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SOME IMPROVEMENTS OF JORDAN-STEČKIN AND BECKER-STARK INEQUALITIES

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The aim of this article is to propose some improvements of the Jordan-Stečkin and Becker-Stark inequalities discussed in L. Debnath, C. Mortici, L. Zhu: *Refinements of Jordan-Stečkin and Becker-Stark inequalities,* Results Math. **67**(1-2)(2015), 207–215.

1. INTRODUCTION

L. Debnath, C. Mortici and L. Zhu discuss in [**1**] Jordan's inequality:

$$\frac{\sin x}{x} \geq \frac{2}{\pi}, \quad x \in (0, \pi/2] \tag{1}$$

and its improvements

$$\frac{2}{\pi} + \frac{1}{\pi^3}\left(\pi^2 - 4x^2\right) \leq \frac{\sin x}{x} \leq \frac{2}{\pi} + \frac{\pi - 2}{\pi^3}\left(\pi^2 - 4x^2\right), \quad x \in (0, \pi/2], \tag{2}$$

and

$$\frac{2}{\pi} + \frac{1}{2\pi^5}\left(\pi^4 - 16x^4\right) \leq \frac{\sin x}{x} \leq \frac{2}{\pi} + \frac{\pi - 2}{\pi^5}\left(\pi^4 - 16x^4\right), \quad x \in (0, \pi/2]. \tag{3}$$

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They conclude that the equalities in (2) and (3) hold if and only if $x = \pi/2$. In the case where $x \to 0_+$, we have equalities on the right-hand side of (2) and (3), and strict inequalities on the left-hand side of (2) and (3).

In [**1**] (*Theorem 1*, *Theorem 2*), the left-hand side of (2) and (3) near zero was improved.

The following inequality:

$$\tan x \geq \frac{4}{\pi} \cdot \frac{x}{\pi - 2x}, \quad x \in [0, \pi/2). \tag{4}$$

well known as STEČKIN's inequality, was also analysed in [**1**].

As noted in [**1**], this inequality becomes an equality for $x = 0$, and

$$\lim_{x \to (\pi/2)_-} \left(\tan x - \frac{4}{\pi} \cdot \frac{x}{\pi - 2x} \right) = \frac{2}{\pi}.$$

Some improvements of (4), in the left neighbourhood of $\pi/2$, were presented in [**1**] (*Theorem 3*, *Theorem 4*).

M. Becker and L. E. Stark present in [**2**] the inequality

$$\frac{8}{\pi^2 - 4x^2} < \frac{\tan x}{x} < \frac{\pi^2}{\pi^2 - 4x^2}, \quad 0 < x < \frac{\pi}{2}. \tag{5}$$

Certain double inequalities of the BECKER-STARK type were proposed in [**1**] (*Theorem 5*, *Theorem 6*).

In this paper, we generalise and improve the inequalities stated in *Theorem 1*, *Theorem 2*, *Theorem 3*, *Theorem 4*, *Theorem 5* and *Theorem 6* from [**1**]. They are cited below for readers' convenience.

Statement 1 ([**1**], *Theorem 1*) *For every $x \in (0, \pi/2)$, it holds that*

$$\begin{aligned} &\frac{2}{\pi} + \frac{1}{\pi^3}\left(\pi^2 - 4x^2\right) + \left(1 - \frac{3}{\pi}\right) - \left(\frac{1}{6} - \frac{4}{\pi^3}\right) x^2 < \\ &< \frac{\sin x}{x} < \\ &< \frac{2}{\pi} + \frac{1}{\pi^3}\left(\pi^2 - 4x^2\right) + \left(1 - \frac{3}{\pi}\right) - \left(\frac{1}{6} - \frac{4}{\pi^3}\right) x^2 + \frac{1}{120} x^4. \end{aligned} \tag{6}$$

Statement 2 ([**1**], *Theorem 2*) *For every $x \in (0, \pi/2)$, it holds that*

$$\begin{aligned} &\frac{2}{\pi} + \frac{1}{2\pi^5}\left(\pi^4 - 16x^4\right) + \left(1 - \frac{5}{2\pi}\right) - \frac{1}{6} x^2 < \\ &< \frac{\sin x}{x} < \\ &< \frac{2}{\pi} + \frac{\pi - 2}{\pi^5}\left(\pi^4 - 16x^4\right) + \left(1 - \frac{5}{2\pi}\right) - \frac{1}{6} x^2 + \left(\frac{8}{\pi^5} + \frac{1}{120}\right) x^4. \end{aligned} \tag{7}$$

