

if and only if $3b = 3a+k+1$, for some k, $0 \leq k \leq b-2$.

Corollary 1.8. *Let a, $b \in \mathbb{N}$, $b \geq 2$, where $(a,b)=1$ and $b \neq 3$. Then, the number of integers a, such that $1 \leq a \leq b-1$ and $S(1,a,b)=-b/2$, is given by the following formula*

$$\#\left\{a \mid S(1,a,b)=-\frac{b}{2}\right\} = \phi\left(b, \left\lceil \frac{2b+1}{3} \right\rceil, \left\lfloor \frac{3b-1}{3} \right\rfloor\right).$$

Proposition 1.9. *Let n, A, $B \in \mathbb{N}$, $n > 1$. Then, we have*

$$\phi(n,A,B) = \frac{B-A}{n}\phi(n) + \delta_{n,A} + O\left(\sum_{d \mid n} \mu(d)^2\right),$$

where $\delta_{n,A} = 1$ if $(n,A)=1$ and 0 otherwise.

2. PRELIMINARIES

Proposition 2.10. *For every a, b, $n \in \mathbb{N}$, $b \geq 2$, such that $b \nmid 3n$, we have*

$$x_{3n} = 3x_n - 1 + \frac{2}{b}S(n,a,b)\ ,$$

where $x_n := \{na/b\}$ and

$$S(n,a,b) := \sum_{m=1}^{b-1} \cot\left(\frac{\pi m}{b}\right) \sin^3\left(2\pi mn\frac{a}{b}\right)\ .$$

Proof. We know that

$$x_n = \frac{1}{2} - \frac{1}{2b}\sum_{m=1}^{b-1} \cot\left(\frac{\pi m}{b}\right) \sin\left(2\pi mn\frac{a}{b}\right)\ ,$$

for every $b \nmid n$. So, we get

$$x_{3n} = \frac{1}{2} - \frac{1}{2b}\sum_{m=1}^{b-1} \cot\left(\frac{\pi m}{b}\right) \sin\left(2\pi m(3n)\frac{a}{b}\right)\ ,$$

for every $b \nmid 3n$. Thus

$$x_{3n} = \frac{1}{2} - \frac{1}{2b}\sum_{m=1}^{b-1} \cot\left(\frac{\pi m}{b}\right) \left(3\sin\left(2\pi mn\frac{a}{b}\right) - 4\sin^3\left(2\pi mn\frac{a}{b}\right)\right)\ ,$$

for every $b \nmid 3n$. So

$$\begin{aligned} x_{3n} &= \frac{1}{2} - \frac{3}{2b}\sum_{m=1}^{b-1} \cot\left(\frac{\pi m}{b}\right) \sin\left(2\pi mn\frac{a}{b}\right) + \frac{4}{2b}\sum_{m=1}^{b-1} \cot\left(\frac{\pi m}{b}\right) \sin^3\left(2\pi mn\frac{a}{b}\right) \\ &= 3\left(\frac{1}{2} - \frac{1}{2b}\sum_{m=1}^{b-1} \cot\left(\frac{\pi m}{b}\right) \sin\left(2\pi mn\frac{a}{b}\right)\right) - 1 + \frac{2}{b}\sum_{m=1}^{b-1} \cot\left(\frac{\pi m}{b}\right) \sin^3\left(2\pi mn\frac{a}{b}\right)\ , \end{aligned}$$

for every $b \nmid 3n$. Hence

$$x_{3n} = 3x_n - 1 + \frac{2}{b}S(n,a,b)\ ,$$

for every $b \nmid 3n$. □

Lemma 2.11. *For every a, b, $n \in \mathbb{N}$, $b \geq 2$ and every $k \in \mathbb{N} \cup \{0\}$, we have*

$$\left\{\frac{na+k}{b}\right\} = x_n + \frac{k}{b} - \frac{1}{b}E(n,k)\ ,$$

where

$$E(n,k) := \sum_{\lambda=na+1}^{na+k} \sum_{m=0}^{b-1} e^{2\pi i m\lambda/b} \,,$$

for $k \in \mathbb{N}$ *and* $E(n,0) := 0$.

