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A q-ANALOGUE OF $\bar{\alpha}$-WHITNEY NUMBERS

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We define the $(q, \bar{\boldsymbol{\alpha}})$-Whitney numbers which are reduced to the $\bar{\boldsymbol{\alpha}}$-Whitney numbers when $q \to 1$. Moreover, we obtain several properties of these numbers such as explicit formulas, recurrence relations, generating functions, orthogonality and inverse relations. Finally, we define the $\bar{\boldsymbol{\alpha}}$-Whitney-Lah numbers as a generalization of the r-Whitney-Lah numbers and we introduce their important basic properties.

1. INTRODUCTION

El-Desouky et al. [**5**] introduced the $\bar{\boldsymbol{\alpha}}$-Whitney numbers of both kinds as a new family of numbers generalizing many types of numbers such as r-Whitney numbers, Whitney numbers, r-Stirling numbers, Jacobi-Stirling numbers and Legendre-Stirling numbers.

The $\bar{\boldsymbol{\alpha}}$-Whitney numbers of the first kind $w_{m,\bar{\boldsymbol{\alpha}}}(n,k)$ and second kind $W_{m,\bar{\boldsymbol{\alpha}}}(n,k)$ are defined by

$$(x; \bar{\boldsymbol{\alpha}}|m)_n = \sum_{k=0}^{n} w_{m,\bar{\boldsymbol{\alpha}}}(n,k) x^k, \tag{1.1}$$

and

$$x^n = \sum_{k=0}^{n} W_{m,\bar{\boldsymbol{\alpha}}}(n,k) (x; \bar{\boldsymbol{\alpha}}|m)_k, \tag{1.2}$$

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where $\bar{\boldsymbol{\alpha}} = (\alpha_0, \alpha_1, \dots, \alpha_{n-1})$, and

$$(x; \bar{\boldsymbol{\alpha}}|m)_n = \prod_{j=0}^{n-1}(x - \alpha_j - jm) \text{ with } (x; \bar{\boldsymbol{\alpha}}|m)_0 = 1.$$

The $\bar{\boldsymbol{\alpha}}$-Whitney numbers of the first and second kind satisfying recurrence relations of the form:

$$w_{m,\bar{\boldsymbol{\alpha}}}(n+1,k) = w_{m,\bar{\boldsymbol{\alpha}}}(n,k-1) - (\alpha_n + nm)w_{m,\bar{\boldsymbol{\alpha}}}(n,k),$$

$$W_{m,\bar{\boldsymbol{\alpha}}}(n+1,k) = W_{m,\bar{\boldsymbol{\alpha}}}(n,k-1) + (\alpha_k + km)W_{m,\bar{\boldsymbol{\alpha}}}(n,k).$$

Note that the $\bar{\boldsymbol{\alpha}}$-Whitney numbers coincide with the r-Whitney numbers and Whitney numbers by setting $\bar{\boldsymbol{\alpha}} = (r, r, \dots, r)$ and $\bar{\boldsymbol{\alpha}} = (1, 1, \dots, 1)$, respectively. Many properties of the $\bar{\boldsymbol{\alpha}}$-Whitney numbers, r-Whitney numbers and Whitney numbers can be found in [**5, 4, 1, 8, 10, 11, 12**].

The organization of this article is as follows. In the next two sections, we define the q-analogue of the $\bar{\boldsymbol{\alpha}}$-Whitney numbers of the first and second kind denoted by $w_{q,m,\bar{\boldsymbol{\alpha}}}(n,k)$ and $W_{q,m,\bar{\boldsymbol{\alpha}}}(n,k)$, respectively, and obtain their recurrence relations, explicit formulas and generating functions. In the third section, we obtain the orthogonality property of the both kinds of the $(q, \bar{\boldsymbol{\alpha}})$-Whitney numbers which yields to the inverse relations. Moreover we give some important special cases. In the fourth section, we define the $\bar{\boldsymbol{\alpha}}$-Whitney-Lah numbers and deduce its recurrence relation, explicit formula and matrix representation.

Let $0 < q < 1$, x a real number and n a non-negative integer. The number $[x]_q = \frac{1-q^x}{1-q}$ is called q-real number and the q-factorial of n is defined by $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$, with $[0]_q! = 1$. Finally, the q-falling factorial of order n is defined by

$$([x]_q)_n = \prod_{j=0}^{n-1}[x-j]_q \text{ and } ([x]_q)_o = 1.$$

Moreover, the following definitions and notation are introduced.

$$([x; \bar{\boldsymbol{\alpha}}]_q)_n = \prod_{j=0}^{n-1}[x - \alpha_j]_q = [x-\alpha_0]_q[x-\alpha_1]_q \cdots [x - \alpha_{n-1}]_q \text{ with } ([x; \bar{\boldsymbol{\alpha}}]_q)_0 = 1,$$

$$([x; \bar{\boldsymbol{\alpha}}|m]_q)_n = \prod_{j=0}^{n-1}[x - \alpha_j - jm]_q, \text{ with } ([x; \bar{\boldsymbol{\alpha}}|m]_q)_0 = 1,$$

and

$$\left\langle [x; \bar{\boldsymbol{\alpha}}|m]_q \right\rangle_n = \prod_{j=0}^{n-1}([x]_q - [\alpha_j + jm]_q), \text{ with } \left\langle [x; \bar{\boldsymbol{\alpha}}|m]_q \right\rangle_0 = 1.$$

1. THE $(q, \bar{\boldsymbol{\alpha}})$-WHITNEY NUMBERS OF THE FIRST KIND

Definition 1. The $(q, \bar{\boldsymbol{\alpha}})$-Whitney numbers of the first kind $w_{q,m,\bar{\boldsymbol{\alpha}}}(n,k)$ are defined by

$$\left\langle [x; \bar{\boldsymbol{\alpha}}|m]_q \right\rangle_n = \sum_{k=0}^{n} w_{q,m,\bar{\boldsymbol{\alpha}}}(n,k)[x]_q^k, \tag{1.3}$$

where $w_{q,m,\bar{\boldsymbol{\alpha}}}(0,0) = 1$ and $w_{q,m,\bar{\boldsymbol{\alpha}}}(n,k) = 0$ for $k > n$ or $k < 0$.