Statement 3 ([**1**], *Theorem 3*) *For every $x \in (0, \pi/2)$, it holds that*

$$\frac{2}{\pi} - \frac{1}{2}\left(\frac{\pi}{2} - x\right) < \tan x - \frac{4}{\pi} \cdot \frac{x}{\pi - 2x} < \frac{2}{\pi} - \frac{1}{3}\left(\frac{\pi}{2} - x\right). \tag{8}$$

Statement 4 ([**1**], *Theorem 4*) *For every $x \in (0, 1)$, it holds that*

$$\left(1 - \frac{4}{\pi^2}\right)x - \frac{8}{\pi^3}x^2 < \tan x - \frac{4}{\pi} \cdot \frac{x}{\pi - 2x} < \left(1 - \frac{4}{\pi^2}x\right). \tag{9}$$

Statement 5 ([**1**], *Theorem 5*) *For every $x \in (0.373, \pi/2)$ on the left-hand side and every $x \in (0.301, \pi/2)$ on the right-hand side, the following inequalities hold true:*

$$\frac{8 + a(x)}{\pi^2 - 4x^2} < \frac{\tan x}{x} < \frac{8 + b(x)}{\pi^2 - 4x^2}, \tag{10}$$

where

$$a(x) = \frac{8}{\pi}\left(\frac{\pi}{2} - x\right) + \left(\frac{16}{\pi^2} - \frac{8}{3}\right)\left(\frac{\pi}{2} - x\right)^2$$

and

$$b(x) = a(x) + \left(\frac{32}{\pi^3} - \frac{8}{3\pi}\right)\left(\frac{\pi}{2} - x\right)^3.$$

Statement 6 ([**1**], *Theorem 6*) *For every real number $x \in (0, 1.371)$, the following inequality holds true:*

$$\frac{\tan x}{x} < \frac{\pi^2 - \left(4 - \frac{1}{3}\pi^2\right)x^2 - \left(\frac{4}{3} - \frac{2}{15}\pi^2\right)x^4}{\pi^2 - 4x^2}. \tag{11}$$

2. PRELIMINARIES

Let $T_n^{\varphi,a}(x)$ be the Taylor polynomial of the order $n \in N$, associated to the function $\varphi(x)$ at the point $x = a$. $\overline{T}_n^{\varphi,a}(x)$ and $\underline{T}_n^{\varphi,a}(x)$ represent the Taylor polynomial of the order $n \in N$, associated to the function $\varphi(x)$ at the point $x = a$, in the case $T_n^{\varphi,a}(x) \geq \varphi(x)$, respectively $T_n^{\varphi,a}(x) \leq \varphi(x)$, for every $x \in (a, b)$. We call $\overline{T}_n^{\varphi,a}(x)$ and $\underline{T}_n^{\varphi,a}(x)$ an upward and a downward approximation of φ on (a, b), respectively.

As discussed in [**4**], for the sine function the following inequalities hold:

$$\begin{aligned} &\underline{T}_3^{\sin,0}(x) < \underline{T}_7^{\sin,0}(x) < \underline{T}_{11}^{\sin,0}(x) < \underline{T}_{15}^{\sin,0}(x) < \ldots < \sin x < \ldots \\ &< \overline{T}_{13}^{\sin,0}(x) < \overline{T}_9^{\sin,0}(x) < \overline{T}_5^{\sin,0}(x) < \overline{T}_1^{\sin,0}(x), \end{aligned} \tag{12}$$

for $x \in (0, \sqrt{20}) = (0, 4.472...)$.

We have the following TAYLOR series of $\operatorname{sinc} x$:

$$\operatorname{sinc} x = \frac{\sin x}{x} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k+1)!} \tag{13}$$

for $x \neq 0$.