Proof. We know (see [**3**], Section 2) that

$$\left\{\frac{a}{b}\right\} = \frac{a}{b} - \frac{1}{b}\sum_{\lambda=1}^{a}\sum_{m=0}^{b-1} e^{2\pi i m\lambda/b} \,.$$

Thus, we can write

$$\begin{aligned}
\left\{\frac{na+k}{b}\right\} &= \frac{na+k}{b} - \frac{1}{b}\sum_{\lambda=1}^{na+k}\sum_{m=0}^{b-1} e^{2\pi i m\lambda/b} \\
&= \frac{na}{b} + \frac{k}{b} - \frac{1}{b}\sum_{\lambda=1}^{na}\sum_{m=0}^{b-1} e^{2\pi i m\lambda/b} - \frac{1}{b}\sum_{\lambda=na+1}^{na+k}\sum_{m=0}^{b-1} e^{2\pi i m\lambda/b} \\
&= \left(\frac{na}{b} - \frac{1}{b}\sum_{\lambda=1}^{na}\sum_{m=0}^{b-1} e^{2\pi i m\lambda/b}\right) + \left(\frac{k}{b} - \frac{1}{b}\sum_{\lambda=na+1}^{na+k}\sum_{m=0}^{b-1} e^{2\pi i m\lambda/b}\right) \\
&= \left\{\frac{na}{b}\right\} + \frac{k}{b} - \frac{1}{b}\sum_{\lambda=na+1}^{na+k}\sum_{m=0}^{b-1} e^{2\pi i m\lambda/b} \,,
\end{aligned}$$

for every $k \in \mathbb{N}$.
For the case when $k = 0$, the result is clear.

□

The following proposition also holds.

Proposition 2.12. *Let* a, $b \in \mathbb{N}$, *with* $b \geq 2$.
If $a \not\equiv 0 \pmod{b}$, *then*

$$\left\{\frac{a}{b}\right\} = \left\{\frac{a-1}{b}\right\} + \frac{1}{b} \,.$$

If $a \equiv 0 \pmod{b}$, *then*

$$\left\{\frac{a-2}{b}\right\} = 1 - \frac{2}{b} \,.$$

Proposition 2.13. *Let* a, $b \in \mathbb{N}$, *where* $(a,b) = 1$, $a \geq b \geq 2$ *and* $b \neq 3$. *Then*

$$(3\nu + 2)b = (3a + k + 1) + 3E(1,k) + 2S(1,a,b) \,,$$

for some integer k, $0 \leq k \leq b-2$, *such that* $3a + k + 1 \equiv 0 \pmod{b}$, *where*

$$\nu := \left\lfloor \frac{a+k}{b} \right\rfloor$$

and

$$S(n,a,b)=\sum_{m=1}^{b-1}\cot\left(\frac{\pi m}{b}\right)\sin^3\left(2\pi mn\frac{a}{b}\right)\ .$$

Proof. Since $b\neq 3$ and $(a,b)=1$, it is clear that b does not divide $3a$. Thus, b should divide one of the consecutive integers

$$3a+1,\ 3a+2,\ \ldots,\ 3a+(b-1)\ .$$

In other words, there exists k, with $0\leq k\leq b-2$, such that

$$3a+k+1\equiv 0\ (\mathrm{mod}\ b)\ .$$

But, then it is obvious that $b\not|\,3a+k$. Hence, by Proposition 5.2 of [**3**], we get

$$\left\{\frac{3a+k}{b}\right\}=\left\{\frac{3a+k-1}{b}\right\}+\frac{1}{b}\ . \tag{1}$$

Also, since $3a+k+1\equiv 0\ (\mathrm{mod}\ b)$, by Proposition 5.2 of [**3**], it follows that