Since for the q-numbers we have $[x-y]_q = q^{-y}([x]_q - [y]_q)$. Then

$$([x; \bar{\boldsymbol{\alpha}}|m]_q)_n = q^{-\sum_{i=0}^{n-1} \alpha_i + im} \prod_{j=0}^{n-1} ([x]_q - [\alpha_j + jm]_q).$$

Thus Eq. (1.3) in Definition 1 can be written in the equivalent form

$$([x; \bar{\boldsymbol{\alpha}}|m]_q)_n = q^{-\sum_{i=0}^{n-1} \alpha_i + im} \sum_{k=0}^{n} w_{q,m,\bar{\boldsymbol{\alpha}}}(n,k)[x]_q^k.$$

In particular, note that $w_{q,m,\bar{\boldsymbol{\alpha}}}(n,k)$ is reduced to the $w_{m,\bar{\boldsymbol{\alpha}}}(n,k)$ when $q \to 1$.

Theorem 1. *The $(q, \bar{\boldsymbol{\alpha}})$-Whitney numbers of the first kind satisfy the recurrence relation*

$$w_{q,m,\bar{\boldsymbol{\alpha}}}(n+1,k) = w_{q,m,\bar{\boldsymbol{\alpha}}}(n,k-1) - [\alpha_n + nm]_q w_{q,m,\bar{\boldsymbol{\alpha}}}(n,k), \tag{1.4}$$

where $n \geq k \geq 1$, and

$$w_{q,m,\bar{\boldsymbol{\alpha}}}(n,0) = (-1)^n \prod_{i=0}^{n-1} [\alpha_i + im]_q. \tag{1.5}$$

Proof. Since $\left\langle [x; \bar{\boldsymbol{\alpha}}|m]_q \right\rangle_{n+1} = \left\langle [x; \bar{\boldsymbol{\alpha}}|m]_q \right\rangle_n ([x]_q - [\alpha_n + nm]_q)$.

Using Eq. (1.3), we get

$$\begin{aligned}
&\sum_{k=0}^{n+1} w_{q,m,\bar{\boldsymbol{\alpha}}}(n+1,k)[x]_q^k = \left\langle [x; \bar{\boldsymbol{\alpha}}|m]_q \right\rangle_n ([x]_q - [\alpha_n + nm]_q) \\
&= \sum_{k=0}^{n} w_{q,m,\bar{\boldsymbol{\alpha}}}(n,k)[x]_q^k ([x]_q - [\alpha_n + nm]_q) \\
&= \sum_{k=0}^{n} w_{q,m,\bar{\boldsymbol{\alpha}}}(n,k)[x]_q^{k+1} - [\alpha_n + nm]_q \sum_{k=0}^{n} w_{q,m,\bar{\boldsymbol{\alpha}}}(n,k)[x]_q^k \\
&= \sum_{k=1}^{n+1} w_{q,m,\bar{\boldsymbol{\alpha}}}(n,k-1)[x]_q^k - [\alpha_n + nm]_q \sum_{k=0}^{n} w_{q,m,\bar{\boldsymbol{\alpha}}}(n,k)[x]_q^k.
\end{aligned}$$

Equating the coefficients of $[x]_q^k$ on both sides yields (1.4).

For $k=0$, we find

$$w_{q,m,\bar{\boldsymbol{\alpha}}}(n+1,0) = -[\alpha_n + nm]_q \, w_{q,m,\bar{\boldsymbol{\alpha}}}(n,0)\ ,\ n = 0,1,2,\ldots,$$

successive application gives (1.5). □

Definition 2. The $(q,\bar{\boldsymbol{\alpha}})$-Whitney matrix of the first kind is the $n \times n$ lower triangular matrix defined by

$$\mathbf{W}_1 := \mathbf{w}_{q,m,\bar{\boldsymbol{\alpha}}}(n) := \big(w_{q,m,\bar{\boldsymbol{\alpha}}}(i,j)\big)_{0\le i,j\le n-1}.$$

For example when $n=4$ the matrix $\mathbf{w}_{q,m,\bar{\boldsymbol{\alpha}}}(n)$ is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -[\alpha_0]_q & 1 & 0 & 0 \\ [\alpha_0]_q[\alpha_1+m]_q & -[\alpha_0]_q-[\alpha_1+m]_q & 1 & 0 \\ -[\alpha_0]_q[\alpha_1+m]_q[\alpha_2+2m]_q & [\alpha_0]_q[\alpha_1+m]_q+([\alpha_0]_q+[\alpha_1+m]_q)[\alpha_2+2m]_q & -[\alpha_0]_q-[\alpha_1+m]_q-[\alpha_2+2m]_q & 1 \end{pmatrix}$$

In particular, we note that $\mathbf{w}_{q,m,\bar{\boldsymbol{\alpha}}}(n)$ is reduced to the $\bar{\boldsymbol{\alpha}}$-Whiyney matrix of the first kind [**5**] when $q \to 1$. In addition at $q \to 1$ and $\bar{\boldsymbol{\alpha}} = (r,r,\ldots,r)$ the $\mathbf{w}_{q,m,\bar{\boldsymbol{\alpha}}}(n)$ is reduced to the r-Whitney matrix of the first kind [**12**].