According to [**6**], for $x \in (0, \pi/2)$ we have the following series representations:

$$\tan x = \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1)}{(2k)!} \, |B_{2k}| \, x^{2k-1} \tag{14}$$

and

$$\cot x = \frac{1}{x} - \sum_{k=1}^{\infty} \frac{2^{2k} \, |B_{2k}|}{(2k)!} x^{2k-1} \tag{15}$$

where B_i $(i \in N)$ are BERNOULLI's numbers.

Suppose that $f(x)$ is a real function on (a, b), and that n is a positive integer such that $f^{(k)}(a+)$, $f^{(k)}(b-)$, $(k \in 0, 1, 2, \ldots, n-1)$ exist. Let us denote:

$$\underline{T}_n^{f;b,a}(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(b-)}{k!}(x-b)^k + \\ + \frac{1}{(a-b)^n}\left(f(a+) - \sum_{k=0}^{n-1} \frac{(a-b)^k f^{(k)}(b-)}{k!}\right)(x-b)^n$$

and

$$\overline{T}_n^{f;a,b}(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a+)}{k!}(x-a)^k + \\ + \frac{1}{(b-a)^n}\left(f(b-) - \sum_{k=0}^{n-1} \frac{(b-a)^k f^{(k)}(a+)}{k!}\right)(x-a)^n.$$

S. Wu and L. Debnath proved the following theorem in [**7**]:

Theorem WD *Suppose that $f(x)$ is a real function on (a, b), and that n is a positive integer such that $f^{(k)}(a+)$, $f^{(k)}(b-)$, $(k \in 0, 1, 2, \ldots, n)$ exist.*

(i) Supposing that $(-1)^{(n)} f^{(n)}(x)$ is increasing on (a, b), then for all $x \in (a, b)$ the following inequality holds :

$$\underline{T}_n^{f;b,a}(x) < f(x) < \overline{T}_n^{f,b}(x) \tag{16}$$

Furthermore, if $(-1)^n f^{(n)}(x)$ is decreasing on (a, b), then the reverse inequality holds.

(ii) Supposing that $f^{(n)}(x)$ is increasing on (a,b), then for all $x \in (a,b)$ the following inequality holds:

$$\overline{\mathbb{T}}_n^{f;a,b}(x) > f(x) > \underline{T}_n^{f,a}(x). \tag{17}$$

Furthermore, if $f^{(n)}(x)$ is decreasing on (a,b), then the reverse inequality holds.

Some interesting applications of the previous theorem can be found in [**5, 19, 20, 32**].

3. MAIN RESULTS

3.1 IMPROVEMENTS OF INEQUALITIES IN STATEMENT 1

According to (12), we can approximate the $\operatorname{sinc} x$ function as follows:

$$\begin{aligned} &\underline{T}_2^{\text{sinc},0}(x) < \underline{T}_6^{\text{sinc},0}(x) < \underline{T}_{10}^{\text{sinc},0}(x) < \underline{T}_{14}^{\text{sinc},0}(x) < \ldots < \operatorname{sinc} x < \ldots \\ &\qquad < \overline{T}_{12}^{\text{sinc},0}(x) < \overline{T}_8^{\text{sinc},0}(x) < \overline{T}_4^{\text{sinc},0}(x) < \overline{T}_0^{\text{sinc},0}(x), \end{aligned} \tag{18}$$

for $x \in (0, \pi/2) \subset (0, \sqrt{20})$.

Based on approximation (18), we have the following theorem

Theorem 1 *For every $x \in (0,\pi/2)$ we have:*

$$\begin{aligned} &\underline{T}_2^{\text{sinc},0}(x) = \frac{2}{\pi} + \frac{1}{\pi^3}\left(\pi^2 - 4x^2\right) + \left(1 - \frac{3}{\pi}\right) - \left(\frac{1}{6} - \frac{4}{\pi^3}\right)x^2 \leq \\ &\leq \underline{T}_{4k_1-2}^{\text{sinc},0}(x) < \operatorname{sinc} x < \overline{T}_{4k_2}^{\text{sinc},0}(x) \leq \frac{2}{\pi} + \frac{1}{\pi^3}\left(\pi^2 - 4x^2\right) + \\ &+ \left(1 - \frac{3}{\pi}\right) - \left(\frac{1}{6} - \frac{4}{\pi^3}\right)x^2 + \frac{1}{120}x^4 = \overline{T}_4^{\text{sinc},0}(x) < \overline{T}_0^{\text{sinc},0}(x), \end{aligned} \tag{19}$$

for $k_1, k_2 \in N$.

