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ON DERIVATIONS AND GENERALIZED DERIVATIONS OF BITONIC ALGEBRAS

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We introduce the notion of bitonic algebras as a generalization of dual BCC-algebras, and define the notion of (r,l)-derivations, (l,r)-derivations and generalized (r,l) and (l,r)-derivations on the bitonic algebras. Then we study the properties of the derivations and the generalized derivations on the bitonic algebras and the commutative bitonic algebras. Finally, we show that every generalized derivation of commutative bitonic algebras is a derivation.

1. INTRODUCTION

The notion of generalized derivation of rings was introduced by H. Brešar, and the algebraic study was initiated by B. Hvala[**3, 6, 14**], and it was applied to logical systems such as BCC-algebras and lattices. The notion of BCC-algebras was introduced by Komori[**17**] and Dudek[**10**] as a generalization of BCK-algebra[**15, 16**], and the generalized derivation and derivation of BCC-algebras was studied in [**2, 4, 20**].

A *dual BCC-algebra* is an algebraic system $(X, *, 1)$ satisfying the following axioms.

(D1) $(x * y) * ((y * z) * (x * z)) = 1$,
(D2) $1 * x = x$,
(D3) $x * 1 = 1$,

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(D4) $x * x = 1$,

(D5) $x * y = 1$ and $y * x = 1$ imply $x = y$.

The notion of dual BCC-algebras is a generalization of DBCK-algebras[**7, 18, 21**], Hilbert algebras[**9, 11, 13, 19**], Heyting algebras(or Brouwerian lattices)[**8, 5**], implication algebras[**1**] and lattice implication algebras[**22, 23**]. All of such algebras satisfy the property: (P) $x \leq y$ implies $z * x \leq z * y$ and $y * z \leq x * z$. So those algebras can be generalized to algebras which have the axiom (P).

In section 2, we define the notion of bitonic algebras as a generalization of dual BCC-algebras, and define the notion of (r,l)-derivations and (l,r)-derivations on the bitonic algebras, and we research the properties of the derivations on the bitonic algebras. In section 3, we define the notion of generalized (r,l)-derivations and generalized (l,r)-derivations on the bitonic algebras, and study the properties of them. Finally, in section 4, we study the properties of the derivations and the generalized derivations on the commutative bitonic algebras, and we show that every generalized derivation of commutative bitonic algebras is a derivation.

2. DERIVATIONS OF BITONIC ALGEBRAS

Definition 2.1. A bitonic algebra is an algebraic system $(A, *, 1)$, where A is a set, 1 an element in A and $*$ a binary operation on A, satisfying the following axioms. For every $a, b, c \in A$,

(B1) $a * 1 = 1$,

(B2) $1 * a = a$,

(B3) $a * b = 1$ and $b * a = 1$ implies $a = b$,

(B4) $a * b = 1$ implies $(c * a) * (c * b) = 1$ and $(b * c) * (a * c) = 1$.

Lemma 2.2. *Let $(A, *, 1)$ be a bitonic algebra. Then for every $a, b, c \in A$,*

(1) $a * a = 1$,

(2) $a * b = b * c = 1$ *implies* $a * c = 1$,

(3) $a * (b * a) = 1$.

Proof. (1) It is clear that $1 * 1 = 1$ by (B2). Hence by (B2) and (B4), $a * a = (1 * a) * (1 * a) = 1$ for every $a \in A$.

(2) Let $a * b = b * c = 1$ for any $a, b, c \in A$. Then $a * c = 1 * (a * c) = (a * b) * (a * c) = 1$ by (B2) and (B4).

(3) Let $a, b \in A$. Then $b * 1 = 1$ by (B1), and this implies $a * (b * a) = (1 * a) * (b * a) = 1$ by (B2) and (B4). □

Let $(A, *, 1)$ be a bitonic algebra. If we define a binary relation "$\leq$" on A by

$$a \leq b \iff a * b = 1$$

for any $a, b \in A$, then $\leq$ is a partial order on A by (B3) and Lemma 2.2. Hence $(A, \leq)$ is a poset, and 1 is the greatest element in A by (B1).

Lemma 2.3. *Let $(A, *, 1)$ be a bitonic algebra. Then for every $a, b, c \in A$,*

(1) $a \leq b$ implies $c * a \leq c * b$ and $b * c \leq a * c$,

(2) $a \leq b * a$.

Proof. It is clear from (B4) and Lemma 2.2(3). □

Example 2.4. *Let $N = \{1, x, y, z, w\}$ be a set. If we define a binary operation $*$ on N by the following table:*

$*$	1	x	y	z	w
1	1	x	y	z	w
x	1	1	y	z	w
y	1	x	1	z	w
z	1	1	1	1	x
w	1	1	1	z	1

Then $(N, *, 1)$ is a bitonic algebra with Hasse diagram in Figure 1.