$$\left\{\frac{3a+k-1}{b}\right\}=1-\frac{2}{b}\ .$$

So, by the above relation and (1), we obtain

$$\left\{\frac{3a+k}{b}\right\}=1-\frac{1}{b}\ . \tag{2}$$

However, by Lemma 2.12 we know that

$$\left\{\frac{na+k}{b}\right\}=x_n+\frac{k}{b}-\frac{1}{b}E(n,k)\ ,$$

for every $n\in\mathbb{N}$. Thus, this yields

$$\left\{\frac{3a+k}{b}\right\}=x_3+\frac{k}{b}-\frac{1}{b}E(3,k)$$

and

$$\left\{\frac{a+k}{b}\right\}=x_1+\frac{k}{b}-\frac{1}{b}E(1,k)\ .$$

Therefore,

$$x_3=\left\{\frac{3a+k}{b}\right\}-\frac{k}{b}+\frac{1}{b}E(3,k) \tag{3.1}$$

and

$$x_1=\left\{\frac{a+k}{b}\right\}-\frac{k}{b}+\frac{1}{b}E(1,k)\ . \tag{3.2}$$

But $E(3,k)=0$, since we have assumed that $3a+k+1\equiv 0 \pmod b$. Thus, $3a+j\not\equiv 0 \pmod b$, $1\le j\le k$. Otherwise $b|(3a+k+1)-(3a+j)\Rightarrow b|(k-j)+1$ and for $k\ge 1$ it holds $0\le k-j\le k-1\le b-3\Rightarrow 1\le (k-j)+1\le b-2$. If $k=0$ then by definition $E(3,0)=0$.
In addition, by Proposition 2.10 we know that

$$x_3 = 3x_1 - 1 + \frac{2}{b}S(1,a,b)\,.$$

Hence, by relations (3.1), (3.2), we obtain

$$\left\{\frac{3a+k}{b}\right\} - \frac{k}{b} + \frac{1}{b}E(3,k) = 3\left\{\frac{a+k}{b}\right\} - \frac{3k}{b} + \frac{3}{b}E(1,k) - 1 + \frac{2}{b}S(1,a,b)$$

or

$$\left\{\frac{3a+k}{b}\right\} = 3\left\{\frac{a+k}{b}\right\} - \frac{2k}{b} + \frac{3}{b}E(1,k) - 1 + \frac{2}{b}S(1,a,b).$$

Hence, by (2) we get

$$1-\frac{1}{b} = 3\left\{\frac{a+k}{b}\right\} - \frac{2k}{b} + \frac{3}{b}E(1,k) - 1 + \frac{2}{b}S(1,a,b). \tag{4}$$

But since $a\ge b$, it is clear that

$$\left\lfloor\frac{a+k}{b}\right\rfloor = \nu, \quad \nu\in\mathbb{N}\,.$$

Therefore, by (4) we get

$$\begin{aligned}1-\frac{1}{b} &= 3\left(\frac{a+k}{b}\right) - 3\nu - \frac{2k}{b} + \frac{3}{b}E(1,k) - 1 + \frac{2}{b}S(1,a,b)\\ &= 3\frac{a}{b} + \frac{k}{b} - (3\nu+1) + \frac{3}{b}E(1,k) + \frac{2}{b}S(1,a,b)\,.\end{aligned}$$

Thus

$$\frac{a}{b} = \frac{3\nu+1}{3} - \frac{k+1}{3b} - \frac{1}{b}E(1,k) - \frac{2}{3b}S(1,a,b) + \frac{1}{3}$$

or

$$3a = (3\nu+1)b - k - 1 - 3E(1,k) - 2S(1,a,b) + b$$

or

$$(3\nu+2)b = (3a+k+1) + 3E(1,k) + 2S(1,a,b)\,.$$

□

Corollary 2.14. *Let a, $b\in\mathbb{N}$, where $(a,b)=1$, $a\ge b\ge 2$ and b is even. Then*

$$2S(1,a,b)\equiv 0 \pmod b$$

and therefore $S(1,a,b)$ is an integer.

Proof. We know that $3a+k+1 \equiv 0 \pmod{b}$. Also, $E(1,k) \equiv 0 \pmod{b}$, since its terms are either b or 0. Hence,

$$2S(1,a,b) \equiv 0 \pmod{b} .$$

Since b is an even integer, it follows that $S(1,a,b) \in \mathbb{Z}$.

□

3. COMPUTING THE VALUES OF $S(1,A,B)$

By Proposition 2.10 and since $0 \le x_3 < 1$, it easily follows that

$$|S(1,a,b)| < b .$$

The above inequality and the fact that $S(1,a,b)$ is always an integer when b is even, lead us to the assumption that the values of this cotangent sum could possibly be very specific. Some numerical experiments revealed that the value of $S(1,a,b)$ was either 0 or $\pm b/2$. Hence, with some further investigation we obtained the following result.

