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LINEAR PROGRAMMING PROBLEMS ON TIME SCALES

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In this work, we study linear programming problems on time scales. This approach unifies discrete and continuous linear programming models and extends them to other cases "in between." After a brief introduction to time scales, we formulate the primal as well as the dual time scales linear programming models. Next, we establish and prove the weak duality theorem and the optimality conditions theorem for arbitrary time scales, while the strong duality theorem is established for isolated time scales. Finally, examples are given in order to illustrate the effectiveness of the presented results.

1. INTRODUCTION

It is well known that discrete-time linear programming problems have numerous applications in areas such as portfolio optimization, crew scheduling, manufacturing, transportation, telecommunication, agriculture, and so on. Continuous-time linear programming problems were first studied by Bellman [3] as a bottleneck process. He established the weak duality theorem and optimality conditions. A computational approach has been presented by Bellman and Dreyfus [4]. The strong duality theorem was studied by Tyndall [27,28] and Levinson [24]. Grinold [21] has established strong duality without discretizing the continuous problem. A numerical solution to continuous-time linear programming was considered by Buie and Abrham [19]. Wen, Lur, and Lai [29] have presented an approximation approach to solve continuous-time problems.

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The theory of time scales, on the other hand, was first introduced by Stefan Hilger in 1988 in his PhD dissertation, see [22]. The purpose of this theory is to unify discrete and continuous analysis and to offer an extension to cases "in between". Many applications in mathematical modelling exist for this theory, e.g., to optimal control [2, 15–18, 25], population biology [6], calculus of variations [5, 7, 10], and economics [1, 8, 11, 12, 26].

In this work, it is therefore meaningful to consider linear programming problems on arbitrary time scales $\mathbb{T}$ as an extension of the continuous-time linear programming problems and to extend the results in [21, 24, 27–29] to arbitrary time scale. The paper is organized as follows. In Section 2, some examples related to time scales calculus are given. In Section 3, the basic structures of the primal linear programming model as well as the dual model on time scales are formulated. The weak duality theorem and the optimality condition theorem are presented and proved in Section 4. Section 5 states and proves the strong duality theorem on isolated time scales. Examples are presented in Section 6 in order to demonstrate our theoretical results. In Section 7, some conclusions are given.

2. TIME SCALES CALCULUS

In this section, instead of introducing the basic definitions, derivative, and integral on time scales, we refer the reader to the monographs [9, 13, 14], in which comprehensive details and complete proofs are given. For readers not familiar with the time scales calculus, we give the following few examples. Throughout, $\mathbb{T}$ is the time scale, σ is the forward jump operator, μ is the graininess, $f : \mathbb{T} \to \mathbb{R}$ is a function, $f^\sigma = f \circ \sigma$ is the advance of f, f^Δ is the delta derivative of f, and $\int_a^b f(t)\Delta t$ is the time scales integral of f between $a, b \in \mathbb{T}$.

Example 2.1. If $\mathbb{T} = \mathbb{R}$, then

$$\sigma(t) = t, \quad \mu(t) \equiv 0, \quad f^\Delta(t) = f'(t) \quad \text{for} \quad t \in \mathbb{T},$$

and

$$\int_a^b f(t)\Delta t = \int_a^b f(t)\mathrm{d}t, \quad \text{where} \quad a, b \in \mathbb{T} \text{ with } a < b,$$

is the usual Riemann integral from calculus.

Example 2.2. If $\mathbb{T} = \{t_k \in \mathbb{R} : k \in \mathbb{N}_0\}$ with $t_k < t_{k+1}$ for all $k \in \mathbb{N}_0$ consists only of isolated points, then

$$\sigma(t_k) = t_{k+1}, \quad \mu(t_k) = t_{k+1} - t_k, \quad f^\Delta(t_k) = \frac{f(t_{k+1}) - f(t_k)}{t_{k+1} - t_k} \quad \text{for} \quad k \in \mathbb{N}_0,$$

and

$$\int_{t_m}^{t_n} f(t)\Delta t = \sum_{k=m}^{n-1} \mu(t_k) f(t_k), \quad \text{where} \quad m, n \in \mathbb{N}_0 \text{ with } m < n. \tag{2.1}$$

The examples in Section 6 are specific cases of Example 2.2 as follows.

