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EXTENSION OF GENERALIZED INTEGRO-EXPONENTIAL FUNCTION AND ITS APPLICATION IN STUDY OF CHEN DISTRIBUTION

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In 2000 Chen introduced a two-parameter lifetime model and has reported only a few mathematical properties moments, quantile and generating functions, among others. In this article, we derive a power series expansion for newly introduced real upper parameter generalized integro–exponential function $E_s^p(z)$ extending certain Milgram's findings. By our novel results we derive closed–form expressions for the moments, generating function, Rényi entropy and power series for the quantile function of the Chen distribution.

1. INTRODUCTION

The lifetime models exhibiting either monotone (increasing and decreasing) or non-monotone (bathtub and upside-down bathtub) failure rate properties have wide applications *inter alia* in the fields of engineering, life-sciences, medicine and health sciences, actuaries. The models exhibiting bathtub-shaped failure rate are very important to study the lifetime of electro-mechanical, electronic and mechanical products since they often show such kind of failure rate characteristic. The gamma, Weibull, linear failure rate and their extensions are amongst the most popular lifetime models. Chen [**1**] also proposed a useful lifetime model that exhibits most of the failure rate behavior.

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A random variable (RV) T defined on a standard probability space $(\Omega, \mathscr{F}, \mathsf{P})$ is said to have the *two-parameter Chen distribution* (abbreviated in the sequel as 'Chen distribution') if its cumulative distribution function (CDF) $F(x)$ and the related probability density function (PDF) $f(x)$ are, respectively, given by

$$F(x) = \left(1 - \exp\left\{a(1 - \mathrm{e}^{x^\beta})\right\}\right) \cdot 1_{\mathbb{R}_+}(x)$$

and

$$f(x) = a\,\beta x^{\beta-1} \exp\left\{x^\beta + a(1 - \mathrm{e}^{x^\beta})\right\} \cdot 1_{\mathbb{R}_+}(x)\,, \tag{1}$$

where $a > 0$ is the scale parameter, $\beta > 0$ is the shape parameter and $1_S(x)$ stands for the indicator of the event $x \in S$. That T follows the Chen distribution we shall denote $T \sim \mathrm{Chen}(a, \beta)$ throughout.

Xie *et al.* considered a slightly extended Chen's model studying the RV $\Xi_1 = \alpha T, \alpha > 0$ which CDF reads [**22**, p. 280, Eq. (3)]

$$F_1(x; \alpha) = F\Big(\frac{x}{\alpha}\Big) = \left(1 - \exp\left\{\lambda\alpha(1 - \mathrm{e}^{\left(\frac{x}{\alpha}\right)^\beta})\right\}\right) \cdot 1_{\mathbb{R}_+}(x)\,,$$

where $\lambda, \beta > 0$. Here, the scaling parameter α gives more flexibility with respect to the Chen model, but choosing $\lambda\alpha = a$ the probabilistic structure does not change substantially; for $\alpha = 1$ these distributions coincide. However, the mathematical properties of the distribution were not studied in detail. Finally, it trivially follows $\mathsf{E}\Xi_1^\gamma = \alpha^\gamma\,\mathsf{E}T^\gamma$ for some general fractional γ, so the general moment identification for $T \sim \mathrm{Chen}(a, \beta)$ uniquely solves the problem for the model by Xie *et al.*

On the other hand, the RV $\Xi_2 = T^{1/\beta}, \beta > 0$ behaves according to the familiar (truncated) Gompertz distribution with the CDF

$$F_2(x; \alpha, \beta) = F\Big[\Big(\frac{x}{\alpha}\Big)^{\frac{1}{\beta}}\Big]\,.$$

Indeed, the RV $T \sim \mathrm{Chen}(a, \beta)$ is actually the βth power of the Gompertz distribution. The integer order moments for the full support, non–truncated Gompertz RV are established by Lenart [**7**, p. 257, Proposition 2] in terms of the so–called *generalized integro–exponential function* considered by Milgram [**11**, p. 444, Eq. (2.3)]

$$E_s^j(z) = \frac{(-1)^j}{j!}\Big(\frac{\partial}{\partial s}\Big)^j z^{s-1}\,\Gamma(1 - s, z) \tag{2}$$

$$= \frac{1}{\Gamma(j+1)}\int_1^\infty (\ln x)^j x^{-s}\mathrm{e}^{-z\,x}\,\mathrm{d}x, \qquad j \in \mathbb{N}_0; s, z \in \mathbb{C}\,, \tag{3}$$

where $\Gamma(a, b) = \int_b^\infty x^{a-1}\,\mathrm{e}^{-x}\,\mathrm{d}x$ stands for the *upper incomplete Gamma* function. Lenart applied the case $s = 1$ for deriving the mean, variance, skewness and kurtosis in terms of generalized hypergeometric function ${}_3F_3$ and Meijer G function, but derives also approximations for these values, see [**7**, p. 259 *et seq.*].

In his introductory paper [**1**] Chen considered interval estimation and joint confidence regions, maximum likelihood estimation, but mentioned that the kth moment is actually expressible in terms of the integral $\Psi(a; k\beta^{-1})$, compare [**1**, p. 161]. On the other hand, Xie *et al.* avoided to give any novel expressions for moments only remarking that numerical integration can be used in calculating the mean and the variance.

Obviously, the RV $T \sim \mathrm{Chen}(a, \beta)$ generates the RVs Ξ_1 and Ξ_2 as well and there are further mathematical properties which are not yet covered.