Remark 1 *It is obvious that Statement 1 is a special case of Theorem 1.*

3.2 IMPROVEMENTS OF INEQUALITIES IN STATEMENT 2

Consider the following polynomials in inequality (7) from *Statement 2*:

$$Q_4(x) = \frac{2}{\pi} + \frac{1}{2\pi^5}\left(\pi^4 - 16x^4\right) + \left(1 - \frac{5}{2\pi}\right) - \frac{1}{6}x^2 = -\frac{8x^4}{\pi^5} - \frac{x^2}{6} + 1$$

and

$$\begin{aligned} R_4(x) &= \frac{2}{\pi} + \frac{\pi - 2}{\pi^5}\left(\pi^4 - 16x^4\right) + \left(1 - \frac{5}{2\pi}\right) - \frac{1}{6}x^2 + \left(\frac{8}{\pi^5} + \frac{1}{120}\right)x^4 \\ &= \left(-\frac{16}{\pi^4} + \frac{40}{\pi^5} + \frac{1}{120}\right)x^4 - \frac{x^2}{6} - \frac{5}{2\pi} + 2. \end{aligned}$$

We have the following theorem:

Theorem 2 *For every $x \in (0, \pi/2)$ we have:*

$$\text{(20)} \qquad Q_4(x) < \underline{T}_6^{\text{sinc},0}(x) \leq \underline{T}_{4k_1-2}^{\text{sinc},0}(x) < \operatorname{sinc} x < \overline{T}_{4k_2}^{\text{sinc},0}(x) \leq \overline{T}_4^{\text{sinc},0}(x) < R_4(x),$$

for $k_1, k_2 \in N$.

Proof In order to prove (20), it is sufficient to prove that for every $x \in (0, \pi/2)$ the inequalities $Q_4(x) < \underline{T}_6^{\text{sinc},0}(x)$ and $\overline{T}_4^{\text{sinc},0}(x) < R_4(x)$ are true.

According to (13) we have:

$$\begin{aligned} \overline{T}_4^{\text{sinc},0}(x) &= 1 - \frac{x^2}{6} + \frac{x^4}{120}, \\ \underline{T}_6^{\text{sinc},0}(x) &= 1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040}. \end{aligned}$$

It is obvious that

$$\begin{aligned} \underline{T}_6^{\text{sinc},0}(x) - Q_4(x) &> \left(1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040}\right) - \left(-\frac{8x^4}{\pi^5} - \frac{x^2}{6} + 1\right) = \\ &= \left(\frac{1}{120} + \frac{8}{\pi^5}\right)x^4 - \frac{x^6}{5040} > 0 \end{aligned}$$

and

$$\begin{aligned} R_4(x) - \overline{T}_4^{\text{sinc},0}(x) &> \left(-\frac{16}{\pi^4} + \frac{40}{\pi^5} + \frac{1}{120}\right)x^4 - \frac{x^2}{6} - \frac{5}{2\pi} + 2 \\ &- \left(1 - \frac{x^2}{6} + \frac{x^4}{120}\right) = \left(-\frac{16}{\pi^4} + \frac{40}{\pi^5}\right)x^4 - \frac{5}{2\pi} + 1 > 0 \end{aligned}$$

hold for $x \in (0, \pi/2)$. ■

Remark 2 *Statement 2 is a special case of Theorem 2.*

3.3 IMPROVEMENTS OF INEQUALITIES IN STATEMENT 3

In a monography [**3**], D. S. Mitrinović discussed about STEČKIN'S inequality:

$$\tan x > \frac{4}{\pi} \cdot \frac{x}{\pi - 2x},$$

for $x \in (0, \pi/2)$. Let us denote:

$$f(x) = \tan x - \frac{4x}{\pi\,(\pi - 2x)}, \tag{21}$$

for $x \in (0, \pi/2)$ and let us notice:

$$\lim_{x \to \pi/2-} f(x) = \frac{\pi}{2}.$$

In [**1**], inequalities (8) are proposed as adequate approximations of the function $f(x)$ in the left neighbourhood of the point $x = \pi/2$.