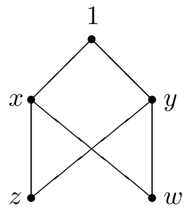

Figure 1: Hasse diagram of the bitonic algebra N in Example 2.4

Lemma 2.5. ([**12**]) *Let $(X, *, 1)$ be a dual BCC-algebra. Then $a * b = 1$ implies $(c * a) * (c * b) = 1$ and $(b * c) * (a * c) = 1$ for every $a, b, c \in X$.*

Theorem 2.6. *If $(X, *, 1)$ be a dual BCC-algebra, then it is a bitonic algebra.*

Proof. It is clear from (D3), (D2) and (D5) in section 1, and Lemma 2.5. □

The converse of Theorem 2.6 does not hold. In fact, the bitonic algebra N in Example 2.4 is not dual BCC-algebra, because $(x * z) * ((z * w) * (x * w)) = z * (x * w) = z * w = x \neq 1$.

Theorem 2.7. *Let $(A, *, 1)$ be a bitonic algebra. Then the following are equivalent:*

(1) $a * (b * c) = b * (a * c)$ *for every* $a, b, c \in A$,

(2) $a * b \leq (b * c) * (a * c)$ *for every* $a, b, c \in A$.

Proof. Suppose that $a*(b*c) = b*(a*c)$ for every $a, b, c \in A$. Then $b*((b*c)*c) = (b*c)*(b*c) = 1$ by the hypothesis and Lemma 2.2(1). This implies $b \leq (b*c)*c$. Hence $a*b \leq a*((b*c)*c) = (b*c)*(a*c)$ by Lemma 2.3(1) and the hypothesis.

Conversely, suppose that $a * b \leq (b * c) * (a * c)$ for every $a, b, c \in A$. Then $b * (a * c) \leq ((a * c) * c) * (b * c)$ by the hypothesis. Also, since $a = 1 * a \leq (a * c) * (1 * c) = (a * c) * c$, we have $((a * c) * c) * (b * c) \leq a * (b * c)$ by Lemma 2.3(1). This implies $b * (a * c) \leq a * (b * c)$. Interchanging the role of a and b, we can show $a * (b * c) \leq b * (a * c)$. Hence $a * (b * c) = b * (a * c)$. □

A *DBCK-algebra* [**7**] is an algebraic system $(X, *, 1)$ satisfying the following axioms. For every $a, b, c \in X$,

(DK1) $(a * b) * ((b * c) * (a * c)) = 1$,
(DK2) $a * ((a * b) * b) = 1$,
(DK3) $a * a = 1$,
(DK4) $a * b = 1$ and $b * a = 1$ imply $a = b$,
(DK5) $a * 1 = 1$.

Lemma 2.8. ([**18, 21**]) *The DBCK-algebra* X *satisfies the following properties: for every* $a, b, c \in X$

(1) $a * (b * c) = b * (a * c)$,

(2) $1 * a = a$,

(3) $a * b = 1$ *implies* $(c * a) * (c * b) = 1$ *and* $(b * c) * (a * c) = 1$.

Theorem 2.9. *An algebra* $(A, *, 1)$ *of type* $(2, 0)$ *is DBCK-algebra if and only if it is a bitonic algebra satisfying the property: (E)* $a * (b * c) = b * (a * c)$.

Proof. If $(A, *, 1)$ be a DBCK-algebra, then it is a bitonic algebra satisfying the property: $a * (b * c) = b * (a * c)$ by (DK5), (DK4), (2) and (3) of Lemma 2.8.

Conversely, suppose that $(A, *, 1)$ is a bitonic algebra satisfying the property (E). Then it satisfies (DK1) by Theorem 2.7(2), and it satisfies (DK2), since $a * ((a * b) * b) = (a * b) * (a * b) = 1$ by (E). Also, it satisfies (DK3), (DK4) and (DK5) by Lemma 2.2(1), (B3) and (B1). Hence $(A, *, 1)$ is a DBCK-algebra. □

Let $(A, *, 1)$ be a bitonic algebra. We will define a binary operation "$\vee$" on A by $a \vee b = (a * b) * b$ for every $a, b \in A$.

Lemma 2.10. *Let* $(A, *, 1)$ *be a bitonic algebra. Then the binary operation* $\vee$ *on* A *satisfies the following properties: for every* $a, b \in A$,

(1) $b \leq a \vee b$,

(2) $a \leq b$ *implies* $a \vee b = b$,

(3) $1 \vee a = 1$ *and* $a \vee 1 = 1$.

Proof. It is clear from the definition of the binary operation $\vee$ on A. □

Definition 2.11. Let $(A, *, 1)$ be a bitonic algebra and $d : A \to A$ a map. Then

(1) d is called a *(r,l)-derivation* of A if $d(a * b) = (a * d(b)) \vee (d(a) * b)$ for every $a, b \in A$,

(2) d is called a *(l,r)-derivation* of A if $d(a * b) = (d(a) * b) \vee (a * d(b))$ for every $a, b \in A$.