The rth moment of Chen RV is defined by

$$\mu'_r = \mathsf{E}\, T^r = \int_0^\infty x^r\, f(x)\,\mathrm{d}x = a\,\beta \int_0^\infty x^r\, x^{\beta-1}\, \exp\left\{x^\beta + a(1-\mathrm{e}^{x^\beta})\right\}\,\mathrm{d}x.$$

Substitute $\exp\{x^\beta\} \mapsto x$. So, the rth moment of T *via* (3) becomes

$$\mu'_r = a\,\mathrm{e}^a \int_1^\infty \mathrm{e}^{-a\,x}\,(\ln x)^{r\beta^{-1}}\,\mathrm{d}x = a\,\mathrm{e}^a\,\Gamma(r\beta^{-1}+1)\,E_0^{r\beta^{-1}}(a), \qquad r>0. \tag{4}$$

For the sake of simplicity, we will write $p = r\beta^{-1} > 0$ in the sequel. [1]

The rth incomplete moment is defined by

$$m_r(w) = \int_0^w x^r\, f(x)\,\mathrm{d}x = a\,\mathrm{e}^a \int_1^w \mathrm{e}^{-a\,x}\,(\ln x)^p\,\mathrm{d}x, \qquad w>0. \tag{5}$$

The moment generating function (MGF) of the Chen distribution is defined by

$$\mathbb{M}_T(s) = \mathsf{E}\,\mathrm{e}^{-sT} = \int_0^\infty \mathrm{e}^{-sT}\, f(x)\,\mathrm{d}x = a\,\mathrm{e}^a \int_1^\infty \mathrm{e}^{-s(\ln\,x)^{\beta^{-1}} - a\,x}\,\mathrm{d}x, \quad s>0.$$

Finally, the quantile function (QF) of T takes the form

$$Q(p; a, \beta) = \left[\ln\left\{1 - \ln(1-p)^{a^{-1}}\right\}\right]^{\beta^{-1}} \cdot \mathsf{1}_{(0,1)}(p)\,.$$

This paper is unfolded as follows. In Section 2, firstly we derive the triple power series expansion of the generalized integro–exponential function $E_s^p(z)$ for real $p > -1$ and complex $s, z \in \mathbb{C}$ significantly extending the region of validity from the nonnegative integer $p \in \mathbb{N}_0$ which is in detail presented and discussed by Milgram [**11**].

One of the by–products of our result upon $E_s^p(z)$ is the expression for non-negative real order moment for the RV T behaving according to the Chen distribution. Secondly, we extend Nadarajah's derivative formula for the integer order moments of RV $T \sim \mathrm{Chen}(a, \beta)$ to the moments of non-negative real order, which results form the Section 3. The incomplete moments are presented in Section 4. We obtain the moment generating function in Section 5, while the Section 6 is devoted

[1] The parameter p denotes not necessarily the same throughout. However, there is no doubt which role is playing in the situation considered.

to the Rényi entropy expression, which was derived by using the results presented upon $E_s^p(z)$ in the Section 2. The computational power series expansion for the quantile function is in the focus of considerations in the Section 7. The exposition is closed in the Conclusion section 8.

The established analytical expressions containing special functions to determine some structural properties for Chen distribution which are suitable for computing them directly instead of the classical approach, which was the numerical integration of the related PDF. Analytical facilities available in programming softwares like Ox, Mathematica, Maple and Matlab can substantially contribute to use these results in practice. We end the exposition in Section 7 with certain concluding remarks.

2. EXTENSION OF GENERALIZED INTEGRO–EXPONENTIAL FUNCTION

Our main goal in this section is giving full consideration to the *generalized integro–exponential function* having upper p parameter

$$E_s^p(z) = \frac{1}{\Gamma(p+1)} \int_1^\infty (\ln x)^p x^{-s} \mathrm{e}^{-z\,x}\,\mathrm{d}x, \qquad \Re(p) > -1; s, z \in \mathbb{C}\,. \tag{6}$$

The derived power series form of this integral which upper parameter satisfies $\Re(p) > -1$, becomes the main tool in yielding the moments and incomplete moments of positive real order for the Chen distributed RV T. By this result we substantially extend the existing Milgram's results for $p \in \mathbb{N}$ (for which compare [**11**] and the references therein).

The first mathematical tool we need here is the Lin–Srivastava generalized Hurwitz–Lerch Zeta (HLZ) function which series definition reads [**8**, p. 727, Eq. (8)], [**20**, p. 489, Eq. (1.10)]

$$\Phi_{\mu,\nu}^{(\rho,\sigma)}(z,s,u) = \sum_{n\geq 0} \frac{(\mu)_{\rho n}}{(\nu)_{\sigma n}} \frac{z^n}{(n+u)^s}\,,$$

where $\mu \in \mathbb{C}$, $\nu, u \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $\rho, \sigma \in \mathbb{R}_+$ and $\rho < \sigma$ when $z, s \in \mathbb{C}$ (for further convergence details see e.g. [**20**, p. 489]). Here

$$(\lambda)_\eta = \frac{\Gamma(\lambda+\eta)}{\Gamma(\lambda)}, \qquad \lambda \in \mathbb{C} \setminus \{0\}$$

denotes the generalized Pochhammer symbol, where by convention $(0)_0 = 1$. Thus,

$$\Phi_{\mu,1}^{(0,1)}(-a, p+1, 1) = \lim_{\rho\to 0} \Phi_{\mu,1}^{(\rho,1)}(-a, p+1, 1) = \sum_{n\geq 0} \frac{(-a)^n}{n!\,(n+1)^{p+1}}\,. \tag{7}$$

Next, we determine the power series of $E_s^p(z)$. By setting $(1-x)^{-1} \mapsto x$ in the integrand of (6), we obtain

$$E_s^p(z) = \frac{(-1)^p}{\Gamma(p+1)} \int_0^1 (1-x)^{-s-2} \ln^p(1-x) \cdot \exp\Big\{ -\frac{z}{1-x} \Big\} \mathrm{d}x.$$