By replacing x with $\pi/2 - t$ in the function $f(x)$, we obtain:

$$g(t) = f\left(\frac{\pi}{2} - t\right) = \cot t - \frac{1}{t} + \frac{2}{\pi},$$

for $t \in (0, \pi/2)$. According to (15), we have that

$$\cot t < \overline{T}_n^{\cot,0}(t) = \frac{1}{t} - \sum_{k=1}^{n} \frac{2^{2k}\,|B_{2k}|}{(2k)!} t^{2k-1}$$

for $t \in (0, \pi/2]$ and $n \in N$. Further, we have the following:

$$g(t) < \overline{T}_n^{\cot,0}(t) - \frac{1}{t} + \frac{2}{\pi} \tag{22}$$

and according to *Theorem WD*

$$\cot t > \mathbb{T}_n^{\cot;0,\pi/2}(t) = T_{n-1}^{\cot,0}(t) + \left(\frac{2}{\pi}\right)^n \left(g\left(\frac{\pi}{2}\right) - T_{n-1}^{\cot,0}\left(\frac{\pi}{2}\right)\right) t^n, \tag{23}$$

for $t \in (0, \pi/2]$ and $n \in N$. According to (22) and (23), we have:

$$g(t) > \mathbb{T}_n^{\cot;0,\pi/2}(t) - \frac{1}{t} + \frac{2}{\pi} \tag{24}$$

for $t \in (0, \pi/2]$. Let us denote:

$$F_n^g(t) = \overline{T}_n^{\cot,0}(t) - \frac{1}{t} + \frac{2}{\pi}$$

and

$$\mathbb{F}_n^g(t) = \mathbb{T}_n^{\cot;0,\pi/2}(t) - \frac{1}{t} + \frac{2}{\pi}.$$

Returning replacement $t = \pi/2 - x$ in (22) and (24), we have the following theorem:

Theorem 3 *For $x \in (0,\pi/2)$ and $n \in N$, we have:*

$$\mathbb{F}_n^g\Big(\frac{\pi}{2}-x\Big) < f(x) < F_n^g\Big(\frac{\pi}{2}-x\Big) \tag{25}$$

Corollary 1 *We have the following improvements for inequality (8) given in Statement 3.*

1. *For $n=1$ and for $x \in (0,\pi/2)$, we have:*

$$Q_1(x) < \mathbb{F}_1^g\Big(\frac{\pi}{2}-x\Big) = \\ = \frac{2}{\pi} - \frac{4}{\pi^2}\Big(\frac{\pi}{2}-x\Big) < f(x) < \frac{2}{\pi} - \frac{1}{3}\Big(\frac{\pi}{2}-x\Big) = \\ = F_1^g\Big(\frac{\pi}{2}-x\Big) = R_1(x).$$

2. *For $n=3$ and for $x \in (0,\pi/2)$, we have:*

$$Q_1(x) < \mathbb{F}_1^g\Big(\frac{\pi}{2}-x\Big) < \mathbb{F}_3^g\Big(\frac{\pi}{2}-x\Big) = \frac{2}{\pi} - \frac{1}{3}\Big(\frac{\pi}{2}-x\Big) - \Big(\frac{2}{\pi}\Big)^3\Big(\frac{2}{\pi}-\frac{\pi}{6}\Big)\Big(\frac{\pi}{2}-x\Big)^3 \\ < f(x) < \frac{2}{\pi} - \frac{1}{3}\Big(\frac{\pi}{2}-x\Big) - \frac{\big(\frac{\pi}{2}-x\big)}{45} = F_3^g\Big(\frac{\pi}{2}-x\Big) < F_1^g\Big(\frac{\pi}{2}-x\Big) = R_1(x).$$