The following examples are a (r,l)-derivation but not a (l,r)-derivation, and a (l,r)-derivation but not a (r,l)-derivation, and both (r,l) and (l,r)-derivation of bitonic algebras.

Example 2.12. *Let* $N = \{1, x, y, z, w\}$ *be the bitonic algebra in Example 2.4. If we define a map* $d : N \to N$ *by*

$$d(1) = 1, d(x) = 1, d(y) = y, d(z) = 1, d(w) = w,$$

then d *is a (r,l)-derivation, but not a (l,r)-derivation, because* $d(z * w) = 1 \neq x = (d(z) * w) \vee (z * d(w))$. *Also, define a map* $\delta : N \to N$ *by*

$$\delta(1) = 1, \delta(x) = x, \delta(y) = 1, \delta(z) = 1, \delta(w) = w.$$

Then δ *is a (l,r)-derivation, but not a (r,l)-derivation, because* $\delta(z * w) = x \neq 1 = (z * \delta(w)) \vee (\delta(z) * w)$.

Example 2.13. *Let* $B = \{1, x, y, 0\}$ *be a set. If we define a binary operation* $*$ *on* B *by the following table:*

$*$	1	x	y	0
1	1	x	y	0
x	1	1	y	y
y	1	x	1	0
0	1	1	1	1

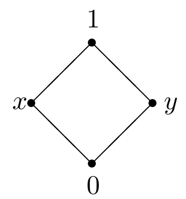

Figure 2: Hasse diagram of the bitonic algebra B in Example 2.13

Then $(B, *, 1)$ is a bitonic algebra with Hasse diagram in Figure 2. If we define a map $d : B \to B$ by

$$d(1) = 1, d(x) = x, d(y) = 1, d(0) = 1,$$

then it is a (r,l)-derivation and a (l,r)-derivation of B.

Lemma 2.14. *Let d be a (r,l)-derivation of a bitonic algebra A. Then for every $a \in A$,*

(1) $d(1) = 1$,

(2) $d(a) = d(a) \vee a$,

(3) *d is expansive, i.e.,* $a \leq d(a)$,

(4) $d(a) = a \vee d(a)$.

Proof. (1) For the element 1 in A, we have $d(1) = d(1*1) = (1*d(1)) \vee (d(1)*1) = d(1) \vee 1 = 1$ by Lemma 2.10(3).

(2) Let $a \in A$. Then we have $d(a) = d(1 * a) = (1 * d(a)) \vee (d(1) * a) = d(a) \vee (1 * a) = d(a) \vee a$.

(3) It is clear from (2) of this lemma and Lemma 2.10(1).

(4) Let $a \in A$. Then $a \leq d(a)$ by (3) of this lemma. Hence $a \vee d(a) = d(a)$ by Lemma 2.10(2). □

From (2) and (4) of Lemma 2.14, if d is a (r,l)-derivation of a bitonic algebra A, then $d(a) = a \vee d(a) = d(a) \vee a$ for every $a \in A$.

Lemma 2.15. *Let d be (r,l)-derivation of a bitonic algebra A. Then $d(a) * b \leq a * d(b)$ for all $a, b \in A$.*

Proof. Let d be (r,l)-derivation of A and $a, b \in A$. Then $a \leq d(a)$ and $b \leq d(b)$ by Lemma 2.14(3). This implies

$$d(a) * b \leq a * b \text{ and } a * b \leq a * d(b)$$

by Lemma 2.3(1). Hence $d(a) * b \leq a * d(b)$. □

Lemma 2.16. *Let δ be a (l,r)-derivation of a bitonic algebra A. Then for every $a \in A$,*

(1) $\delta(1) = 1$,

(2) $\delta(a) = a \vee \delta(a)$,

Proof. It can be shown in similar way to Lemma 2.14. □

Theorem 2.17. *Let A be a bitonic algebra and $\delta : A \to A$ a map. Then the following are equivalent:*

(1) δ is an expansive (l,r)-derivation,

(2) $\delta(a * b) = a * \delta(b)$ for every $a, b \in A$.

Proof. ($\Rightarrow$) Let δ be an expansive (l,r)-derivation and $a, b \in A$. Then $a \leq \delta(a)$ and $b \leq \delta(b)$. This implies

$$\delta(a) * b \leq a * b \leq a * \delta(b)$$

by Lemma 2.3(1). Hence $\delta(a * b) = (\delta(a) * b) \vee (a * \delta(b)) = a * \delta(b)$ by Lemma 2.10(2).