Example 2.3. Let $h > 0$. If $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$, then

$$\sigma(t) = t + h, \quad \mu(t) \equiv h, \quad f^{\Delta}(t) = \frac{f(t+h) - f(t)}{h} \quad \text{for} \quad t \in \mathbb{T},$$

and

$$\int_a^b f(t)\Delta t = h \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh), \quad \text{where} \quad a, b \in \mathbb{T} \text{ with } a < b.$$

Example 2.4. If $\mathbb{T} = \mathbb{Z}$, then

$$\sigma(t) = t + 1, \quad \mu(t) \equiv 1, \quad f^{\Delta}(t) = \Delta f(t) = f(t+1) - f(t) \quad \text{for} \quad t \in \mathbb{T},$$

and

$$\int_a^b f(t)\Delta t = \sum_{k=a}^{b-1} f(k), \quad \text{where} \quad a, b \in \mathbb{T} \text{ with } a < b.$$

Example 2.5. Let $q > 1$. If $\mathbb{T} = q^{\mathbb{N}_0} = \{q^n : n \in \mathbb{N}_0\}$, then

$$\sigma(t) = qt, \quad \mu(t) = (q-1)t, \quad f^{\Delta}(t) = \frac{f(qt) - f(t)}{(q-1)t} \quad \text{for} \quad t \in \mathbb{T},$$

and

$$\int_{q^m}^{q^n} f(t)\Delta t = (q-1) \sum_{k=m}^{n-1} q^k f(q^k), \quad \text{where} \quad m, n \in \mathbb{N}_0 \text{ with } m < n.$$

3. PRIMAL AND DUAL MODELS

Throughout this paper, $\mathbb{T}$ stands for a time scale, we assume $0 \in \mathbb{T}$, we let $T \in \mathbb{T}$, and we use $\mathcal{I}$ to denote the time scales interval

$$\mathcal{I} = [0, T] \cap \mathbb{T}.$$

By E_k, we denote the space of all rd-continuous functions from $\mathcal{I}$ into $\mathbb{R}^k$. We also put $E_k^{\top} = \{f : f^{\top} \in E_k\}$. The primal time scales linear programming model is formulated as

$$\text{(P)} \qquad \begin{cases} \text{Maximize} \quad U(x) = \displaystyle\int_0^{\sigma(T)} f(t)x(t)\Delta t \\ \text{subject to} \quad B(t)x(t) \le g(t) + \displaystyle\int_0^t K(t,s)x(s)\Delta s, \quad t \in \mathcal{I} \\ \text{and} \quad x \in E_n, \quad x(t) \ge 0, \quad t \in \mathcal{I}, \end{cases}$$

where $f \in E_n^\top$, $g \in E_m$, and B and K are rd-continuous matrix-valued functions of size $m \times n$. The dual time scales linear programming model is formulated as

$$\text{(D)} \qquad \begin{cases} \text{Minimize} \quad V(z) = \displaystyle\int_0^{\sigma(T)} z(t)g(t)\Delta t \\ \text{subject to} \quad z(t)B(t) \geq f(t) + \displaystyle\int_{\sigma(t)}^{\sigma(T)} z(s)K(s,t)\Delta s, \quad t \in \mathcal{I} \\ \text{and} \quad z \in E_m^\top, \quad z(t) \geq 0, \quad t \in \mathcal{I}. \end{cases}$$

A feasible solution of (P) (or (D)) is any one that satisfies the given constraints. An optimal solution to (P) (or (D)) is a feasible solution with the largest (or smallest) objective function value.