Proposition 3.15. *Let $a,\ b \in \mathbb{N}$, where $(a,b)=1$, $a \ge b \ge 2$ and $b \ne 3$. Then*

$$S(1,a,b) = 0 \ \text{or}\ \pm b/2 .$$

Proof. By Proposition 2.13 we know that

$$2S(1,a,b) = (3\nu+2)b - (3a+k+1) - 3E(1,k) , \tag{5}$$

We can consider a, such that $1 \le a \le b-1$ due to the periodicity of $S(1,a,b)$ with period a. Thus, since $0 \le k \le b-2$ we get

$$\frac{1}{b} \le \frac{a+k}{b} \le 2 - \frac{3}{b} .$$

Therefore

$$\left\lfloor \frac{a+k}{b} \right\rfloor = 0 \text{ or } 1 .$$

In other words, $\nu = 0$ or 1. Hence, we can consider the following cases.
Case 1. If $\nu = 0$, we have

$$2S(1,a,b) = 2b - (3a+k+1) - 3E(1,k) .$$

That is

$$S(1,a,b) = b - \frac{3a+k+1}{2} - \frac{3E(1,k)}{2} .$$

Set

$$\frac{3a+k+1}{2}+\frac{3E(1,k)}{2}:=m\in\mathbb{Q}^{+}\,.$$

Then, we can write

$$S(1,a,b)=b-m\,.$$

However, we know that $|S(1,a,b)|<b$ and thus

$$|m-b|<b$$

or

$$0<m<2b\,. \tag{6}$$

But, since both $3a+k+1$ and $3E(1,k)$ are divisible by b, it follows that $2m$ is divisible by b. Therefore, we obtain

$$\frac{2m}{b}=r,\ \text{where } r\in\mathbb{N}$$

or equivalently

$$m=\frac{b}{2}\cdot r \tag{7}$$

By (6), (7) it follows that the only possible values for m are

$$m=\frac{b}{2},\ b,\ \frac{3b}{2}\,.$$

Consequently, the only possible values that $S(1,a,b)$ may obtain, in the case when $\nu=0$, are

$$S(1,a,b)=0,\ \pm\frac{b}{2}\,.$$

Case 2. If $\nu=1$, by (5) we have

$$2S(1,a,b)=5b-(3a+k+1)-3E(1,k)$$

or equivalently

$$S(1,a,b)=\frac{5b}{2}-m\,, \tag{8}$$

where m is defined as in Case 1. Thus, similarly to the case when $\nu=0$, we get

$$\left|m-\frac{5b}{2}\right|<b\,,$$

from which it follows that

$$\frac{3b}{2}<m<\frac{7b}{2}\,,\ m\in\mathbb{N}\,.$$

Additionally, we have

$$m = \frac{b}{2} \cdot r,\ r \in \mathbb{N}\,.$$

Hence, the possible values of m are

$$m = 2b,\ \frac{5b}{2},\ 3b\,.$$

Therefore, by (8) it follows that the only possible values that $S(1, a, b)$ may obtain, in the case when $\nu = 1$, are

$$S(1, a, b) = 0,\ \pm\frac{b}{2}\,.$$

□

Now that we have specified the only values which the cotangent sum $S(1, a, b)$ can obtain, an interesting question is to investigate when does this sum obtain these values. Thus, in the following we will determine the values of the integer a, for fixed b, for which $S(1, a, b) = 0,\ \pm\, b/2$, respectively.

4. THE DISTRIBUTION OF THE VALUES OF $S(1, A, B)$

The set of integer values a for which $S(1, a, b) = 0$.

By Proposition 2.13, for $S(1, a, b) = 0$ we obtain

$$(3\nu + 2)b = (3a + k + 1) + 3E(1, k)\,. \tag{9}$$

As we have illustrated in the previous sections, $\nu = 0$ or 1 and $E(1, k) = 0$ or b. Thus, we can distinguish the following cases.
Case 1 If $\nu = 0$, by (9) we get

$$2b = (3a + k + 1) + 3E(1, k)\,.$$

Hence, if $E(1, k) = 0$ then $2b = 3a + k + 1$. On the other hand, if $E(1, k) = b$ then $3a + k + 1 = -b < 0$, which is a contradiction.
Case 2. If $\nu = 1$, by (9) we obtain

$$5b = (3a + k + 1) + 3E(1, k)\,.$$

Thus, if $E(1, k) = 0$ then $5b = 3a+k+1$. But, since $1 \le a \le b-1$ and $0 \le k \le b-2$, it follows that $4 \le 3a+k+1 \le 4b-4$, which is a contradiction. If $E(1, k) = b$ then $2b = 3a + k + 1$.
Therefore, we obtain the following proposition

Proposition 4.16. *Let a, $b \in \mathbb{N}$, $b \geq 2$, where $(a,b)=1$ and $b \neq 3$. Then*

$$S(1,a,b)=0$$

if and only if $2b = 3a+k+1$, *for some* k, $0 \leq k \leq b-2$.