Mansour et al. [**9**] derived a closed formula for all sequences satisfying a certain recurrence relation as follows:

Theorem 2. [**9**, Theorem 1.1]. *Suppose $(a_i)_{i\ge 0}$ and $(b_i)_{i\ge 0}$ are sequences of numbers with $b_i \ne b_j$ when $i \ne j$ and*

$$u(n,k) = u(n-1,k-1) + (a_{n-1}+b_k)u(n-1,k), \tag{1.6}$$

with boundary conditions $u(n,0) = \prod_{i=0}^{n-1}(a_i+b_0)$ and $u(0,k) = \delta_{0k}$, where δ_{jk} is the Kronecker delta function, then

$$u(n,k) = \sum_{j=0}^{k} \left(\frac{\prod_{i=0}^{n-1}(b_j+a_i)}{\prod_{i=0,i\ne j}^{k}(b_j-b_i)} \right), \quad \forall\, n,k \in \mathbb{N}. \tag{1.7}$$

Remark 1. [**9**, p. 25]

1. The recurrence for $u(n,k)$ is given by

$$u(n,k) = \sum_{j=k}^{n} u(j-1,k-1) \prod_{i=j}^{n-1}(a_i+b_k). \tag{1.8}$$

2. In the case when $b_i = 0$ for all i, then $u(n,k)$ is the $(n-k)$th elementary symmetric function of $a_0,\ a_1,\ \ldots,\ a_{n-1}$. The elementary symmetric function σ_k is defined by

$$\sigma_k(z_1,z_2,\ldots,z_n) = \sum_{1\le j_1<j_2<\cdots<j_k\le n} \prod_{i=1}^{k} z_{j_i},$$

where $\sigma_0 = 1$ and $\sigma_k = 0$ when $n < k$ or $k < 0$.

Theorem 3. *The $(q,\bar{\boldsymbol{\alpha}})$-Whitney numbers of the first kind are given by*

$$
\begin{aligned}
w_{q,m,\bar{\boldsymbol{\alpha}}}(n,k) &= (-1)^{n-k}\,\sigma_{n-k}([\alpha_0]_q,\ [\alpha_1+m]_q,\ \ldots,\ [\alpha_{n-1}+(n-1)m]_q)\\
&= (-1)^{n-k}\sum_{0\le j_1<\cdots<j_{n-k}\le n-1}\ \prod_{i=1}^{n-k}[\alpha_{j_i}+j_i\,m]_q.
\end{aligned}
\tag{1.9}
$$

and the following recurrence relation holds:

$$
w_{q,m,\bar{\boldsymbol{\alpha}}}(n,k) = \sum_{j=k}^{n}(-1)^{n-j}\,w_{q,m,\bar{\boldsymbol{\alpha}}}(j-1,k-1)\prod_{i=j}^{n-1}[\alpha_i+im]_q. \tag{1.10}
$$

Proof. Taking $a_i = -[\alpha_i+im]_q$ and $b_i = 0$, one can use Remark 1 to obtain Eq. (1.9) and Eq. (1.10). □

From Eq. (1.9), we can obtain the generating function

$$
\sum_{k=0}^{n} w_{q,m,\bar{\boldsymbol{\alpha}}}(n,k)x^k = \prod_{j=0}^{n-1}(x-[\alpha_j+jm]_q). \tag{1.11}
$$

As $q\to 1$, Eq. (1.10) reduces to a new recurrence relation for the $\bar{\boldsymbol{\alpha}}$-Whitney numbers of the first kind given by

$$
w_{m,\bar{\boldsymbol{\alpha}}}(n,k) = \sum_{j=k}^{n}(-1)^{n-j}\,w_{m,\bar{\boldsymbol{\alpha}}}(j-1,k-1)\prod_{i=j}^{n-1}(\alpha_i+im).
$$

2. THE $(q,\bar{\boldsymbol{\alpha}})$-WHITNEY NUMBERS OF THE SECOND KIND

Definition 3. The $(q,\bar{\boldsymbol{\alpha}})$-Whitney numbers of the second kind $W_{m,\bar{\boldsymbol{\alpha}}}(n,k)$ are defined by

$$
[x]_q^n = \sum_{k=0}^{n} W_{q,m,\bar{\boldsymbol{\alpha}}}(n,k)\big\langle [x;\bar{\boldsymbol{\alpha}}|m]_q\big\rangle_k, \tag{1.12}
$$

where $W_{q,m,\bar{\boldsymbol{\alpha}}}(0,0)=1$ and $W_{q,m,\bar{\boldsymbol{\alpha}}}(n,k)=0$ for $k>n$ or $k<0$.