4. WEAK DUALITY AND OPTIMALITY CONDITION

In this section, we state and prove the weak duality theorem and the optimality condition theorem. Based on the objective functions and the constraints in (P) and (D), we define the bilinear form $\langle \cdot, \cdot \rangle_1$ on $E_m^\top \times E_m$ by

$$\langle z, g \rangle_1 = \int_0^{\sigma(T)} z(t)g(t)\Delta t, \tag{4.2}$$

the bilinear form $\langle \cdot, \cdot \rangle_2$ on $E_n^\top \times E_n$ by

$$\langle f, x \rangle_2 = \int_0^{\sigma(T)} f(t)x(t)\Delta t, \tag{4.3}$$

the linear operator $A : E_n \to E_m$ by

$$Ax(t) = B(t)x(t) - \int_0^t K(t,s)x(s)\Delta s, \tag{4.4}$$

and the linear operator $A' : E_m^\top \to E_n^\top$ by

$$A'z(t) = z(t)B(t) - \int_{\sigma(t)}^{\sigma(T)} z(s)K(s,t)\Delta s. \tag{4.5}$$

Using these operators, we can rewrite (P) and (D) as

$$\begin{cases} \text{Maximize} \quad \langle f, x \rangle_2 \\ \text{subject to} \quad Ax \leq g \quad \text{and} \quad x \geq 0 \end{cases} \tag{4.6}$$

and

$$\begin{cases} \text{Minimize} \quad \langle z, g \rangle_1 \\ \text{such that} \quad A'z \geq f \quad \text{and} \quad z \geq 0, \end{cases} \tag{4.7}$$

respectively.

Lemma 4.6. *We have*

$$\langle z, Ax\rangle_1 = \langle A'z, x\rangle_2 . \tag{4.8}$$

Proof. Using standard properties of the time scales integral (see [13, Section 1.4]), we calculate

$$\begin{aligned}
\langle z, Ax\rangle_1 &\overset{(4.2)}{=} \int_0^{\sigma(T)} z(t)Ax(t)\Delta t \\
&\overset{(4.4)}{=} \int_0^{\sigma(T)} z(t)\left[B(t)x(t) - \int_0^t K(t,s)x(s)\Delta s\right]\Delta t \\
&= \int_0^{\sigma(T)} z(t)B(t)x(t)\Delta t - \int_0^{\sigma(T)}\int_0^t z(t)K(t,s)x(s)\Delta s\Delta t \\
&\overset{(*)}{=} \int_0^{\sigma(T)} z(s)B(s)x(s)\Delta s - \int_0^{\sigma(T)}\int_{\sigma(s)}^{\sigma(T)} z(t)K(t,s)x(s)\Delta t\Delta s \\
&= \int_0^{\sigma(T)}\left[z(s)B(s) - \int_{\sigma(s)}^{\sigma(T)} z(t)K(t,s)\Delta t\right]x(s)\Delta s \\
&\overset{(4.5)}{=} \int_0^{\sigma(T)} A'z(s)x(s)\Delta s \\
&\overset{(4.3)}{=} \langle A'z, x\rangle_2 ,
\end{aligned}$$

where in $(*)$ we have used the time scales change of integration order formula presented in [23, Lemma 1]. This shows (4.8) and completes the proof. □

Theorem 4.7 (Weak Duality Theorem)**.** *If x and z are arbitrary feasible solutions of* (P) *and* (D)*, respectively, then $U(x) \le V(z)$.*

Proof. By the constraints in (4.6) and (4.7), respectively, we have

$$Ax(t) \le g(t) \quad \text{and} \quad A'z(t) \ge f(t) \quad \text{for} \quad t \in \mathcal{I}. \tag{4.9}$$