By expanding the exponential and the binomial terms and interchange the sums and the integral it follows

$$E_s^p(z) = \frac{(-1)^p}{\Gamma(p+1)} \sum_{n,k\geq 0} \frac{(-1)^k (-z)^n}{n!} \binom{-n-s-2}{k} \underbrace{\int_0^1 x^k \ln^p(1-x)\,\mathrm{d}x}_{J_k(p)}.$$

This moves are enabled with the dominated convergence theorem having in mind

$$\big|(-1)^p J_k(p)\big| \leq \int_0^1 \Big(\ln \frac{1}{1-x}\Big)^{\Re(p)} \mathrm{d}x = \Gamma\left(\Re(p)+1\right), \qquad \Re(p)+1>0\,.$$

The inner integral $J_k(p)$ can be handled by substituting $1-\mathrm{e}^{-x} \mapsto x$:

$$J_k(p) = \int_0^\infty (1-\mathrm{e}^{-x})^k \,\mathrm{e}^{-x}\,(-x)^p\,\mathrm{d}x = (-1)^p \sum_{m=0}^k (-1)^m \binom{k}{m} \int_0^\infty \mathrm{e}^{-(m+1)x}\,x^p\,\mathrm{d}x$$

$$= (-1)^p \sum_{m=0}^k \binom{k}{m} \frac{(-1)^m \Gamma(p+1)}{(m+1)^{p+1}} = (-1)^p \sum_{m=0}^k \frac{\Gamma(p+1)(-k)_m}{m!\,(m+1)^{p+1}}\,, \tag{8}$$

where both sums in (8) can be used for exact numerical evaluation. Moreover, considering the last equation, we arrive at the more compactly written expression

$$J_k(p) = (-1)^p \Gamma(p+1)\,\Phi_{\mu,1}^{(0,1)}(-k, p+1, 1), \qquad \mu \in \mathbb{C}, \tag{9}$$

where $\Phi_{\mu,1}^{(0,1)}(-k,p+1,1)$ is the k–partial sum of the HL Zeta function (7).

Remark 2.1. *It is worth to mention that the case of non-negative integer p for $J_k(p)$ is well documented. Namely, the formula (transformed into our setting)*

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^p \frac{1}{(x+1)_{k-1}} = \frac{(-1)^p\,p!}{k!} \sum_{m=0}^k \frac{(-k)_m}{m!\,(m+x+1)^{p+1}}, \qquad p \in \mathbb{N}_0$$

occurs in the monograph by the father of Hungarian probabilistic school Charles (Károly) Jordan [**5**, p. 337]. *Also consult* [**4**, p. 139, Eq. (6.7.47)]. ■

The relation (9) leads to

$$\begin{aligned} E_s^p(z) &= \sum_{n,k\geq 0} \frac{(-1)^k(-z)^n}{n!}\binom{-n-s-2}{k}\Phi_{\mu,1}^{(0,1)}(-k,p+1,1) \\ &= \sum_{n,k\geq 0} \frac{(s+2)_{n+k}}{k!\,(s+2)_k}\frac{(-z)^n}{n!}\,\Phi_{\mu,1}^{(0,1)}(-k,p+1,1) \\ &= \sum_{k\geq 0} \frac{(s+2)_k}{k!}\,\Phi_{\mu,1}^{(0,1)}(-k,p+1,1)\sum_{n\geq 0}\frac{(s+2+k)_n}{(s+2)_n}\,\frac{(-z)^n}{n!} \\ &= \sum_{k\geq 0} \frac{(s+2)_k}{k!}\,\Phi_{\mu,1}^{(0,1)}(-k,p+1,1)\,{}_1F_1(s+k+2;s+2;-z)\,, \end{aligned}$$

where the *confluent hypergeometric function - Kummer's function* [**6**, p. 29, Eq. (1.6.14)]

$${}_1F_1(a;b;x) = \sum_{n\geq 0}\frac{(a)_n}{(b)_n}\frac{x^n}{n!}, \qquad x,a\in\mathbb{C};\ b\in\mathbb{C}\setminus\mathbb{Z}_0^-$$

is employed. Hence the desired result.

Theorem 2.2. *For all $\Re(p) > -1, s, z \in \mathbb{C}$, there hold true the triple power series expansion formulae of the generalized integro–exponential function*

$$E_s^p(z) = \sum_{k\geq 0} \frac{(s+2)_k}{k!}\,\Phi_{\mu,1}^{(0,1)}(-k,p+1,1)\,{}_1F_1(s+k+2;s+2;-z)\,. \tag{10}$$

Remark 2.3. *We point out that the closed form expression of $E_s^p(z)$ in the case of nonnegative integer p is reported by* Milgram [**11**, p. 444, (2.7b)] *in terms of the Meijer G function* [**10**], [**6**, p. 63, Eq. (1.12.51)]*; also consult the adequate formulae in* [**16**, pp. 100–103, §2.5.1–2]. *However, to the best of our knowledge, results for general real positive p are not yet published.* ■

3. MOMENTS

Consider the random variable X defined on a standard probability space $(\Omega, \mathscr{F}, \mathsf{P})$. Let $X \sim \mathscr{E}(a), a > 0$, that is, let X be exponentially distributed. Define now the RV

$$Y = (\ln X)_+^p, \qquad p > 0; \quad (u)_+ := \max\{0, u\}.$$

Then, the expectation and the variance of Y are written as

$$\mathsf{E}(Y) = a\int_1^\infty \mathrm{e}^{-ax}\,(\ln x)^p\,\mathrm{d}x = a\,E_0^p(a)\,,$$

and

$$\begin{aligned}\mathsf{Var}(Y) &= a\int_1^\infty \mathrm{e}^{-ax}\,(\ln x)^{2p}\,\mathrm{d}x - a^2\left(\int_1^\infty \mathrm{e}^{-ax}\,(\ln x)^{p}\,\mathrm{d}x\right)^2\\ &= a\,E_0^{2p}(a) - a^2\left[E_0^p(a)\right]^2,\end{aligned}$$

where both expressions can be presented as power series based on Theorem 2.2.