($\Leftarrow$) Let $\delta : A \to A$ be a map satisfying $\delta(a * b) = a * \delta(b)$ for every $a, b \in A$. Then $\delta(1) = \delta(\delta(1) * 1) = \delta(1) * \delta(1) = 1$, and $1 = \delta(1) = \delta(a * a) = a * \delta(a)$ for every $a \in A$. This implies $a \leq \delta(a)$ for every $a \in A$. That is, d is expansive. Let $a, b \in A$. Since $a \leq \delta(a)$ and $b \leq \delta(b)$, we have $\delta(a) * b \leq a * b \leq a * \delta(b)$ by Lemma 2.3(1). This implies

$$\delta(a * b) = a * \delta(b) = (\delta(a) * b) \vee (a * \delta(b))$$

by Lemma 2.10(2). Hence δ is a (l,r)-derivation which is expansive. □

Theorem 2.18. *Let $\delta_1, \delta_2, ..., \delta_n$ be expansive (l,r)-derivations of a bitonic algebra A. Then the composition $\delta_1 \circ \delta_2 \circ \cdots \circ \delta_n$ of δ_i is a (l,r)-derivation.*

Proof. Let $\delta_1, \delta_2, ..., \delta_n$ be expansive (l,r)-derivations and $a, b \in A$. Then by Theorem 2.17, $(\delta_1 \circ \delta_2 \circ \cdots \circ \delta_n)(a * b) = a * (\delta_1 \circ \delta_2 \circ \cdots \circ \delta_n)(b)$. That is, $\delta = \delta_1 \circ \delta_2 \circ \cdots \circ \delta_n$ is a map satisfying $\delta(a * b) = a * \delta(b)$. Hence δ is a (l,r)-derivation by Theorem 2.17. □

For any map $f : A \to A$ on a bitonic algebra A, f is said to be *commutative* if it satisfies $f(a) \vee a = a \vee f(a)$ for every $a \in A$, and *implicative* if it satisfies $(f(a) * a) * f(a) = f(a)$ for every $a \in A$.
We define a set $Kerf$ by $Kerf = \{a \in A \mid f(a) = 1\}$.

Lemma 2.19. *Let δ be a (l,r)-derivation of a bitonic algebra A. If δ is commutative, then δ is expansive.*

Proof. Let $a \in A$. Then $\delta(a) = a \vee \delta(a)$ by Lemma 2.16. Since δ is commutative, $\delta(a) = \delta(a) \vee a$. Hence $a \leq d(a)$ by Lemma 2.10(1). □

Lemma 2.20. *Let d be a (r,l)(or (l,r))-derivation of a bitonic algebra A. Then d satisfies the following properties.*

(1) $a * d(a) \in Ker\, d$ for every $a \in A$,

(2) $Ker\, d = \{d(a) * a \mid a \in A\}$.

Proof. (1) Let d be a (r,l)-derivation of A and $a \in A$. By the definition of (r,l)-derivation and Lemma 2.10(3),

$$d(a * d(a)) = (a * d(d(a))) \vee (d(a) * d(a)) = (a * d(d(a))) \vee 1 = 1.$$

Therefore, $a * d(a) \in Ker\, d$. Similarly we can show the case of a (l,r)-derivation.

(2) Let d be (r,l)-derivation of A and $a \in A$. Then by the definition of (r,l)-derivation and Lemma 2.10(3),

$$d(d(a) * a) = (d(a) * d(a)) \vee (d(d(a)) * a) = 1 \vee (d(d(a)) * a) = 1.$$

Therefore $d(a) * a \in Ker\, d$. That is, $\{d(a) * a \mid a \in A\} \subseteq Ker\, d$. Let $a \in Ker\, d$. Then

$$a = 1 * a = d(a) * a \in \{d(a) * a \mid a \in A\}.$$

This implies $Ker\, d \subseteq \{d(a) * a \mid a \in A\}$. Hence $Ker\, d = \{d(a) * a \mid a \in A\}$. Similarly we can show the case of a (l,r)-derivation. □

Lemma 2.21. *Let d be a (r,l)-derivation of a bitonic algebra A. If $Ker\, d = \{1\}$, then d is identity map.*

Proof. Let $a \in A$. Then $d(a) * a \in Ker\, d = \{1\}$ by Lemma 2.20(2). This implies $d(a) * a = 1$, and $d(a) \leq a$. Hence $d(a) = a$ by Lemma 2.14(3). □

Lemma 2.22. *Let δ be a (l,r)-derivation of a bitonic algebra A. If $Ker\, \delta = \{1\}$, then $\delta(a) \leq a$ for every $a \in A$.*

Proof. Let $a \in A$. Then $\delta(a) * a \in Ker\, \delta = \{1\}$ by Lemma 2.20(2). Hence $\delta(a) * a = 1$, and $\delta(a) \leq a$. □

Theorem 2.23. *Let d be a (r,l)-derivation of a bitonic algebra A. If d is implicative, then $d^2 = d \circ d = d$.*

Proof. Let d be an implicative (r,l)-derivation of A. Then $(d(a) * a) * d(a) = d(a)$ and by Lemma 2.20(2), $d(d(a) * a) = 1$. This implies

$$\begin{aligned} d^2(a) &= d(d(a) \vee a) \\ &= d((d(a) * a) * a) \\ &= ((d(a) * a) * d(a)) \vee ((d(d(a) * a)) * a) \\ &= d(a) \vee (1 * a) \\ &= d(a) \vee a = d(a) \end{aligned}$$

for every $a \in A$. Hence $d^2 = d$. □

Theorem 2.24. *Let δ be a (l,r)-derivation of a bitonic algebra A. If δ is commutative and implicative, then $\delta^2 = \delta \circ \delta = \delta$.*