By the above proposition it follows that the only values of a which can be zeros of $S(1,a,b)$ are the ones for which $(a,b)=1$ and

$$\left\lceil \frac{b+1}{3} \right\rceil \leq a \leq \left\lfloor \frac{2b-1}{3} \right\rfloor .$$

Hence, we obtain the following corollary.

Corollary 4.17. *Let a, $b \in \mathbb{N}$, $b \geq 2$, where $(a,b)=1$ and $b \neq 3$. Then, the number of integers a, such that $1 \leq a \leq b-1$ and $S(1,a,b)=0$, is given by the following formula*

$$\#\{a \mid S(1,a,b)=0\} = \phi\left(b, \left\lceil \frac{b+1}{3} \right\rceil, \left\lfloor \frac{2b-1}{3} \right\rfloor\right),$$

where

$$\phi(n,A,B) = \sum_{\substack{A \leq k \leq B \\ (n,k)=1}} 1 .$$

The set of integer values a for which $S(1,a,b)=b/2$.

We shall now investigate the case when $S(1,a,b)=-b/2$, $1 \leq a \leq b-1$, $(a,b)=1$. More specifically, by Proposition 2.13 we obtain

$$(3\nu+2)b = (3a+k+1) + 3E(1,k) + b .$$

Case 1. If $\nu = 0$ we have

$$b = (3a+k+1) + 3E(1,k) .$$

Thus, if $E(1,k)=0$ then $b=3a+k+1$. If $E(1,k)=b$ then $3a+k+1=-2b<0$, which is a contradiction.
Case 2. If $\nu=1$ we have

$$4b = (3a+k+1) + 3E(1,k) .$$

Thus, if $E(1,k)=0$ then $4b = 3a+k+1 \leq 4b-4$ which is a contradiction. If $E(1,k)=b$ then $b=3a+k+1$. Therefore, from the above we obtain the following proposition.

Proposition 4.18. *Let $a,\ b \in \mathbb{N}$, $b \geq 2$, where $(a,b) = 1$ and $b \neq 3$. Then*

$$S(1,a,b) = \frac{b}{2}$$

if and only if $b = 3a + k + 1$, for some k, $0 \leq k \leq b-2$.

By the above proposition it follows that the only values of a for which $S(1,a,b) = b/2$ are the ones for which $(a,b) = 1$ and

$$1 \leq a \leq \left\lfloor \frac{b-1}{3} \right\rfloor .$$

Hence, we obtain the following corollary.

Corollary 4.19. *Let $a,\ b \in \mathbb{N}$, $b \geq 2$, where $(a,b) = 1$ and $b \neq 3$. Then, the number of integers a, such that $1 \leq a \leq b-1$ and $S(1,a,b) = b/2$, is given by the following formula*

$$\#\left\{a \mid S(1,a,b) = \frac{b}{2}\right\} = \phi\left(b, 1, \left\lfloor \frac{b-1}{3} \right\rfloor\right).$$

The set of integer values a for which $S(1,a,b) = -b/2$.

We shall now investigate the final case when $S(1,a,b) = -b/2$, $1 \leq a \leq b-1$, $(a,b) = 1$. Again by Proposition 2.13 we obtain

$$(3\nu + 2)b = (3a + k + 1) + 3E(1,k) - b\,.$$

Case 1. If $\nu = 0$ we have

$$3b = (3a + k + 1) + 3E(1,k)\,.$$

Thus, if $E(1,k) = 0$ then $3b = 3a + k + 1$. If $E(1,k) = b$ then $3a + k + 1 = 0$, which is a contradiction.
Case 2. If $\nu = 1$ we have

$$6b = (3a + k + 1) + 3E(1,k)\,.$$

Thus, if $E(1,k) = 0$ then $6b = 3a + k + 1 \leq 4b - 4$ from which we get $2b \leq -4$ which is a contradiction. If $E(1,k) = b$ then $3b = 3a + k + 1$. Therefore, from the above we obtain the following proposition.