Our next step is to obtain the rth moment of the random variable $T \sim \mathrm{Chen}(a,\beta)$. Bearing in mind the related PDF (1) and substituting $\exp\{x^\beta\} \mapsto x$, we have

$$\begin{aligned}\mathsf{E}\,T^r &= a\,\beta\int_0^\infty x^{r+\beta-1}\,\exp\left\{x^\beta + a(1-\mathrm{e}^{x^\beta})\right\}\mathrm{d}x\\ &= a\,\mathrm{e}^a\int_1^\infty (\ln x)^{r\beta^{-1}}\,\mathrm{e}^{-ax}\,\mathrm{d}x = a\,\mathrm{e}^a\,E_0^{r\beta^{-1}}(a)\,.\end{aligned}$$

By virtue of Theorem 2.2. the following result is inferred.

Theorem 3.4. *Let the* RV $T \sim \mathrm{Chen}(a,\beta)$; $a,\beta>0; \mu\in\mathbb{C}$. *Then, the rth positive order moment reduces to*

$$\mu_r' = \mathsf{E}T^r = a\,\mathrm{e}^a\sum_{k\geq 0}\frac{(2)_k}{k!}\,\Phi_{\mu,1}^{(0,1)}\big(-k, r\beta^{-1}+1, 1\big)\,{}_1F_1(k+2;2;-a)\,. \tag{11}$$

3.1 The Grünwald–Letnikov approach

In this section, we complement to the any positive real order the derivative formula result of Nadarajah [**14**, p. 115, Theorem 1] by deriving explicit algebraic formulae for the kth moment of the $\mathrm{Chen}(a,\beta)$ distribution *viz.*

$$\mathsf{E}\,\Xi_1^k = n\alpha^k\,\mathrm{e}^{\lambda\alpha}\left(\frac{\partial}{\partial s}\right)^{n-1}\frac{\Gamma(s,\lambda\alpha)}{(\lambda\alpha)^s}\Big|_{s=0},$$

for $k/n=\beta$ positive rational number, being $k,n\in\mathbb{N}$.[2] Our below exposed Grünwald–Letnikov fractional derivative results fill the gap between positive rational and real powers in moments' order.

A more elegant approach in calculating $I_1(a,p)$ (and *a fortiori* the depending moment) can be offered by including the Grünwald–Letnikov fractional derivative operator of the order $p>0$, which definition reads [**17**, §20], [**6**, p. 121 *et seq.*]

$$\mathbb{D}_x^p\big[f\big] = \lim_{n\to\infty}\left(\frac{n}{x-a}\right)^p\sum_{m=0}^n\frac{(p)_m}{m!}\,f\left(x - m\,\frac{x-a}{n}\right), \qquad x>a.$$

[2] The traces of the derivation formula employed in [**14**] we follow back to McLachlan *et al.* [**9**, p. 26]; it concerns the Laplace–transform of the function $x\mapsto x^{\nu-1}(\ln x)^m\cdot\mathbf{1}_{(1,\infty)}(x)$; $m\in\mathbb{N},\nu>0$.

We point out that several numerical algorithms are available for the direct computation of fractional expressions: see, for instance Diethelm *et al.* [**3**], Murio [**13**] and Sousa [**19**].

As it is well–known [**15**] the Grünwald–Letnikov fractional derivative $\mathbb{D}_x^p$ of order $p > 0$ of the exponential function is

$$\mathbb{D}_x^p\left[\mathrm{e}^{\alpha x}\right] = \alpha^p \mathrm{e}^{\alpha x}\,;$$

therefore

$$\mathbb{D}_t^p\left[\mathrm{e}^{t\ln x}\right]_{t=0} = (\ln x)^p\, x^t\big|_{t=0} = (\ln x)^p, \qquad x > 1.$$

So, we obtain

$$\begin{aligned} I_1(a,p) &= \int_1^\infty \mathrm{e}^{-ax}\,(\ln x)^p\,\mathrm{d}x = \int_1^\infty \mathrm{e}^{-ax}\,\mathbb{D}_t^p\left[\mathrm{e}^{t\ln x}\right]_{t=0}\mathrm{d}x \\ &= \mathbb{D}_t^p\left[\int_1^\infty x^t\,\mathrm{e}^{-ax}\mathrm{d}x\right]_{t=0} = \mathbb{D}_t^p\left[\frac{\Gamma(t+1,a)}{a^{t+1}}\right]_{t=0}. \end{aligned}$$

Immediately, we have

$$E_0^p(a) = \frac{1}{\Gamma(p+1)}\,\mathbb{D}_t^p\left[\frac{\Gamma(t+1,a)}{a^{t+1}}\right]_{t=0}.$$

The by-product is the next fractional derivative representation formula of independent interest.

Proposition 3.5. *Let $a > 0, p > -1$ and $\mu \in \mathbb{C}$. Then, we obtain*

$$\mathbb{D}_t^p\left[\frac{\Gamma(t+1,a)}{a^{t+1}}\right]_{t=0} = \Gamma(p+1)\sum_{k\geq 0}\frac{(2)_k}{k!}\,\Phi_{\mu,1}^{(0,1)}(-k,p+1,1)\cdot {}_1F_1(k+2;2;-a).$$

Using Proposition 3.5 and equation (4) the another rth moment expression for Chen distribution follows.