Since $x(t)$ and $z(t)$ are nonnegative, we obtain

$$z(t)Ax(t) \le z(t)g(t) \quad \text{for} \quad t \in \mathcal{I} \tag{4.10}$$

and

$$A'z(t)x(t) \ge f(t)x(t) \quad \text{for} \quad t \in \mathcal{I}. \tag{4.11}$$

Hence,

$$\begin{aligned}
U(x) &\overset{(\mathrm{P})}{=} \int_0^{\sigma(T)} f(t)x(t)\Delta t \overset{(4.11)}{\le} \int_0^{\sigma(T)} A'z(t)x(t)\Delta t \\
&\overset{(4.3)}{=} \langle A'z, x\rangle_2 \overset{(4.8)}{=} \langle z, Ax\rangle_1 \overset{(4.2)}{=} \int_0^{\sigma(T)} z(t)Ax(t)\Delta t
\end{aligned}$$

$$\overset{(4.10)}{\leq} \int_0^{\sigma(T)} z(t)g(t)\Delta t \overset{(\mathrm{D})}{=} V(z),$$

completing the proof. □

Theorem 4.8 (Optimality Condition)**.** *If there exist feasible solutions x^* and z^* of* (P) *and* (D)*, respectively, such that $U(x^*) = V(z^*)$, then x^* and z^* are optimal solutions of their respective problems.*

Proof. By Theorem 4.7, if x is an arbitrary feasible solution x of (P), then

$$U(x) \leq V(z^*) = U(x^*),$$

and thus, x^* is an optimal solution of (P). Similarly, again by Theorem 4.7, if z is an arbitrary feasible solution of (D), then

$$V(z) \geq U(x^*) = V(z^*),$$

and so, z^* is an optimal solution of (D). □

5. STRONG DUALITY THEOREM

In this section, we consider the isolated time scale

$$\mathbb{T} = \{t_k \in \mathbb{R} : \ k \in \mathbb{N}_0\} \quad \text{with} \quad t_i < t_{i+1} \quad \text{for all} \quad i \in \mathbb{N}_0,$$

where

$$0 = t_0 < t_1 < t_2 < \ldots < t_N = T < t_{N+1} = \sigma(t_N)$$

in both (P) and (D).

Theorem 5.9 (Strong Duality Theorem)**.** *If* (P) *has an optimal solution x^*, then* (D) *has an optimal solution z^* such that $U(x^*) = V(z^*)$.*

Proof. Let x^* be an optimal solution of (P), i.e., by (2.1), of

$$\begin{cases} \text{Maximize} \quad U(x) = \displaystyle\sum_{k=0}^{N} \mu(t_k) f(t_k) x(t_k) \\ \text{subject to} \quad x(t_k) \geq 0 \quad \text{for} \quad k = 0, 1, \ldots, N \quad \text{and} \\ B(t_0)x(t_0) \leq g(t_0), \\ B(t_1)x(t_1) \leq g(t_1) + \mu(t_0)K(t_1, t_0)x(t_0), \\ B(t_2)x(t_2) \leq g(t_2) + \mu(t_0)K(t_2, t_0)x(t_0) + \mu(t_1)K(t_2, t_1)x(t_1), \\ \qquad \vdots \\ B(t_N)x(t_N) \leq g(t_N) + \mu(t_0)K(t_N, t_0)x(t_0) + \cdots \\ \qquad + \mu(t_{N-1})K(t_N, t_{N-1})x(t_{N-1}), \end{cases}$$

i.e.,

$$
\text{(5.12)} \qquad \begin{cases} \text{Maximize} \quad U(x) = \displaystyle\sum_{k=0}^{N} \mu(t_k) f(t_k) x(t_k) \\ \text{subject to} \quad x(t_k) \geq 0 \quad \text{for} \quad k = 0, 1, \ldots, N \\ \text{and} \quad \mathcal{A} \begin{pmatrix} x(t_0) \\ x(t_1) \\ x(t_2) \\ \vdots \\ x(t_N) \end{pmatrix} \leq \begin{pmatrix} g(t_0) \\ g(t_1) \\ g(t_2) \\ \vdots \\ g(t_N) \end{pmatrix}, \end{cases}
$$