Proof. Let δ be a commutative and implicative (l,r)-derivation of A. Then δ is an expansive (l,r)-derivation by Lemma 2.19, and by Theorem 2.17, we have

$$\delta^2(a) = \delta(a \vee \delta(a)) = \delta(\delta(a) \vee a) = \delta((\delta(a) * a) * a) = (\delta(a) * a) * \delta(a)$$

for every $a \in A$. Since $(\delta(a) * a) * \delta(a) = \delta(a)$,

$$\delta^2(a) = (\delta(a) * a) * \delta(a) = \delta(a)$$

for every $a \in A$. Hence $\delta^2 = \delta$. □

Let A be a bitonic algebra. A nonempty subset S of A is called a *bitonic subalgebra* of A if $x * y \in S$ for every $x, y \in S$, and a nonempty subset F of A is called a *filter* of A if it satisfies the following:

(F1) $1 \in F$,

(F2) $a \in F$ and $a * b \in F$ imply $b \in F$ for any $a, b \in A$.

Theorem 2.25. *Let d be a (r,l)-derivation of a bitonic algebra A. Then every filter F of A is d-invariant, i.e., $d(F) = \{d(a) \mid a \in F\} \subseteq F$.*

Proof. Let F be a filter of A and $d(a) \in d(F)$ with $a \in F$. Then $a \leq d(a)$ by Lemma 2.14(3). This implies $a * d(a) = 1 \in F$. Since $a \in F$, we have $d(a) \in F$. Hence $d(F) \subseteq F$. □

Theorem 2.26. *Let δ be a (l,r)-derivation of a bitonic algebra A. If δ is commutative, implicative and monotone, then $Ker\,\delta$ is a filter.*

Proof. It is clear that $1 \in Ker\,\delta$. Let $a \in Ker\,\delta$ and $a * b \in Ker\,\delta$. Then $1 = \delta(a*b) = a*\delta(b)$ implies $a \leq \delta(b)$. Since δ is monotone, we have $1 = \delta(a) \leq \delta(\delta(b))$, and $\delta(\delta(b)) = \delta(b)$ by Theorem 2.24. Hence $\delta(b) = 1$ and $b \in Ker\,\delta$. □

Theorem 2.27. *Let d be a (r,l)(or (l,r))-derivation of a bitonic algebra A. Then $Ker\,d$ is a bitonic subalgebra of A.*

Proof. Let d be a (r,l)-derivation of A and $a, b \in Ker\,d$. Then we have $d(a) = 1$ and $d(b) = 1$. By the definition of (r,l)-derivation, we have

$$d(a * b) = (a * d(b)) \vee (d(a) * b) = (a * 1) \vee (1 * b) = 1 \vee b = 1.$$

Therefore we have $a*b \in Ker\,d$. Hence $Ker\,d$ is a bitonic subalgebra of A. Similarly we can show the case of (l,r)-derivations. □

For any bitonic algebra A, we will denote the class of all expansive (l,r)-derivations by $ELRD(A)$ and the class of all bitonic subalgebras of A by $Sub(A)$. Then $ELRD(A)$ is a poset with a partial order defined by

$$\delta_1 \leq \delta_2 \iff \delta_1(a) \leq \delta_2(a) \text{ for every } a \in A,$$

and $Sub(A)$ is a lattice with $X \vee Y = sup\{X, Y\} = \langle X \cup Y \rangle$ and $X \wedge Y = inf\{X, Y\} = X \cap Y$ for every $X, Y \in Sub(A)$, where $\langle Z \rangle$ is the smallest bitonic subalgebra containing Z for any subset Z of A.

Theorem 2.28. *Let A be a bitonic algebra. If we define a map $\varphi : ELRD(A) \to Sub(A)$ by $\varphi(\delta) = Ker\,\delta$, then φ is an order-embedding, i.e., $\delta_1 \leq \delta_2$ if and only if $\varphi(\delta_1) \subseteq \varphi(\delta_2)$ for any $\delta_1, \delta_2 \in ELRD(A)$.*

Proof. Let $\delta_1 \leq \delta_2$ in $ELRD(A)$ and $a \in \varphi(\delta_1) = Ker\,\delta_1$. Then $1 = \delta_1(a) \leq \delta_2(a)$. Hence $\delta_2(a) = 1$ and $a \in Ker\,\delta_2 = \varphi(\delta_2)$.