We notice that Eq. (1.12) in Definition 3 can be written in the equivalent form

$$
[x]_q^n = \sum_{k=0}^{n} q^{\sum_{i=0}^{k-1}\alpha_i+im}W_{q,m,\bar{\boldsymbol{\alpha}}}(n,k)([x;\bar{\boldsymbol{\alpha}}|m]_q)_k.
$$

Theorem 4. *The $(q,\bar{\boldsymbol{\alpha}})$-Whitney numbers of the second kind satisfy the recurrence relation*

$$
W_{q,m,\bar{\boldsymbol{\alpha}}}(n+1,k) = W_{q,m,\bar{\boldsymbol{\alpha}}}(n,k-1) + [\alpha_k+km]_q W_{q,m,\bar{\boldsymbol{\alpha}}}(n,k), \tag{1.13}
$$

where $n\ge k\ge 1$, for $k=0$ we have

$$
W_{q,m,\bar{\boldsymbol{\alpha}}}(n,0) = [\alpha_0]_q^n. \tag{1.14}
$$

Proof. Since we have $[x]_q^{n+1} = [x]_q^n\left([x]_q - [\alpha_k + km]_q + [\alpha_k + km]_q\right)$. Using (1.12), we get

$$\begin{aligned}
&\sum_{k=0}^{n+1} W_{q,m,\bar{\boldsymbol{\alpha}}}(n+1,k)\big\langle [x;\bar{\boldsymbol{\alpha}}|m]_q\big\rangle_k \\
&= \sum_{k=0}^{n} W_{q,m,\bar{\boldsymbol{\alpha}}}(n,k)\big\langle [x;\bar{\boldsymbol{\alpha}}|m]_q\big\rangle_k([x]_q - [\alpha_k + km]_q + [\alpha_k + km]_q) \\
&= \sum_{k=0}^{n} W_{q,m,\bar{\boldsymbol{\alpha}}}(n,k)\big\langle [x;\bar{\boldsymbol{\alpha}}|m]_q\big\rangle_{k+1} + \sum_{k=0}^{n}[\alpha_k + km]_q W_{q,m,\bar{\boldsymbol{\alpha}}}(n,k)\big\langle [x;\bar{\boldsymbol{\alpha}}|m]_q\big\rangle_k \\
&= \sum_{k=1}^{n+1} W_{q,m,\bar{\boldsymbol{\alpha}}}(n,k-1)\big\langle [x;\bar{\boldsymbol{\alpha}}|m]_q\big\rangle_k + \sum_{k=0}^{n}[\alpha_k + km]_q W_{q,m,\bar{\boldsymbol{\alpha}}}(n,k)\big\langle [x;\bar{\boldsymbol{\alpha}}|m]_q\big\rangle_k.
\end{aligned}$$

Equating the coefficients of $\big\langle [x;\bar{\boldsymbol{\alpha}}|m]_q\big\rangle_k$ on both sides, we obtain Eq. (1.13).

When $k = 0$, we get $W_{q,m,\bar{\boldsymbol{\alpha}}}(n+1,0) = [\alpha_0]_q\, W_{q,m,\bar{\boldsymbol{\alpha}}}(n,0),\ n = 0,1,2,\dots$. Thus $W_{q,m,\bar{\boldsymbol{\alpha}}}(n,0) = [\alpha_0]_q^n\, W_{q,m,\bar{\boldsymbol{\alpha}}}(0,0) = [\alpha_0]_q^n$. □

Definition 4. The $(q,\bar{\boldsymbol{\alpha}})$-Whitney matrix of the second kind is the $n\times n$ lower triangular matrix defined by

$$\mathbf{W}_2 := \mathbf{W}_{q,m,\bar{\boldsymbol{\alpha}}}(n) := \big(W_{q,m,\bar{\boldsymbol{\alpha}}}(i,j)\big)_{0\le i,j\le n-1}.$$

For example when $n = 4$, the matrix $\mathbf{W}_{q,m,\bar{\boldsymbol{\alpha}}}(n)$ is given by

$$\begin{pmatrix}
1 & 0 & 0 & 0 \\
[\alpha_0]_q & 1 & 0 & 0 \\
[\alpha_0]_q^2 & [\alpha_0]_q + [\alpha_1 + m]_q & 1 & 0 \\
[\alpha_0]_q^3 & [\alpha_0]_q^2 + [\alpha_0]_q[\alpha_1 + m]_q + [\alpha_1 + m]_q^2 & [\alpha_0]_q + [\alpha_1 + m]_q + [\alpha_2 + 2m]_q & 1
\end{pmatrix}$$

When $q \to 1$ the matrix $\mathbf{W}_{q,m,\bar{\boldsymbol{\alpha}}}(n)$ is reduced to the $\bar{\boldsymbol{\alpha}}$-Whiyney matrix of the second kind [**5**], also at $q \to 1$ and $\bar{\boldsymbol{\alpha}} = (r,r,\dots,r)$ the $\mathbf{W}_{q,m,\bar{\boldsymbol{\alpha}}}(n)$ is reduced to the r-Whitney matrix of the second kind [**4, 12**].

Theorem 5. *The $(q,\bar{\boldsymbol{\alpha}})$-Whitney numbers of the second kind $W_{q,m,\bar{\boldsymbol{\alpha}}}(n,k)$ have the explicit formula*

$$W_{q,m,\bar{\boldsymbol{\alpha}}}(n,k) = \sum_{j=0}^{k} \frac{([\alpha_j + jm]_q)^n}{\prod_{i=0,i\ne j}^{k}([\alpha_j + jm]_q - [\alpha_i + im]_q)}, \tag{1.15}$$

and satisfy the recurrence relation

$$\begin{aligned} W_{q,m,\bar{\boldsymbol{\alpha}}}(n,k) &= \sum_{j=k}^{n} W_{q,m,\bar{\boldsymbol{\alpha}}}(j-1,k-1) \prod_{i=j}^{n-1} [\alpha_k + km]_q \\ &= \sum_{j=k}^{n} W_{q,m,\bar{\boldsymbol{\alpha}}}(j-1,k-1)\big([\alpha_k + km]_q\big)^{n-j}. \end{aligned} \tag{1.16}$$

Proof. Taking $a_i = 0$ and $b_i = [\alpha_i + im]_q$ in (1.7) and (1.8), yield (1.15) and (1.16), respectively. □