where $x : \mathcal{I} \to \mathbb{R}^n$ and

$$
\mathcal{A} = \begin{pmatrix} B(t_0) & 0 & \cdots & 0 \\ -\mu(t_0)K(t_1, t_0) & B(t_1) & \cdots & 0 \\ -\mu(t_0)K(t_2, t_0) & -\mu(t_1)K(t_2, t_1) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -\mu(t_0)K(t_N, t_0) & -\mu(t_1)K(t_N, t_1) & \cdots & B(t_N) \end{pmatrix}.
$$

By the standard strong duality theorem of discrete linear programming [20, Theorem 5.1 on page 58] applied to (5.12), there exists an optimal solution y^* of

$$
\text{(5.13)} \qquad \begin{cases} \text{Minimize} \quad W(y) = \displaystyle\sum_{k=0}^{N} y(t_k) g(t_k) \\ \text{subject to} \quad y(t_k) \geq 0 \quad \text{for} \quad k = 0, 1, \ldots, N \\ \text{and} \quad \begin{pmatrix} y(t_0) & y(t_1) & y(t_2) & \cdots & y(t_N) \end{pmatrix} \mathcal{A} \\ \qquad \geq \begin{pmatrix} \mu(t_0) f(t_0) & \mu(t_1) f(t_1) & \mu(t_2) f(t_2) & \cdots & \mu(t_N) f(t_N) \end{pmatrix}, \end{cases}
$$

where $y^\top : \mathcal{I} \to \mathbb{R}^m$, and we have

$$
U(x^*) = W(y^*). \tag{5.14}
$$

Now we put

$$
z^*(t_k) = \frac{y^*(t_k)}{\mu(t_k)} \quad \text{for} \quad k = 0, 1, \ldots, N. \tag{5.15}
$$

Using (5.15) in (5.14), we get

$$
U(x^*) = W(y^*) = \sum_{k=0}^{N} y^*(t_k) g(t_k) = \sum_{k=0}^{N} \mu(t_k) z^*(t_k) g(t_k) = V(z^*). \tag{5.16}
$$

Since y^* is feasible, it satisfies the nonnegativity constraints in (5.13), and thus (5.15) yields

$$z^*(t) \geq 0 \quad \text{for} \quad t \in \mathcal{I}, \tag{5.17}$$

Moreover, y^* also satisfies the other contraints in (5.13), and by writing them as $N+1$ inequalities and dividing the first one by $\mu(t_0)$, the second one by $\mu(t_1)$, $\ldots$, and the last one by $\mu(t_N)$, (5.15) gives

$$\begin{cases} z^*(t_0)B(t_0) & \geq f(t_0) + z^*(t_1)\mu(t_1)K(t_1,t_0) + \cdots + z^*(t_N)\mu(t_N)K(t_N,t_0) \\ & = f(t_0) + \displaystyle\int_{\sigma(t_0)}^{\sigma(t_N)} z^*(s)K(s,t_0)\Delta s, \\ z^*(t_1)B(t_1) & \geq f(t_1) + z^*(t_2)\mu(t_2)K(t_2,t_1) + \cdots z^*(t_N)\mu(t_N)K(t_N,t_1) \\ & = f(t_1) + \displaystyle\int_{\sigma(t_1)}^{\sigma(t_N)} z^*(s)K(s,t_1)\Delta s, \\ z^*(t_2)B(t_2) & \geq f(t_2) + \cdots z^*(t_N)\mu(t_N)K(t_N,t_2) \\ & = f(t_2) + \displaystyle\int_{\sigma(t_2)}^{\sigma(t_N)} z^*(s)K(s,t_2)\Delta s, \\ & \vdots \\ z^*(t_N)B(t_N) & \geq f(t_N) \\ & = f(t_N) + \displaystyle\int_{\sigma(t_N)}^{\sigma(t_N)} z^*(s)K(s,t_N)\Delta s, \end{cases}$$

i.e.,

$$z^*(t)B(t) \geq f(t) + \int_{\sigma(t)}^{\sigma(T)} z(s)K(s,t)\Delta s \quad \text{for} \quad t \in \mathcal{I}. \tag{5.18}$$