Proposition 4.20. *Let $a,\ b \in \mathbb{N}$, $b \geq 2$, where $(a,b) = 1$ and $b \neq 3$. Then*

$$S(1,a,b) = -\frac{b}{2}$$

if and only if $3b = 3a + k + 1$, for some k, $0 \leq k \leq b-2$.

By the above proposition it follows that the only values of a for which $S(1,a,b) = -b/2$ are the ones for which $(a,b)=1$ and

$$\left\lceil \frac{2b+1}{3} \right\rceil \le a \le \left\lfloor \frac{3b-1}{3} \right\rfloor .$$

Hence, we obtain the following corollary.

Corollary 4.21. *Let a, $b \in \mathbb{N}$, $b \ge 2$, where $(a,b)=1$ and $b \ne 3$. Then, the number of integers a, such that $1 \le a \le b-1$ and $S(1,a,b) = -b/2$, is given by the following formula*

$$\#\left\{a \mid S(1,a,b) = -\frac{b}{2}\right\} = \phi\left(b, \left\lceil \frac{2b+1}{3} \right\rceil, \left\lfloor \frac{3b-1}{3} \right\rfloor\right).$$

5. THE FUNCTION $\phi(N,A,B)$

Lemma 5.22. *Let $A, B \in \mathbb{N}$ and*

$$\phi(n,A,B) := \sum_{\substack{A \le k \le B \\ (n,k)=1}} 1 .$$

Then, we have

$$\phi(n,A,B) = \sum_{d \mid n} \mu(d) \left(\left\lfloor \frac{B}{d} \right\rfloor - \left\lceil \frac{A}{d} \right\rceil \right) ,$$

where $\mu(n)$ is the Möbius function.

Proof. We know that

$$\sum_{d \mid N} \mu(d) = \left\lfloor \frac{1}{N} \right\rfloor .$$

Thus, we get

$$\phi(n,A,B) = \sum_{k=A}^{B} \sum_{d \mid (n,k)} \mu(d) = \sum_{k=A}^{B} \sum_{\substack{d \mid n \\ d \mid k}} \mu(d) .$$

Hence, $k = md$, for some $m \in \mathbb{N}$. But, since $A \le k \le B$ it follows that

$$\frac{A}{d} \le m \le \frac{B}{d} .$$

Therefore, we obtain

$$\begin{aligned}\phi(n,A,B) &= \sum_{d|n}\sum_{m=\lceil A/d\rceil}^{\lfloor B/d\rfloor}\mu(d)\\ &= \sum_{d|n}\mu(d)\sum_{m=\lceil A/d\rceil}^{\lfloor B/d\rfloor}1\\ &= \sum_{d|n}\mu(d)\left(\left\lfloor\frac{B}{d}\right\rfloor-\left\lceil\frac{A}{d}\right\rceil\right).\end{aligned}$$

□

Proposition 5.23. *Let $n,\ A,\ B\in\mathbb{N},\ n>1$. Then, we have*

$$\phi(n,A,B)=\frac{B-A}{n}\phi(n)+\delta_{n,A}+O\left(\sum_{d|n}\mu(d)^2\right),$$

where $\delta_{n,A}=1$ if $(n,A)=1$ and 0 otherwise.

Proof. By Lemma 5.22 we get

$$\phi(n,A,B) = \sum_{d|n}\mu(d)\left\lfloor\frac{B}{d}\right\rfloor-\sum_{d|n}\mu(d)\left\lceil\frac{A}{d}\right\rceil.$$

However, since

$$\left\lceil\frac{A}{d}\right\rceil=\begin{cases}\left\lfloor\frac{A}{d}\right\rfloor+1\,, & \text{if } d\nmid A\\ \left\lfloor\frac{A}{d}\right\rfloor\,, & \text{otherwise}\,,\end{cases}$$

it follows that

$$\begin{aligned}\phi(n,A,B) &= \sum_{d|n}\mu(d)\left\lfloor\frac{B}{d}\right\rfloor-\sum_{d|n}\mu(d)\left(\left\lfloor\frac{A}{d}\right\rfloor+1\right)+\sum_{\substack{d|n\\ d|A}}\mu(d)\\ &= \phi(n,1,B)-\phi(n,1,A)-\sum_{d|n}\mu(d)+\sum_{\substack{d|n\\ d|A}}\mu(d).\end{aligned}$$