As $q \to 1$, the recurrence relation (1.16) reduces to a new recurrence relation for the $\bar{\boldsymbol{\alpha}}$-Whitney numbers of the second kind given by

$$\begin{aligned} W_{m,\bar{\boldsymbol{\alpha}}}(n,k) &= \sum_{j=k}^{n} W_{m,\bar{\boldsymbol{\alpha}}}(j-1,k-1) \prod_{i=j}^{n-1} (\alpha_k + km) \\ &= \sum_{j=k}^{n} W_{m,\bar{\boldsymbol{\alpha}}}(j-1,k-1)(\alpha_k + km)^{n-j}. \end{aligned}$$

Using (1.15) we obtain the exponential generating function of the $(q, \bar{\boldsymbol{\alpha}})$-Whitney numbers of the second kind $W_{q,m,\bar{\boldsymbol{\alpha}}}(n,k)$

$$\begin{aligned} \sum_{n=0}^{\infty} W_{q,m,\bar{\boldsymbol{\alpha}}}(n,k) \frac{t^n}{[n]_q!} &= \sum_{j=0}^{k} \frac{1}{\prod_{i=0, i\neq j}^{k} ([\alpha_j + jm]_q - [\alpha_i + im]_q)} \sum_{n=0}^{\infty} \frac{([\alpha_j + jm]_q\, t)^n}{[n]_q!} \\ &= \sum_{j=0}^{k} \frac{1}{\prod_{i=0, i\neq j}^{k} ([\alpha_j + jm]_q - [\alpha_i + im]_q)} e_q([\alpha_j + jm]_q\, t). \end{aligned} \tag{1.17}$$

Theorem 6. *The generating function of* $W_{q,m,\bar{\boldsymbol{\alpha}}}(n,k)$ *is given by*

$$Y_{k,q}(t) = \sum_{n=k}^{\infty} W_{q,m,\bar{\boldsymbol{\alpha}}}(n,k)\, t^n = t^k \prod_{j=0}^{k} (1 - [\alpha_j + jm]_q t)^{-1}, \quad t < \frac{1}{[\alpha_k + km]_q} \tag{1.18}$$

where $k = 1, 2, 3, \ldots,$ *and*

$$Y_{k,q}(0) = 0 \; for \; k \geq 1 \; and \; Y_{0,q}(t) = (1 - [\alpha_0]_q t)^{-1}. \tag{1.19}$$

Proof. Equation (1.19) can easily obtained from the definition of generating function

$$Y_{0,q}(t) = \sum_{n=0}^{\infty} W_{q,m,\bar{\boldsymbol{\alpha}}}(n,0)\, t^n = \sum_{n=0}^{\infty} [\alpha_0]_q^n\, t^n = \sum_{n=0}^{\infty} ([\alpha_0]_q\, t)^n = (1 - [\alpha_0]_q\, t)^{-1}.$$

From (1.13), we get

$$\sum_{n=k}^{\infty} W_{q,m,\bar{\boldsymbol{\alpha}}}(n,k)\, t^n = \sum_{n=k}^{\infty} W_{q,m,\bar{\boldsymbol{\alpha}}}(n-1,k-1)\, t^n + (\alpha_k + km) \sum_{n=k}^{\infty} W_{q,m,\bar{\boldsymbol{\alpha}}}(n-1,k)\, t^n.$$

Thus we obtain the recurrence relation for the generating function $Y_{k,q}(t)$

$$Y_{k,q}(t) = t\, Y_{k-1,q} + [\alpha_k + km]_q\, t\, Y_{k,q}(t), \quad k = 1, 2, \dots .$$

Hence

$$Y_{k,q}(t) = \frac{t}{(1 - [\alpha_k + km]_q\, t)}\, Y_{k-1,q}(t), \quad k = 1, 2, \dots . \tag{1.20}$$

Applying successively this recurrence, we get Eq. (1.18). □

The previous theorem shows that the numbers $W_{q,m,\bar{\boldsymbol{\alpha}}}(n,k)$ are the complete symmetric function of the numbers $[\alpha_0]_q$, $[\alpha_1 + m]_q$, $\dots$, $[\alpha_k + km]_q$ of order $n - k$.

We obtain from Eq. (1.18)

$$\sum_{n=k}^{\infty} W_{q,m,\bar{\boldsymbol{\alpha}}}(n,k)\, t^{n-k} = \prod_{j=0}^{k} (1 - [\alpha_j + jm]_q t)^{-1}, \quad t < \frac{1}{[\alpha_k + km]_q}.$$

Expanding the right hand side and comparing the coefficients of t^{n-k} yields

$$W_{q,m,\bar{\boldsymbol{\alpha}}}(n,k) = \sum_{0 \le j_1 \le \cdots \le j_{n-k} \le k} \prod_{i=1}^{n-k} [\alpha_{j_i} + j_i\, m]_q.$$

3. ORTHOGONALITY AND INVERSE RELATIONS

The orthogonality and the inverse relations for the $\bar{\boldsymbol{\alpha}}$-Whitney numbers of both kinds were obtained in [**5**]. In this section, we establish analogous properties for the $(q, \bar{\boldsymbol{\alpha}})$-Whitney numbers of both kinds.