Conversely, let $\varphi(\delta_1) = Ker\,\delta_1 \subseteq Ker\,\delta_2 = \varphi(\delta_2)$ and $a \in A$. Then $\delta_1(a) * a \in Ker\,\delta_1$ by Lemma 2.20(2). This implies $\delta_1(a) * a \in Ker\,\delta_2$ by hypothesis, and

$$1 = \delta_2(\delta_1(a) * a) = \delta_1(a) * \delta_2(a)$$

by Theorem 2.17. That is, $\delta_1(a) \leq \delta_2(a)$ for ever $a \in A$. Hence $\delta_1 \leq \delta_2$. □

Theorem 2.29. *Let d be a (r,l)-derivation and a (l,r)-derivation of a bitonic algebra. If d is implicative and monotone, then $Ker\,d$ is a filter.*

Proof. Let d be implicative and monotone. Then d is commutative since d is a (r,l)-derivation. Hence d is a commutative, implicative and monotone (l,r)-derivation of A. Therefore $Ker\,d$ is a filter by Theorem 2.26. □

3. GENERALIZED DERIVATIONS OF BITONIC ALGEBRAS

We can consider four types of the generalized derivations on a bitonic algebra A as follows:

- $D_1(a * b) = (a * D_1(b)) \vee (d(a) * b)$,
- $D_2(a * b) = (a * d(b)) \vee (D_2(a) * b)$,
- $D_3(a * b) = (D_3(a) * b) \vee (a * \delta(b))$,
- $D_4(a * b) = (\delta(a) * b) \vee (a * D_4(b))$,

where d and δ are the (r,l)-derivation and the (l,r)-derivation on A respectively. For the maps D_i, $i = 2, 3$, $D_i(1) = 1$ obviously. Hence we can show that $D_2 = d$ and $D_3 = \delta$. In fact,

$$D_2(a) = D_2(1 * a) = (1 * d(a)) \vee (D_2(1) * a) = d(a) \vee a.$$

for every $a \in A$. Since d is a (r,l)-derivation and $d(a) = d(a) \vee a$ by Lemma 2.14(1), we have $D_2(a) = d(a)$, Similarly, we can show $D_3 = \delta$. Therefore D_2 and D_3 are trivially a (r,l)-derivation and a (l,r)-derivation of A respectively. Thus we will define two types of the generalized derivations of A.

Definition 3.30. Let $(A, *, 1)$ be a bitonic algebra and $D : A \to A$ a map. Then

(1) D is called a *generalized (r,l)-derivation* of A if there is a (r,l)-derivation d such that $D(a * b) = (a * D(b)) \vee (d(a) * b)$for every $a, b \in A$,

(2) D is called a *(l,r)-derivation* of A if there is a (l,r)-derivation δ such that $D(a * b) = (\delta(a) * b) \vee (a * D(b))$for every $a, b \in A$.

The following example is a generalized (r,l) and (l,r)-derivation which is neither (r,l)-derivation nor (l,r)-derivation, and it is a generalized derivation of a bitonic algebra.

Example 3.31. *On the bitonic algebra B in Example 2.13, if we define a map $D : B \to B$ by*

$$D(1) = 1, D(x) = x, D(y) = 1, D(0) = 0,$$

then D is a generalized (r,l)-derivation with respect to the (r,l)-derivation d defined by $d(1) = 1$, $d(x) = 1$, $d(y) = 1$ and $d(0) = 0$, but D is not (r,l)-derivation, because $D(x * 0) = 1 \neq y = (x * D(0)) \vee (D(x) * 0)$. Also, D is a generalized (l,r)-derivation with respect to the (l,r)-derivation δ defined by $\delta(1) = 1$, $\delta(x) = 0$, $\delta(y) = 1$ and $\delta(0) = 1$, but D is not (l,r)-derivation, because $D(x * 0) = 1 \neq y = (D(x) * 0) \vee (x * D(0))$.

Theorem 3.32. *Every (r,l)(resp. (l,r))-derivation of a bitonic algebra A is a generalized (r,l)(resp. (l,r))-derivation.*

Proof. Let d be a (r,l)(resp. (l,r))-derivation of A. Then it is a generalized (r,l)(resp. (l,r))-derivation with respect to (r,l)(resp. (l,r))-derivation d. □

Lemma 3.33. *Let D be a generalized (r,l)-derivation of a bitonic algebra A. Then for every $a \in A$,*

(1) $D(1) = 1$,

(2) $D(a) = D(a) \vee a$,

(3) $a \leq D(a)$,

(4) $D(a) = a \vee D(a)$.

Proof. It can be shown in similar way to Lemma 2.14. □

Lemma 3.34. *Let $\mathcal{D}$ be a generalized (l,r)-derivation of a bitonic algebra A. Then for every $a \in A$,*

(1) $\mathcal{D}(1) = 1$,

(2) $\mathcal{D}(a) = a \vee \mathcal{D}(a)$.