However, it is a well known fact that for $n>1$ it holds $\sum_{d|n}\mu(d)=0$. Additionally, one can easily show that

$$\sum_{\substack{d|n\\ d|A}}\mu(d)=\begin{cases}1\,, & \text{if } (n,A)=1\\ 0\,, & \text{otherwise}\,.\end{cases}$$

Therefore, we obtain

$$\phi(n,A,B) = \phi(n,1,B) - \phi(n,1,A) + \delta_{n,A}\,. \tag{10}$$

The function $\phi(n,1,x)$, $x \in \mathbb{R}^+$ is exactly the so-called Legendre totient function . Generally, we have

$$\begin{aligned}
\phi(n,1,x) &= \sum_{d|n} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor = \sum_{d|n} \mu(d) \left(\frac{x}{d} + O(1) \right) \\
&= x \sum_{d|n} \frac{\mu(d)}{d} + O\left(\sum_{d|n} \mu(d)^2 \right) \\
&= x \frac{\phi(n)}{n} + O\left(\sum_{d|n} \mu(d)^2 \right).
\end{aligned}$$

Hence, by (10) we obtain the desired result.

□

The above proposition presents an approximation formula for the generalized totient function $\phi(n,A,B)$ up to the error

$$\sum_{d|n} \mu(d)^2 = 2^{\omega(n)}.$$

However, it is a known fact that for every positive integer n and every $\epsilon > 0$, we have

$$2^{\omega(n)} \leq d(n) \ll_\epsilon n^\epsilon,$$

where $d(n)$ denotes the number of positive divisors of n (for relevant properties of $d(n)$ cf. [**1**]).
This demonstrates that the error term in Proposition 5.23 is relatively small.

Based just on the definition of the function $\phi(n,A,B)$ we can also prove the following two propositions.

Proposition 5.24. *For every n, A, $B \in \mathbb{N}$, we have*

$$\sum_{d|n} \phi\left(d, \frac{A}{d}, \frac{B}{d}\right) = B - A + 1\,.$$

Proof. We consider the sets

$$N(A,B) := \{A, A+1, \ldots, B-1, B\}$$

and

$$R(n,d\,;A,B) := \{m \,:\, (m,n) = d, A \leq m \leq B\}\,.$$

It is evident that each set $R(n,d\,;A,B)$ is a subset of $N(A,B)$, containing those elements which have greatest common divisor d with n. Since the sets $R(n,d;A,B)$ are mutually disjoint for different values of d, it follows that

$$\sum_{d|n} |R(n,d\,;A,B)| = B - A + 1. \tag{11}$$

However, since $(m,n) = d$ is equivalent to $(m/d, n/d) = 1$ and the inequality $A \le m \le B$ is equivalent to $A/d \le m/d \le B/d$, by setting $r := m/d$ it follows that

$$|R(n,d\,;A,B)| = \left| \left\{ r \,:\, \left(r, \frac{n}{d}\right) = 1, \frac{A}{d} \le r \le \frac{B}{d} \right\} \right| = \phi\left(\frac{n}{d}, \frac{A}{d}, \frac{B}{d}\right).$$

Therefore, we obtain

$$\sum_{d|n} |R(n,d\,;A,B)| = \sum_{d|n} \phi\left(\frac{n}{d}, \frac{A}{d}, \frac{B}{d}\right) = \sum_{d|n} \phi\left(d, \frac{A}{d}, \frac{B}{d}\right)$$

and hence, by (11) the desired result follows.

□

Proposition 5.25. *For every n, A, $B \in \mathbb{N}$, we have*

$$\sum_{\substack{A \le k \le B \\ (k,n)=1}} k = \frac{n}{2}\,\phi(n,A,B).$$

Proof. Let $k_1, k_2, \ldots, k_{\phi(n,A,B)}$ be the integers such that $A \le k_i \le B$, $(k_i, n) = 1$. Since $(k_i, n) = 1$ is equivalent to $(n - k_i, n) = 1$, it is evident that

$$\begin{aligned} k_1 + k_2 + \cdots + k_{\phi(n,A,B)} &= (n - k_1) + (n - k_2) + \cdots + (n - k_{\phi(n,A,B)}) \\ &= n\phi(n,A,B) - (k_1 + k_2 + \cdots + k_{\phi(n,A,B)}), \end{aligned}$$

from which the desired result follows. □

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