Theorem 7. *The $(q, \bar{\boldsymbol{\alpha}})$-Whitney numbers of the first and second kind satisfy the following orthogonality relations:*

$$\sum_{k=j}^{n} W_{q,m,\bar{\boldsymbol{\alpha}}}(n,k)\, w_{q,m,\bar{\boldsymbol{\alpha}}}(k,j) = \delta_{nj}, \tag{1.21}$$

and

$$\sum_{k=j}^{n} w_{q,m,\bar{\boldsymbol{\alpha}}}(n,k)\, W_{q,m,\bar{\boldsymbol{\alpha}}}(k,j) = \delta_{nj}. \tag{1.22}$$

Proof. Using (1.3) and (1.12) give

$$\begin{aligned}[x]_q^n &= \sum_{k=0}^{n} W_{q,m,\bar{\boldsymbol{\alpha}}}(n,k)\big\langle [x;\bar{\boldsymbol{\alpha}}|m]_q\big\rangle_k \\ &= \sum_{k=0}^{n} W_{q,m,\bar{\boldsymbol{\alpha}}}(n,k)\sum_{j=0}^{k} w_{q,m,\bar{\boldsymbol{\alpha}}}(k,j)[x]_q^j \\ &= \sum_{j=0}^{n}\{\sum_{k=j}^{n} W_{q,m,\bar{\boldsymbol{\alpha}}}(n,k)w_{q,m,\bar{\boldsymbol{\alpha}}}(k,j)\}[x]_q^j.\end{aligned}$$

Comparing the coefficients of $[x]_q^j$ gives

$$\sum_{k=j}^{n} W_{q,m,\bar{\boldsymbol{\alpha}}}(n,k)w_{q,m,\bar{\boldsymbol{\alpha}}}(k,j) = \delta_{nj}.$$

The second relation can be proved similarly. □

The orthogonality properties give the following identities

$$\mathbf{W}_2\ \mathbf{W}_1 = \mathbf{W}_1\ \mathbf{W}_2 = \mathbf{I}.\ \text{Thus}\ \mathbf{W}_2^{-1} = \mathbf{W}_1 \text{ and } \mathbf{W}_1^{-1} = \mathbf{W}_2.$$

The following theorem can easily be deduced from Theorem 7.

Theorem 8. *The $(q,\bar{\boldsymbol{\alpha}})$-Whitney numbers of the first and second kind satisfy the following inverse relations*

$$f_n = \sum_{k=0}^{n} W_{q,m,\bar{\boldsymbol{\alpha}}}(n,k)\ g_k \iff g_n = \sum_{k=0}^{n} w_{q,m,\bar{\boldsymbol{\alpha}}}(n,k)\ f_k. \tag{1.23}$$

Proof. If the condition

$$f_n = \sum_{k=0}^{n} W_{q,m,\bar{\boldsymbol{\alpha}}}(n,k)\ g_k$$

holds, then

$$\begin{aligned}\sum_{k=0}^{n} w_{q,m,\bar{\boldsymbol{\alpha}}}(n,k)\ f_k &= \sum_{k=0}^{n} w_{q,m,\bar{\boldsymbol{\alpha}}}(n,k)\sum_{m=0}^{k} W_{q,m,\bar{\boldsymbol{\alpha}}}(k,m)\ g_m \\ &= \sum_{m=0}^{k}(\sum_{k=m}^{n} w_{q,m,\bar{\boldsymbol{\alpha}}}(n,k)W_{q,m,\bar{\boldsymbol{\alpha}}}(k,m))g_m.\end{aligned}$$

By Theorem 7, we get

$$\sum_{k=0}^{n} w_{q,m,\bar{\boldsymbol{\alpha}}}(n,k)\ f_k = \sum_{m=0}^{n} \delta_{mn}\ g_m = g_n.$$

The converse can be shown similarly. □

Special cases:

1. Setting $m = 1$ and $\bar{\boldsymbol{\alpha}} = (r, r, \ldots, r) := \mathbf{r}$, then (1.3) and (1.12), respectively, give
$$w_{q,1,\mathbf{r}}(n,k) = s_q(n,k,r) \text{ and } W_{q,1,\mathbf{r}}(n,k) = S_q(n,k,r),$$
where $s_q(n,k,r)$ and $S_q(n,k,r)$ are the non-central q-Stirling numbers of the first and second kind, respectively, see [**3**].

2. Setting $m = 1$ and $\bar{\boldsymbol{\alpha}} = (0, 0, \ldots, 0) := \mathbf{0}$, hence (1.3) and (1.12), respectively, give
$$w_{q,1,\mathbf{0}}(n,k) = s_q(n,k), \text{ and } W_{q,1,\mathbf{0}}(n,k) = S_q(n,k),$$
where $s_q(n,k)$ and $S_q(n,k)$ are the q-Stirling numbers of the first and second kind, respectively, see [**2, 7**].

3. Setting $m = 1$ and $\alpha_j + j = \beta_j$, for $j = 0, 1, \ldots, n-1$, then (1.3) and (1.12), respectively, give
$$w_{q,1,\bar{\boldsymbol{\alpha}}}(n,k) = s_{q,\bar{\boldsymbol{\beta}}}(n,k) \text{ and } W_{q,1,\bar{\boldsymbol{\alpha}}}(n,k) = S_{q,\bar{\boldsymbol{\beta}}}(n,k),$$
where $s_{q,\bar{\boldsymbol{\beta}}}(n,k)$ and $S_{q,\bar{\boldsymbol{\beta}}}(n,k)$ are the generalized q-Stirling numbers of the first and second kind (q-Comtet numbers), respectively, see [**6**].