Proof. It can be shown in similar way to Lemma 2.14. □

Lemma 3.35. *Let A be a bitonic algebra. If D is a generalized (r,l)(resp. (l,r))-derivation with respect to (r,l)(resp. (l,r))-derivation d, then $a * d(a) \in Ker\, D$ for every $a \in A$.*

Proof. Let $a \in A$. Then $D(a*d(a)) = (a*D(d(a)) \vee (d(a)*d(a)) = (a*D(d(a)) \vee 1 = 1$. Similarly we can show the case of generalized (l,r)-derivation. □

Lemma 3.36. *Let D be a generalized (r,l)(or (l,r))-derivation of a bitonic algebra A. Then $Ker\, D = \{D(a) * a \mid a \in A\}$.*

Proof. Let D be a generalized (r,l)-derivation of A with respect to a (r,l)-derivation d and $a \in A$. Then

$$D(D(a) * a) = (D(a) * D(a)) \vee (d(D(a)) * a) = 1 \vee (d(D(a)) * a) = 1.$$

Hence $D(a)*a \in Ker\, D$. Let $a \in Ker\, D$. Then $a = 1*a = D(a)*a \in \{D(a)*a \mid a \in A\}$. Hence $Ker\, D = \{D(a) * a \mid a \in A\}$. Similarly we can show the case of generalized (l,r)-derivation. □

Theorem 3.37. *Let D be a generalized (r,l)(or (l,r))-derivation of a bitonic algebra A. Then $Ker\, D$ is a bitonic subalgebra of A.*

Proof. Let D be a generalized (r,l)-derivation of A with respect to a (r,l)-derivation d and $a, b \in Ker\, D$. Then we have $D(a) = 1$ and $D(b) = 1$. By the definition of a generalized (r,l)-derivation we get

$$D(a * b) = (a * D(b)) \vee (d(a) * b) = (a * 1) \vee (d(a) * b) = 1 \vee (d(a) * b) = 1.$$

Therefore we have $a * b \in Ker\, D$. Hence $Ker\, D$ is a bitonic subalgebra of A. □

Theorem 3.38. *Let D be a generalized (r,l)-derivation of a bitonic algebra A. Then every filter F of A is D-invariant.*

Proof. Let F be a filter of A and $D(a) \in D(F)$ with $a \in F$. Then $a \leq D(a)$ by Lemma 3.33(3). This implies $a * D(a) = 1 \in F$. Since $a \in F$, we have $D(a) \in F$. Hence $D(F) \subseteq F$. □

Theorem 3.39. *Let $\mathcal{D}$ be a generalized (l,r)-derivation of a bitonic algebra A with respect to a (l,r)-derivation δ. If $\mathcal{D}$ and δ are expansive, then $\mathcal{D}$ is a (l,r)-derivation of A.*

Proof. Let $a, b \in A$. Then $a \leq \delta(a)$, $a \leq \mathcal{D}(a)$ and $b \leq \mathcal{D}(b)$. This implies

$$\delta(a) * b \leq a * b \leq a * \mathcal{D}(b) \quad \text{and} \quad \mathcal{D}(a) * b \leq a * b \leq a * \mathcal{D}(b).$$

Hence we have $\mathcal{D}(a * b) = (\delta(a) * b) \vee (a * \mathcal{D}(b)) = a * \mathcal{D}(b) = (\mathcal{D}(a) * b) \vee (a * \mathcal{D}(b))$. That is, $\mathcal{D}$ is a (l,r)-derivation of A. □

Theorem 3.40. *If A is a bitonic algebra satisfying $a \leq a \vee b = (a * b) * b$ for every $a, b \in A$, then every generalized (l,r)-derivation of A is a (l,r)-derivation.*

Proof. Let $\mathcal{D}$ be a generalized (l,r)-derivation of A with respect to a (l,r)-derivation δ. Then

$$\delta(a) = a \vee \delta(a) \geq a \quad \text{and} \quad \mathcal{D}(a) = a \vee \mathcal{D}(a) \geq a$$

for every $a \in A$. Hence $\mathcal{D}$ is a (l,r)-derivation of A by Theorem 3.39. □

Corollary 3.41. *Every generalized (l,r)-derivation of DBCK-algebras is a (l,r)-derivation.*

Proof. Let A be a DBCK-algebra. Then A is a bitonic algebra satisfying: $a \leq (a * b) * b$ for every $a, b \in A$. Hence every generalized (l,r)-derivation of A is a (l,r)-derivation by Theorem 3.40. □

4. DERIVATIONS AND GENERALIZED DERIVATIONS OF COMMUTATIVE BITONIC ALGEBRAS

A bitonic algebra $(A, *, 1)$ is said to be *commutative* if $(a * b) * b = (b * a) * a$ for every $a, b \in A$.

Lemma 4.42. *An algebra $(A, *, 1)$ of type (2,0) is a commutative bitonic algebra if and only if it satisfies the following properties;*

$(B1')$ $a * 1 = 1$,

$(B2')$ $1 * a = a$,

$(B3')$ $a * b = 1$ *implies* $(c * a) * (c * b) = 1$ *and* $(b * c) * (a * c) = 1$.

$(B4')$ $(a * b) * b = (b * a) * a$.