Theorem 3.6. *Let the situation be the same as in* Theorem 2. *Then, we have*

$$\mu_r' = a\,\mathrm{e}^a\,\mathbb{D}_t^{r\beta^{-1}}\left[\frac{\Gamma(t+1,a)}{a^{t+1}}\right]_{t=0}. \tag{12}$$

Equations (11) and (12) are the main results of this section. Finally, the central moments (μ_s) and cumulants (κ_s) for the Chen distribution are

$$\mu_s = \sum_{k=0}^{s}(-1)^k\binom{s}{k}\mu_1'^s\,\mu_{s-k}'; \quad \kappa_s = \mu_s' - \sum_{k=1}^{s-1}\binom{s-1}{k-1}\kappa_k\,\mu_{s-k}', \qquad s\in\mathbb{N},$$

respectively, where $\kappa_1 = \mu_1'$. Thus, $\kappa_2 = \mu_2' - \mu_1'^2$, $\kappa_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$, $\kappa_4 = \mu_4' - 4\mu_3'\mu_1' - 3\mu_2'^2 + 12\mu_2'\mu_1'^2 - 6\mu_1'^4$, etc. The skewness $\gamma_1 = \kappa_3\,\kappa_2^{-3/2}$ and kurtosis $\gamma_2 = \kappa_4\,\kappa_2^{-2}$ can be calculated from the third and fourth standardized cumulants.

4. INCOMPLETE MOMENTS

In order to establish the Chen distribution's rth incomplete moment (5), we follow the lines of method used in derivation of the auxiliary integral $I_1(a,p)$. Now, we have to determine

$$H(q;a,p)=\int_1^q \mathrm{e}^{-ax}\,(\ln x)^p\,\mathrm{d}x,\qquad q>1.$$

Setting $x=(1-y)^{-1}$ gives

$$H(q;a,p)=(-1)^p\int_0^{1-q^{-1}} \mathrm{e}^{-\frac{a}{1-y}}\,\ln^p(1-y)\,\frac{\mathrm{d}y}{(1-y)^2}\,.$$

Expanding the exponential term in power series, exchanging the order of integration and summation, and then expanding the binomial in power series, we obtain

$$H(q;a,p)=(-1)^p\sum_{n,k\geq 0}\frac{(2)_{n+k}}{(2)_n}\,\frac{(-a)^n}{n!\,k!}\underbrace{\int_0^{1-q^{-1}} y^k\,\ln^p(1-y)\,\mathrm{d}y}_{A_k(q;p)}.$$

Letting $y=1-\mathrm{e}^{-u}$ in $A_k(q;p)$ yields

$$\begin{aligned}A_k(q;p)&=(-1)^p\int_0^{\infty}(1-\mathrm{e}^{-u})^k\,\mathrm{e}^{-u}\,u^p\,\mathrm{d}u\\&=(-1)^p\sum_{m=0}^{k}(-1)^m\binom{k}{m}\int_0^{1-1/q}u^p\,\mathrm{e}^{-(m+1)u}\,\mathrm{d}u\\&=(-1)^p\sum_{m=0}^{k}\frac{(-1)^m}{(m+1)^{p+1}}\binom{k}{m}\gamma(p,(q-1)(m+1)/q)\,,\end{aligned}$$

where the *lower incomplete gamma* function $\gamma(p,y)=\Gamma(p)-\Gamma(p,y)$ is used. Hence,

$$H(q;a,p)=\sum_{n,k\geq 0}\sum_{m=0}^{k}\frac{(2)_{n+k}}{(2)_n}\,\frac{(-1)^{m+n}\,a^n}{n!\,k!(m+1)^{p+1}}\binom{k}{m}\gamma\left(p,(1-q^{-1})(m+1)\right).\tag{13}$$

Finally, by (5) and (13) we deduce the desired result.

Theorem 4.7. *For all $r,\beta,w>0$ the rth general order incomplete moment for Chen distribution* Chen(a,β) *reads as follows:*

$$m_r(w)=a\,\mathrm{e}^a\sum_{n,k\geq 0}\sum_{m=0}^{k}\frac{(2)_{n+k}}{(2)_n}\,\frac{(-1)^{m+n}\,a^n\binom{k}{m}}{n!\,k!(m+1)^{r\beta^{-1}+1}}\,\gamma(r\beta^{-1},(1-w^{-1})(m+1)).\tag{14}$$

The first incomplete moment $m_1(w)$ of $T \sim \mathrm{Chen}(a, \beta)$ has several applications. The first one is related to the mean residual life and the mean waiting time (also known as mean inactivity time) given by $v_1(t) = [1 - m_1(t)][1 - F(t)]^{-1} - t$ and $V_1(t) = t - m_1(t)(F(t))^{-1}$, respectively. The mean residual life $v_1(t)$represents the expected additional life length for a unit which is alive at age t, whereas the the mean waiting time $V_1(t)$ represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0, t)$.

The second application of $m_1(w)$ refers to the mean deviations about the mean ($\delta_1 = \mathsf{E}|T - \mu_1'|$) and about the median ($\delta_2 = \mathsf{E}|T - M|$) of T given by

$$\delta_1 = 2\mu_1' F(\mu_1') - 2m_1(\mu_1') \qquad \text{and} \qquad \delta_2 = \mu_1' - 2m_1(M),$$

respectively, where $= \left[\ln\left\{1 + \ln 2^{a^{-1}}\right\}\right]^{\beta^{-1}}$ is the median of T, $\mu_1' = \mathsf{E}(T)$ can follow from (11) and (12), and $m_1(w)$ can be determined from (14) with $r = 1$.

The third application of the first incomplete moment is to obtain the Bonferroni and Lorenz curves, which are very useful in economics, reliability, demography, insurance and medicine. For a given probability p, the Bonferroni and Lorenz curves are given by $B(p) = m_1(\pi)/(\pi\mu_1')$ and $L(\pi) = m_1(\pi)/\mu_1'$, and the $\pi = Q(p; a, \beta)$ is given in Section 1.