4. THE $\bar{\alpha}$-WHITNEY-LAH NUMBERS

The signless Lah numbers $L(n,k) = \frac{n!}{k!}\binom{n-1}{k-1}$ were first studied by Lah [**13**] and they expressed in terms of the signless stirling numbers $s(n,k)$ of the first kind, and the stirling numbers $S(n,k)$ of the second kind

$$L(n,k) = \sum_{j=k}^{n} s(n,j)S(j,k).$$

Choen and Jung [**4**] defined the r-Whitney-Lah numbers $L_{m,r}(n,k)$ by

$$L_{m,r}(n,k) = \sum_{j=k}^{n} (-1)^{n-j} w_{m,r}(n,j) W_{m,r}(j,k).$$

Analogously, we define the $\bar{\boldsymbol{\alpha}}$-Whitney-Lah numbers $L_{m,\bar{\boldsymbol{\alpha}}}(n,k)$ as follows:

$$L_{m,\bar{\boldsymbol{\alpha}}}(n,k) = \sum_{j=k}^{n} (-1)^{n-j}\, w_{m,\bar{\boldsymbol{\alpha}}}(n,j)\, W_{m,\bar{\boldsymbol{\alpha}}}(j,k), \tag{1.24}$$

where $L_{m,\bar{\boldsymbol{\alpha}}}(0,0) = 1$ and $L_{m,\bar{\boldsymbol{\alpha}}}(n,k) = 0$ for $n < k$ or $k < 0$.

Theorem 9. *The $\bar{\boldsymbol{\alpha}}$-Whitney-Lah numbers $L_{m,\bar{\boldsymbol{\alpha}}}(n,k)$ may be obtained from*

$$\prod_{i=0}^{n-1}(x+\alpha_i+im)=\sum_{k=0}^{n} L_{m,\bar{\boldsymbol{\alpha}}}(n,k)(x;\bar{\boldsymbol{\alpha}}|m)_k. \tag{1.25}$$

Proof. Replacing x by $-x$ in Eq. (1.1), we get

$$\prod_{i=0}^{n-1}(x+\alpha_i+im)=\sum_{j=0}^{n}(-1)^{n-j}w_{m,\bar{\boldsymbol{\alpha}}}(n,j)\,x^j. \tag{1.26}$$

Hence

$$\begin{aligned}\sum_{k=0}^{n} L_{m,\bar{\boldsymbol{\alpha}}}(n,k)(x;\bar{\boldsymbol{\alpha}}|m)_k &= \sum_{k=0}^{n}\sum_{j=k}^{n}(-1)^{n-j}w_{m,\bar{\boldsymbol{\alpha}}}(n,j)W_{m,\bar{\boldsymbol{\alpha}}}(j,k)(x;\bar{\boldsymbol{\alpha}}|m)_k\\ &= \sum_{j=0}^{n}\sum_{k=0}^{j}(-1)^{n-j}w_{m,\bar{\boldsymbol{\alpha}}}(n,j)\,W_{m,\bar{\boldsymbol{\alpha}}}(j,k)(x;\bar{\boldsymbol{\alpha}}|m)_k\\ &= \sum_{j=0}^{n}(-1)^{n-j}w_{m,\bar{\boldsymbol{\alpha}}}(n,j)x^j=\prod_{i=0}^{n-1}(x+\alpha_i+im)\end{aligned}$$

□

Theorem 10. *The $\bar{\boldsymbol{\alpha}}$-Whitney-Lah numbers satisfy the recurrence relation*

$$L_{m,\bar{\boldsymbol{\alpha}}}(n+1,k)=L_{m,\bar{\boldsymbol{\alpha}}}(n,k-1)+(\alpha_k+\alpha_n+(k+n)m)L_{m,\bar{\boldsymbol{\alpha}}}(n,k), \tag{1.27}$$

where $n\ge k\ge 1$, for $k=0$ we have

$$L_{m,\bar{\boldsymbol{\alpha}}}(n,0)=\prod_{i=0}^{n-1}(\alpha_0+\alpha_i+im). \tag{1.28}$$

Proof. We can write

$$\prod_{i=0}^{n}(x+\alpha_i+im)=\prod_{i=0}^{n-1}(x+\alpha_i+im)(x-\alpha_k-km+\alpha_k+km+\alpha_n+nm).$$

Using (1.25), we get

$$\begin{aligned}&\sum_{k=0}^{n+1} L_{m,\bar{\boldsymbol{\alpha}}}(n+1,k)(x;\bar{\boldsymbol{\alpha}}|m)_k\\ &=\sum_{k=0}^{n} L_{m,\bar{\boldsymbol{\alpha}}}(n,k)(x;\bar{\boldsymbol{\alpha}}|m)_k\big((x-\alpha_k-km)+(\alpha_k+km+\alpha_n+nm)\big)\\ &=\sum_{k=0}^{n} L_{m,\bar{\boldsymbol{\alpha}}}(n,k)(x;\bar{\boldsymbol{\alpha}}|m)_{k+1}+\sum_{k=0}^{n} L_{m,\bar{\boldsymbol{\alpha}}}(n,k)(x;\bar{\boldsymbol{\alpha}}|m)_k(\alpha_k+km+\alpha_n+nm)\\ &=\sum_{k=1}^{n+1} L_{m,\bar{\boldsymbol{\alpha}}}(n,k-1)(x;\bar{\boldsymbol{\alpha}}|m)_k+\sum_{k=0}^{n} L_{m,\bar{\boldsymbol{\alpha}}}(n,k)(x;\bar{\boldsymbol{\alpha}}|m)_k(\alpha_k+km+\alpha_n+nm).\end{aligned}$$

Equating the coefficients of $(x;\bar{\boldsymbol{\alpha}}|m)_k$ on both sides, we obtain (1.27).