Proof. If $(A, *, 1)$ is a commutative bitonic algebra, then it satisfies the properties $(B1')$–$(B4')$. Conversely, suppose that an algebra $(A, *, 1)$ of type (2,0) satisfies the properties $(B1')$–$(B4')$. Then we need to show that it satisfies the axiom (B3). Let $a * b = b * a = 1$. Then we have

$$a = 1 * a = (b * a) * a = (a * b) * b = 1 * b = b.$$

Hence $(A, *, 1)$ is a bitonic algebra, and it is commutative by $(B4')$. □

Theorem 4.43. *Let $(A, *, 1)$ be a commutative bitonic algebra. Then it is a join-semilattice in which $a \vee b$ is the least upper bound of a and b for every $a, b \in A$.*

Proof. Let $a, b \in A$. Then $a \leq (b*a)*a = (a*b)*b = a \vee b$ and $b \leq (a*b)*b = a \vee b$. This implies $a \vee b$ is an upper bound of a and b. Suppose that u is an upper bound of a and b. Then $a \leq u$ and $b * u = 1$. This implies $u * b \leq a * b$ and $(a * b) * b \leq (u * b) * b$ by Lemma 2.3(1). Here $(u * b) * b = (b * u) * u = 1 * u = u$. This implies $a \vee b = (a * b) * b \leq u$. Hence $a \vee b$ is the least upper bound of a and b. □

In a commutative bitonic algebra $(A, *, 1)$, $a \vee b = b \vee a$ for every $a, b \in A$. Hence every (r,l)-derivation of A is a (l,r)-derivation of A, and vice versa. In fact, if d is a (r,l)-derivation of A, then

$$d(a * b) = (a * d(b)) \vee (d(a) * b) = (d(a) * b) \vee (a * d(b)).$$

Definition 4.44. Let $(A, *, 1)$ be a commutative bitonic algebra. A map $d : A \to A$ is called a *derivation* of A if $d(a * b) = (a * d(b)) \vee (d(a) * b)$ for every $a, b \in A$.

Theorem 4.45. *Let A be a commutative bitonic algebra and $d : A \to A$ a map. Then the following are equivalent:*

(1) d is a derivation,

(2) $d(a * b) = a * d(b)$ for every $a, b \in A$.

Proof. Let d be a derivation on a commutative bitonic algebra A. Then d is both (l,r) and (r,l)-derivation of A, and this implies that d is an expansive (l,r)-derivation by Lemma 2.14(3). Hence the properties (1) and (2) are equivalent by Theorem 2.17. □

Lemma 4.46. *Let d be a derivation on a commutative bitonic algebra A. Then $d(a) * d(b) \leq d(a * b)$ for all $a, b \in A$.*

Proof. Let d be a derivation on a commutative bitonic algebra A and $a, b \in A$. Then $a \leq d(a)$ by Lemma 2.14(3). This implies $d(a) * d(b) \leq a * d(b) = d(a * b)$ by Lemma 2.3(1) and Lemma 4.45. □

Lemma 4.47. *Let d be a derivation on a commutative bitonic algebra A. Then d is monotone.*

Proof. Let $a \leq b$. Then $d(a) \leq (b * a) * d(a) = d((b * a) * a) = d((a * b) * b) = d(1 * b) = d(b)$ by Lemma 2.3(2) and Theorem 4.45. □

Theorem 4.48. *Let d be a implicative derivation on a commutative bitonic algebra A. Then $Ker\, d$ is a filter of A.*

Proof. Let d be a implicative derivation on a commutative bitonic algebra A. Since A is commutative, d is commutative and monotone by Lemma 4.47. Also, since d is implicative by hypothesis, $Ker\, d$ is a filter by Theorem 2.26. □

Definition 4.49. Let $(A, *, 1)$ be a commutative bitonic algebra. A map $D : A \to A$ is called a *generalized derivation* on A if there is a derivation d on A such that $D(a * b) = (a * D(b)) \vee (d(a) * b)$ for every $a, b \in A$.

Lemma 4.50. *Let $(A, *, 1)$ be a commutative bitonic algebra. Then every derivation on A is a generalized derivation.*

Proof. If d be a derivation of A, then d is trivially a generalized derivation of A with respect to itself d. □

Theorem 4.51. *Let $(A, *, 1)$ be a commutative bitonic algebra. Then every generalized derivation of A is a derivation.*

Proof. Let D be a generalized derivation of A with respect to a derivation d. Since A is commutative, D is a generalized (r,l)-derivation of A and d is also a (r,l)-derivation. So D and d are expansive by Lemma 2.14(3) and 3.33(3). Since D is also a generalized (l,r)-derivation with respect to the (l,r)-derivation d, D is a (l,r)-derivation by Theorem 3.39. Hence D is a derivation since A is commutative. □

Corollary 4.52. *Every generalized derivation of a commutative DBCK-algebra A is a derivation.*

Proof. It is clear from Theorem 4.51 □

As the implication algebras, the lattice implication algebras, the commutative Hilbert algebras and the commutative Heyting algebras are the commutative bitonic algebras, every generalized derivation of those algebras is a derivation by Theorem 4.51.

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