5. GENERATING FUNCTION

Recall that the MGF of Y is defined in the form

$$\mathbb{M}_Y(s) = \mathsf{E}(\mathrm{e}^{-sY}), \qquad s > 0\,.$$

Obviously

$$\mathbb{M}_Y(s) = a \int_1^\infty \mathrm{e}^{-s\,(\ln x)^p - ax}\,\mathrm{d}x, \qquad p := \beta^{-1}\,,$$

where

$$\mathbb{M}_Y(0+) = a; \qquad \frac{\mathrm{d}}{\mathrm{d}s}\left[\mathbb{M}_Y(s)\right]_{s=0} = a\,I_1(a, p)\,.$$

However, in deriving a computational sum representation for the MGF the restricted parameter range $p > 1$ should be used. Indeed, because

$$\mathbb{M}_Y(s) = a \sum_{n\geq 0} \frac{(-a)^n}{n!} \int_1^\infty x^n \mathrm{e}^{-s\,(\ln x)^p}\,\mathrm{d}x = a \sum_{n\geq 0} \frac{(-a)^n}{n!} \int_1^\infty \mathrm{e}^{n\ln x - s\,(\ln x)^p}\,\mathrm{d}x\,,$$

the integral expression in the addend converges for all $n \in \mathbb{N}_0$ only for $p > 1$. The substitution $\ln x = t$ transforms the MGF of Y into

$$\mathbb{M}_Y(s) = a \sum_{n\geq 0} \frac{(-a)^n}{n!} \int_0^\infty \mathrm{e}^{(n+1)t - s\,t^p}\,\mathrm{d}t \tag{15}$$

$$= \frac{a}{p\,\sqrt[p]{s}} \sum_{k\geq 0} \frac{\Gamma\left(p^{-1}(1+k)\right)}{k!\,(\sqrt[p]{s}\,)^k} \sum_{n\geq 0} \frac{(n+1)^k\,(-a)^n}{n!}\,. \tag{16}$$

Observe the formula [**4**, p. 149, Eq. 6.12.3.] (also see [**21**, p. 336, 48])

$$\sum_{n\geq 0}\sum_{k\geq 0}\frac{(n+\alpha)^k\,x^n}{n!}=\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k\left(\,\mathrm{e}^{\alpha t+x\mathrm{e}^t}\,\right)_{t=0}=\mathrm{e}^x\,P_k(x),\qquad k\in\mathbb{N}_0\,,$$

where $P_k(x)$ is a polynomial in x of degree $\deg(P_k)=k$, see [**4**, p. 149, Eq. 6.12.3.]. Hence,

$$\mathbb{M}_Y(s)=\frac{a}{p\,\sqrt[p]{s}}\sum_{k\geq 0}\frac{\Gamma\left(p^{-1}(1+k)\right)}{k!\,(\,\sqrt[p]{s}\,)^k}\,\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k\left(\,\mathrm{e}^{t-a\mathrm{e}^t}\,\right)_{t=0}.$$

Moreover, re–writing (16) into

$$\mathbb{M}_Y(s)=\frac{a}{p\,\sqrt[p]{s}}\sum_{n\geq 0}\frac{(-a)^n}{n!}\sum_{k\geq 0}\frac{\Gamma\left(p^{-1}(1+k)\right)}{k!}\left(\frac{n+1}{\sqrt[p]{s}}\right)^k,$$

we arrive at the double series formula

$$\mathbb{M}_Y(s)=\frac{a}{p\,\sqrt[p]{s}}\sum_{n\geq 0}\frac{(-a)^n}{n!}\,{}_1\Psi_0\left[(p^{-1},p^{-1});-;\frac{n+1}{\sqrt[p]{s}}\right],$$

where the generalized Fox–Wright function notation is used [**6**, p. 56, Eq. (1.11.14)]

$${}_1\Psi_0\big[(a,b);-;z\big]=\sum_{n\geq 0}\frac{\Gamma(a+bn)\,z^n}{n!}\,,\qquad z,a\in\mathbb{C},b>0\,.$$

The series converges [**6**, p. 56, Eq. (1.11.15)] for all $b>-1$ which obviously holds true.

Theorem 5.8. *For all $a,\beta,s>0$, the* MGF *of the* RV $T\sim\mathrm{Chen}(a,\beta)$ *possesses the series representation*:

$$\mathbb{M}_T(s)=a\beta\,\mathrm{e}^a\,s^{-\beta}\sum_{n\geq 0}\frac{(-a)^n}{n!}\,{}_1\Psi_0\left[(\beta,\beta);-;\frac{n+1}{s^\beta}\right].$$

Remark 5.9. *The same derivation method is senseless for $s<0$ since the convergence issues in* (15). *On the other hand for rational values of the parameter b there exists a representation of ${}_1\Psi_0$ in terms of generalized hypergeometric ${}_pF_q$ functions, see* [**12**]. ■

6. RÉNYI ENTROPY

The entropy of a random variable X with density function $f(x)$ is a measure of variation of the uncertainty. For any real parameter $\lambda>0$ and $\lambda\neq 1$, the Rényi entropy for the random variable $T\sim\mathrm{Chen}(a,\beta)$ is given by

$$I_R(\lambda)=\frac{1}{1-\lambda}\ln\left(\int_0^\infty f^\lambda(x)\,\mathrm{d}x\right). \tag{17}$$

According to (1) we write

$$f^{\lambda}(x) = (a\beta)^{\lambda}\,\mathrm{e}^{a\lambda}\,x^{(\beta-1)\lambda}\,\exp\left\{\lambda x^{\beta} - \lambda a\,\mathrm{e}^{x^{\beta}}\right\}.$$