For $k=0$, we find

$$L_{m,\bar{\boldsymbol{\alpha}}}(n+1,0)=L_{m,\bar{\boldsymbol{\alpha}}}(n,0)\,(\alpha_0+\alpha_n+nm),\ n=0,1,2,\ldots .$$

Consequently, we get

$$L_{m,\bar{\boldsymbol{\alpha}}}(n,0)=L_{m,\bar{\boldsymbol{\alpha}}}(0,0)\,2\alpha_0\,(\alpha_0+\alpha_1+m)\cdots(\alpha_0+\alpha_{n-1}+(n-1)m).$$

□

Special cases:

1. The $L_{m,\bar{\boldsymbol{\alpha}}}(n,k)$ is reduced to $L(n,k)$ when $m=1$ and $\bar{\boldsymbol{\alpha}}=(0,0,\ldots,0)$.
2. The $L_{m,\bar{\boldsymbol{\alpha}}}(n,k)$ is reduced to $L_{m,r}(n,k)$ when $\bar{\boldsymbol{\alpha}}=(r,r,\ldots,r)$.
3. The $L_{m,\bar{\boldsymbol{\alpha}}}(n,k)$ is reduced to the r-Lah numbers $L_r(n+r,k+r)$ when $m=1$ and $\bar{\boldsymbol{\alpha}}=(r,r,\ldots,r)$, see [**14**].

Defining the $\bar{\boldsymbol{\alpha}}$-Whitney-Lah matrix as

$$\mathbf{L}:=\mathbf{L}_{m,\bar{\boldsymbol{\alpha}}}(n):=\big(L_{m,\bar{\boldsymbol{\alpha}}}(i,j)\big)_{0\le i,j\le n-1}.$$

For example when $n=4$ the matrix $\mathbf{L}_{m,\bar{\boldsymbol{\alpha}}}(n)$ is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0\\ 2\alpha_0 & 1 & 0 & 0\\ 2\alpha_0(\alpha_0+\alpha_1+m) & 2(\alpha_0+\alpha_1+m) & 1 & 0\\ 2\alpha_0(\alpha_0+\alpha_1+m)(\alpha_0+\alpha_2+2m) & 2(\alpha_0+\alpha_1+m)(\alpha_0+\alpha_1+\alpha_2+3m) & 2(\alpha_0+\alpha_1+\alpha_2+3m) & 1 \end{pmatrix}$$

In particular, when $\bar{\boldsymbol{\alpha}}=(r,r,\ldots,r)$ the $\mathbf{L}_{m,\bar{\boldsymbol{\alpha}}}(n)$ is reduced to the r-Whitney-Lah matrix [**12**].

Theorem 11. *The $\bar{\boldsymbol{\alpha}}$-Whitney-Lah numbers $L_{m,\bar{\boldsymbol{\alpha}}}(n,k)$ have the explicit formula*

$$L_{m,\bar{\boldsymbol{\alpha}}}(n,k)=\sum_{j=0}^{k}\frac{\prod_{i=0}^{n-1}(\alpha_j+jm+\alpha_i+im)}{\prod_{i=0,i\neq j}^{k}(\alpha_j+jm-\alpha_i-im)}, \tag{1.29}$$

and the recurrence relation

$$L_{m,\bar{\boldsymbol{\alpha}}}(n,k)=\sum_{j=k}^{n}L_{m,\bar{\boldsymbol{\alpha}}}(j-1,k-1)\prod_{i=j}^{n-1}(\alpha_i+im+\alpha_k+km). \tag{1.30}$$

Proof. The proof follows by setting $a_i=b_i=\alpha_i+im$ in (1.7) and (1.8). □

In particular, by setting $\bar{\boldsymbol{\alpha}}=(r,r,\ldots,r)$ we obtain the explicit formula and recurrence relation for $L_{m,r}(n,k)$ as follows:

Corollary 1. *The r-Whitney-Lah numbers $L_{m,r}(n,k)$ satisfy the following:*

$$L_{m,r}(n,k) = \frac{1}{m^k k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\prod_{i=0}^{n-1}(2r+jm+im). \tag{1.31}$$

$$L_{m,r}(n,k) = \sum_{j=k}^{n} L_{m,r}(j-1,k-1)\prod_{i=j}^{n-1}(2r+km+im). \tag{1.32}$$

Choen and Jung [**4**] showed that

$$L_{m,r}(n,k) = \binom{n}{k}\prod_{i=0}^{n-k-1}(2r+km+im). \tag{1.33}$$

Thus from (1.31) and (1.33) we obtain the following combinatorial identity

$$\frac{1}{m^k k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\prod_{i=0}^{n-1}(2r+jm+im) = \binom{n}{k}\prod_{i=0}^{n-k-1}(2r+km+im). \tag{1.34}$$

4.1 MATRIX REPRESENTATIONS

Let $\mathbf{w}, \mathbf{W}$ and $\mathbf{L}$ denote infinite lower triangular matrices whose (n,k)-th entries are $w_{m,\bar{\alpha}}(n,k)$, $W_{m,\bar{\alpha}}(n,k)$, and $L_{m,\bar{\alpha}}(n,k)$, respectively. Furthermore, let $\mathbf{D}$ be the infinite diagonal matrix whose (n,k)-th entry is $D(n,k) = (-1)^n \delta nk$, hence $\mathbf{D}^{-1} = \mathbf{D}$, and $\mathbf{D}\mathbf{D}^{-1} = \mathbf{I}$. Equation (1.24) can be written in the matrix form

$$\mathbf{L} = \mathbf{DwDW}.$$

El-Desouky et al. [**5**] showed that $\mathbf{w}^{-1} = \mathbf{W}$, $\mathbf{W}^{-1} = \mathbf{w}$. Thus

$$\mathbf{L}^{-1} = \mathbf{W}^{-1}\mathbf{D}\mathbf{w}^{-1}\mathbf{D} = \mathbf{wDWD} = \mathbf{DDwDWD} = \mathbf{DLD}.$$

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