By (5.17) and (5.18), z^* is a feasible solution of (D). By (5.16), z^* is a feasible solution of (D) satisfying $V(z^*) = U(x^*)$. By the optimality condition, Theorem 4.8, we conclude that z^* is an optimal solution of (D). □

6. EXAMPLES

In this section, three examples are given in order to illustrate our duality theorems on isolated time scales.

Example 6.10. Let $\mathbb{T} = \mathbb{Z}$ and $\mathcal{I} = \{0,1,2,3,4\}$. Then, we consider the isolated time scales linear programming primal model

$$\begin{cases} \text{Maximize} \quad U(x) = \displaystyle\int_0^{\sigma(4)} tx(t)\Delta t = \sum_{t=0}^{4} tx(t) \\ \text{subject to} \quad 6x(t) \leq t + \displaystyle\int_0^t x(s)\Delta s = t + \sum_{s=0}^{t-1} x(s), \quad t \in \mathcal{I} \\ \text{and} \quad x(t) \geq 0, \quad t \in \mathcal{I}, \end{cases}$$

where we have used σ and the integral given in Example 2.4. Using MATLAB command linprog or LINDO solver, we have

$$x^*(0) = 0.000000, \qquad x^*(1) = 0.166667, \qquad x^*(2) = 0.361111,$$
$$x^*(3) = 0.587963, \qquad x^*(4) = 0.852623, \qquad U(x^*) = 6.063272.$$

On the other hand, the isolated time scales linear programming dual model is

$$\begin{cases} \text{Minimize} \quad V(z) = \displaystyle\int_0^{\sigma(4)} tz(t)\Delta t = \sum_{t=0}^{4} tz(t) \\ \text{subject to} \quad 6z(t) \geq t + \displaystyle\int_{\sigma(t)}^{\sigma(T)} z(s)\Delta s = t + \sum_{s=t+1}^{4} z(s), \quad t \in \mathcal{I} \\ \text{and} \quad z(t) \geq 0, \quad t \in \mathcal{I}, \end{cases}$$

where we have used again Example 2.4. Using MATLAB command linprog or LINDO solver, we have

$$z^*(0) = 0.382459, \qquad z^*(1) = 0.470679, \qquad z^*(2) = 0.546296,$$
$$z^*(3) = 0.611111, \qquad z^*(4) = 0.666667, \qquad V(z^*) = 6.063272,$$

confirming $U(x^*) = V(z^*)$.

Example 6.11. Let $\mathbb{T} = 5\mathbb{Z}$ and $\mathcal{I} = \{0, 5, 10, 15, 20\}$. Then, we consider the isolated time scales linear programming primal model

$$\begin{cases} \text{Maximize} \quad U(x) = \displaystyle\int_0^{\sigma(20)} tx(t)\Delta t = 25 \sum_{k=0}^{4} kx(5k) \\ \text{subject to} \quad 6x(t) \leq t + \displaystyle\int_0^{t} x(s)\Delta s = t + 5 \sum_{k=0}^{\frac{t}{5}-1} x(5k), \quad t \in \mathcal{I} \\ \text{and} \quad x(t) \geq 0, \quad t \in \mathcal{I}, \end{cases}$$

where we have used σ and the integral given in Example 2.3. Using MATLAB command linprog or LINDO solver, we have

$$x^*(0) = 0.000000, \qquad x^*(5) = 0.833333, \qquad x^*(10) = 2.361111,$$
$$x^*(15) = 5.162037, \qquad x^*(20) = 10.297068, \qquad U(x^*) = 1555.748.$$