By setting $y = \mathrm{e}^{x^{\beta}}$, the integral in (17) reduces to

$$A = \beta^{-1}(a\beta)^{\lambda}\,\mathrm{e}^{a\lambda}\int_{1}^{\infty} y^{\lambda-1}\,(\ln y)^{(1-\beta^{-1})(\lambda-1)}\,\mathrm{e}^{-a\lambda y}\mathrm{d}y.$$

Using the notation (2) of generalized integro–exponential function by Milgram, we obtain

$$A = \beta^{-1}\big(a\beta\,\mathrm{e}^{a}\big)^{\lambda}\,E_{1-\lambda}^{(1-\beta^{-1})(\lambda-1)}(a\lambda)\,.$$

Having in mind Theorem 2.2, we deduce:

Theorem 6.10. *For all* $a, \beta, \lambda > 0$; $\lambda \neq 1$; $\mu \in \mathbb{C}$, *the Rényi entropy for* $\mathrm{Chen}(a, \beta)$ *distribution is*

$$I_R(\lambda) = \frac{1}{1-\lambda}\,\ln\Big(\beta^{-1}\left(a\beta\mathrm{e}^{a}\right)^{\lambda}\cdot E_{1-\lambda}^{(1-\beta^{-1})(\lambda-1)}(a\lambda)\Big),$$

where the power series form of $E(\cdot)$ *is described in* (10).

7. QUANTILE FUNCTION

Let us present the QF $Q(p) := Q(p; a, \beta)$ given in the display () in the form of a Maclaurin series, modestly narrowing the range of p. The power series for the QF is very important to obtain a power series for the cumulant generating function and then the saddle-point approximations for the sum and mean of independent and identically distributed (IID) $\mathrm{Chen}(a, \beta)$ random variables.

To achieve the desired formula we expand the first the power of the QF in several steps. Firstly, being $p \in (0, 1)$ we have

$$Q^{\beta}(p) = \ln\left(1 + \frac{1}{a}\sum_{k\geq 1}\frac{p^k}{k}\right) = \sum_{n\geq 1}\frac{(-1)^{n-1}}{n\,a^n}\left(\sum_{k\geq 1}\frac{p^k}{k}\right)^n;$$

here the second expansion holds true for any $p \in \big(0, 1-\mathrm{e}^{-a}\big)$. It is known that the power series raised positive integer power n implies the structure

$$\left(\sum_{k\geq 1}\frac{p^k}{k}\right)^n = \sum_{m\geq n}\left(\sum_{\substack{k_1,\cdots,k_n\geq 1\\ k_1+\cdots+k_n=m}}\frac{1}{k_1\cdots k_n}\right)p^m =: \sum_{m\geq n} c_m\,p^m\,,$$

whence

$$Q^{\beta}(p) = \sum_{n,m\geq 0}\frac{(-1)^n\,c_{m+n+1}}{(n+1)\,a^{n+1}}\,p^{m+n+1} = \frac{p}{a}\sum_{r\geq 0}\frac{(-1)^r\,c_{r+1}}{a^r}\sum_{j=0}^{r}\frac{(-1)^j\,a^j}{r-j+1}\,p^r.$$

Thus, we establish the following result.

Theorem 7.11. *For all $a, \beta > 0$ and for all $p \in \left(0, 1 - \mathrm{e}^{-a}\right)$, there holds*

$$Q(p; a, \beta) = \left(\frac{p}{a}\right)^{\frac{1}{\beta}} \left(1 + \sum_{r \geq 1} \frac{(-1)^r \, c_{r+1}}{a^r} \sum_{j=0}^{r} \frac{(-1)^j \, a^j}{r - j + 1} \, p^r\right)^{\frac{1}{\beta}}, \tag{18}$$

where

$$c_\ell = \sum_{\substack{k_1, \cdots, k_n \geq 1 \\ k_1 + \cdots + k_n = \ell}} \frac{1}{k_1 \cdots k_n}, \qquad \ell \geq n \,.$$

Now, it remains to derive the β^{-1} power of the sum in (18) for which we suggest numerical in-built routines. In turn, having more elastic formula, writing down the first few terms in the power series of the $a^{-\frac{1}{\beta}} \, p^{\frac{1}{\beta}} \, Q(p; a, \beta)$, one yields

$$Q(p; a, \beta) = \left(\frac{p}{a}\right)^{\frac{1}{\beta}} \left[1 + \frac{a - 1}{2a\beta} \, p + \frac{(5\beta + 3)a^2 - 6(\beta + 1)a + 5\beta + 3}{24a^2\beta^2} \, p^2 + \mathscr{O}(p^3)\right].$$

On the other hand to earn the power series expansion, the Bürmann–Lagrange formula can also be used, see e.g. [**18**, Eq. (1.1) *et seq.*].

Further, the following power series can be obtained in Mathematica for the QF $Q(p) = Q(p; a, \beta)$ in (7), which holds for all $p \in (0, 1)$,

$$Q(p) = \sum_{j \geq 0} f_j \, p^{j + 1/\beta}, \tag{19}$$

where $f_0 = a^{-1/\beta}$, $f_1 = (1 - a)a^{-1-1/\beta}/(2\beta)$, $f_2 = [3\beta^{-1}(1 - a)^2 + 5a^2 - 6a + 5]a^{-2-1/\beta}/(24\beta)$, etc.

Equation (19) reveals that the QF of the Chen distribution can be expressed as a power series. Then, several mathematical quantities of this distribution can be given in terms of integrals over $(0, 1)$. In fact, if $W(\cdot)$ be any integrable function in the real line, we can write

$$\int_0^\infty W(x) \, f(x) \mathrm{d}x = \int_0^1 W\left(\sum_{j \geq 0} f_j \, p^{j + 1/\beta}\right) \mathrm{d}p. \tag{20}$$

Equations (19) and (20) are the main results of this section since we can obtain from them various structural properties for Chen distribution using the integral on the right-hand side for special $W(\cdot)$ functions, which can be simpler than if they are based on the left-hand integral.