On the other hand, the isolated time scales linear programming dual model is

$$\begin{cases} \text{Minimize} \quad V(z) = \displaystyle\int_0^{\sigma(20)} tz(t)\Delta t = 25 \sum_{k=0}^{4} kz(5k) \\ \text{subject to} \quad 6z(t) \geq t + \displaystyle\int_{\sigma(t)}^{\sigma(T)} z(s)\Delta s = t + 5 \sum_{k=\frac{t}{5}+1}^{4} z(5k), \quad t \in \mathcal{I} \\ \text{and} \quad z(t) \geq 0, \quad t \in \mathcal{I}. \end{cases}$$

where we have used again Example 2.3. Using MATLAB command linprog or LINDO solver, we have

$$z^*(0) = 27.359825, \qquad z^*(5) = 15.378086, \qquad z^*(10) = 8.842592,$$
$$z^*(15) = 5.277778, \qquad z^*(20) = 3.333333, \qquad V(z^*) = 1555.748,$$

confirming $U(x^*) = V(z^*)$.

Example 6.12. Let $\mathbb{T} = 2^{\mathbb{N}_0}$ and $\mathcal{I} = \{1, 2, 4\}$. Then, we consider the isolated time scales linear programming primal model

$$\begin{cases} \text{Maximize} \quad U(x) = \displaystyle\int_1^{\sigma(2^2)} tx(t)\Delta t = \sum_{k=0}^{2} 4^k x(2^k) \\ \text{subject to} \quad 8x(t) \leq 6t + 3\displaystyle\int_1^t x(s)\Delta s = 6t + 3 \sum_{k=0}^{\log_2 t - 1} 2^k x(2^k), \quad t \in \mathcal{I} \\ \text{and} \quad x(t) \geq 0, \quad t \in \mathcal{I}, \end{cases}$$

where we have used σ and the integral given in Example 2.5. Using MATLAB command linprog or LINDO solver, we have

$$x^*(1) = 0.750000, \qquad x^*(2) = 1.781250,$$
$$x^*(4) = 4.617188, \qquad U(x^*) = 81.75000.$$

On the other hand, the isolated time scales linear programming dual model is

$$\begin{cases} \text{Minimize} \quad V(z) = 6\displaystyle\int_1^{\sigma(2^2)} tz(t)\Delta t = 6\sum_{k=0}^{2} 4^k z(2^k) \\ \text{subject to} \quad 8z(t) \geq t + 3\displaystyle\int_{\sigma(t)}^{\sigma(4)} z(s)\Delta s = t + 3 \sum_{k=1+\log_2 t}^{2} 2^k z(2^k), \quad t \in \mathcal{I} \\ \text{and} \quad z(t) \geq 0, \quad t \in \mathcal{I}, \end{cases}$$

where we have used again Example 2.5. Using MATLAB command linprog or LINDO solver, we have

$$z^*(1) = 1.625000, \qquad z^*(2) = 1.000000,$$
$$z^*(2) = 0.500000, \qquad V(z^*) = 81.75000,$$

confirming $U(x^*) = V(z^*)$.

7. CONCLUSIONS

In this work, duality theorems for linear programming problems on time scales have been presented. For this novel result, a formulation of the primal and the dual linear models on time scales has been established. Furthermore, the weak duality theorem and the optimality condition theorem were given for arbitrary time scales, while the strong duality theorem has been established for arbitrary isolated time scales. This ensures that our formulation of primal and dual models on time scales presented in this paper is indeed the "correct" formulation. Since all efficient algorithms in the literature can only solve continuous-time linear programming problems by a discrete approximation method (i.e., the optimal solution obtained by this method is a numerical solution with an estimate of the error bound), another contribution of this paper is to solve linear programming problems on arbitrary isolated time scales by efficient exact methods (i.e., the optimal solution obtained by these methods is exact).

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