Provide a power series for the Chen MGF from (20) given by

$$\mathbb{M}_T(s) = \mathsf{E}(\mathrm{e}^{-sT}) = \int_0^1 \exp\left\{ - s \sum_{j \geq 0} f_j \, p^{j + 1/\beta} \right\} \mathrm{d}p,$$

rewriting it into

$$\mathbb{M}_T(s) = \int_0^1 \exp\left\{ - s\, f_0\, p^{\beta^{-1}} \sum_{j\geq 1} g_j\, p^j \right\} \mathrm{d}p, \tag{21}$$

where $g_j = f_j\, f_0^{-1}$ for $j \geq 1$.

Next, we use the exponential partial Bell polynomials [**2**] given by

$$\exp\left\{ u \sum_{j\geq 1} x_j\, \frac{t^j}{j!} \right\} = \sum_{n,k\geq 0} \frac{B_{n,k}}{n!}\, t^n\, u^k, \tag{22}$$

where

$$B_{n,k} = B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum \frac{n!}{c_1!\, c_2! \cdots (1!)^{c_1} (2!)^{c_2} \cdots}\, x_1^{c_1}\, x_2^{c_2} \cdots,$$

and the summation takes place over all integers $c_1, c_2, \ldots \geq 0$, which verify $c_1 + 2c_2 + 3c_3 + \cdots = n$ and $c_1 + c_2 + c_3 + \cdots = k$. These polynomials can be computed in Mathematica using the BellY[n,k,{x1,...,n-k+1}] function and in Maple using the IncompleteBellB(n, k, z[1], z[2], ..., z[n = k+1]) function.

Using (22) and then integrating in (21), we can obtain an infinite power series for the MGF of T given by

$$\mathbb{M}_T(s) = \sum_{k\geq 0} (-f_0)^k \left\{ \sum_{n\geq 0} \frac{B_{n,k}(g_1, 2g_2, \cdots, (n-k+1)!\, g_{n-k+1})}{(n + k\,\beta^{-1} + 1) n!} \right\} s^k. \tag{23}$$

For asymptotic applications, we require in (23) a polynomial of fourth degree.

We define the cumulant generating function (CGF) of T by $K_T(s) = \ln \mathbb{M}_T(-s)$. The saddle–point approximations are the main applications of the CGF in statistics and provide highly accurate approximation formulae for the density of the sum and mean of IID random variables. Let $T_1, \cdots, T_n$ be IID Chen distributed random variables with common CGF $K_T(s)$. Let $S_n = \sum_{j=1}^n T_j$ and

$$K_T^{(j)}(\lambda) = \left(\frac{\mathrm{d}}{\mathrm{d}\lambda} \right)^j K_T(\lambda), \qquad j \geq 1.$$

We define $\widehat{\lambda}$ from the (usual nonlinear) equation $K_T^{(1)}(\widehat{\lambda}) = x/n$ and

$$y = \frac{x - n K_T^{(1)}(\lambda)}{\sqrt{n K_T^{(2)}(\lambda)}}.$$

The density functions of S_n and $\overline{Z}_n = S_n/n$ follow from Daniels' saddle–point approximations as

$$f_{S_n}(x) = \frac{\exp\{n K_T(\widehat{\lambda}) - x\widehat{\lambda}\}}{\sqrt{2n\pi K_T^{(2)}(\widehat{\lambda})}} \left\{ 1 + \mathbb{M}_T(\widehat{\lambda}) + \mathscr{O}(n^{-2}) \right\}$$

and

$$f_{\overline{Y}_n}(y) = \left\{ \frac{n}{2\pi K_T^{(2)}(\widehat{\lambda})} \right\}^{1/2} \exp[n\{K_T(\widehat{\lambda}) - \widehat{\lambda}y\}] \left\{1 + A(\widehat{\lambda}) + \mathscr{O}(n^{-2})\right\},$$

where $A(\lambda) = (3\gamma_2(\lambda) - 5\gamma_1(\lambda)^2)\,(24n)^{-1}$ and

$$\gamma_1(\lambda) = \frac{K_T^{(3)}(\lambda)}{\left[K_T^{(2)}(\lambda)\right]^{3/2}}, \qquad \gamma_2(\lambda) = \frac{K_T^{(4)}(\lambda)}{\left[K_T^{(2)}(\lambda)\right]^{2}},$$

are the third and fourth standardized cumulants of T, respectively.

8. CONCLUSIONS

We provide explicit expressions for the moments of positive real order, incomplete moments of the same range, moment generating function, mean deviations, Bonferroni and Lorenz curves, and Rényi entropy for the two-parameter lifetime model introduced by Chen in 2000 [**1**]. The related RV Ξ_1 by Xie *et al.* [**22**] turns out to be only linearly connected to the Chen model, so establishing its statistical characteristics is a trivial procedure.

Finally, Nadarajah's moment related differential formula [**14**] for positive rational order moments related to Chen's model is complemented to positive real order moments by using either real analytical methods or Grünwald–Letnikov fractional derivative. Besides, our formulae are obtained by using *inter alia* Lin–Srivastava generalized Hurwitz-Lerch Zeta, confluent hypergeometric (Kummer's), and Fox–Wright generalized confluent hypergeometric functions. They are manageable with the use of up-to-date in built computer routines with analytic and numerical capabilities to which several of our formulae are prepared.

Finally, the following question occurs (posed by the unknown referee): "*Is it possible to compare the values of moments of a random variable with Chen distribution with the values obtained by Monte-Carlo simulations for some different values of the distribution's parameters*?" However, in this stage of the research we cannot answering without further study.

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