3.4 IMPROVEMENTS OF INEQUALITIES IN STATEMENT 4

For the function $f(x)$ defined in (21), and according to the TAYLOR series of the $\tan x$ function in (14) and the binomial expansion of $\frac{1}{1-(\frac{2}{\pi}x)}$ over the interval $(0,\pi/2)$, we have:

$$\begin{aligned} f(x) &= \tan x - \frac{4}{\pi}\cdot\frac{x}{\pi-2x} \\ &= \sum_{i=1}^{\infty} \frac{2^{2i}\left(2^{2i}-1\right)|B_{2i}|}{(2i)!}x^{2i-1} - \frac{4}{\pi^2}\cdot\frac{x}{1-\left(\frac{2}{\pi}x\right)} \\ &= \sum_{i=1}^{\infty} \frac{2^{2i}\left(2^{2i}-1\right)|B_{2i}|}{(2i)!}x^{2i-1} - \sum_{j=1}^{\infty}\frac{2^{j+1}}{\pi^{j+1}}x^j \\ &= \sum_{k=1}^{\infty}(-1)^{k-1}\alpha_k x^k, \end{aligned} \tag{26}$$

where

$$\alpha_k = \begin{cases} \dfrac{2^{k+1}}{\pi^{k+1}} & : \quad k=2\ell \\ \dfrac{2^{2k+1}\left(2^{2k+1}-1\right)|B_{k+1}|}{(k+1)!} - \dfrac{2^{k+1}}{\pi^{k+1}} & : \quad k=2\ell-1 \end{cases}$$

for $\ell \in N$. It is not hard to check that:

$$\alpha_k > 0, \quad \lim_{k\to\infty} \alpha_k = 0 \quad \text{and} \quad (\alpha_k)\!\downarrow, \tag{27}$$

for $k \in N$. Finally, based on (26) and (27) and based on LEIBNITZ theorem, we have the following theorem:

Theorem 4 *For every $x\in(0,1)$ and $\ell\in N$, the following holds:*

$$\underline{T}^{f,0}_{2\ell}(x) < f(x) < \overline{T}^{f,0}_{2\ell-1}(x)\,. \tag{28}$$

Remark 3 *Inequality (28) for $\ell = 1$ represents inequality (9) from Statement 4.*

3.5 IMPROVEMENTS OF INEQUALITIES IN STATEMENT 5

Consider the following function:

$$\varphi(x) = \left(\pi^2 - 4x^2\right)\frac{\tan x}{x},$$

for $x \in (0, \pi/2)$.

By replacing x with $\pi/2 - t$ in the function $\varphi(x)$, we obtain:

$$\psi(t) = \varphi\Big(\frac{\pi}{2} - t\Big) = \frac{8\,t\,(\pi - t)\cot t}{\pi - 2\,t}$$

for $t \in (0, \pi/2)$. The improvement or inequalities from (10) are given with the following theorem:

Theorem 5 *For every $x\in(0,\pi/2)$, the following holds*:

$$\begin{aligned}
&\underline{T}^{\psi,0}_4\Big(\frac{\pi}{2}-x\Big) = 8 + \frac{8}{\pi}\Big(\frac{\pi}{2}-x\Big) + \Big(\frac{16}{\pi^2} - \frac{8}{3}\Big)\Big(\frac{\pi}{2}-x\Big)^2 + \Big(\frac{32}{\pi^3} - \frac{8}{3\pi}\Big)\Big(\frac{\pi}{2}-x\Big)^3\\
&+ \Big(\frac{64}{\pi^4} - \frac{16}{3\pi^2} - \frac{8}{45}\Big)\Big(\frac{\pi}{2}-x\Big)^4 <\\
&< \varphi(x) <\\
&< \overline{T}^{\psi,0}_5\Big(\frac{\pi}{2}-x\Big) = 8 + \frac{8}{\pi}\Big(\frac{\pi}{2}-x\Big) + \Big(\frac{16}{\pi^2} - \frac{8}{3}\Big)\Big(\frac{\pi}{2}-x\Big)^2 + \Big(\frac{32}{\pi^3} - \frac{8}{3\pi}\Big)\Big(\frac{\pi}{2}-x\Big)^3\\
&+ \Big(\frac{64}{\pi^4} - \frac{16}{3\pi^2} - \frac{8}{45}\Big)\Big(\frac{\pi}{2}-x\Big)^4 + \Big(\frac{128}{\pi^5} - \frac{32}{3\pi^3} - \frac{8}{45\pi}\Big)\Big(\frac{\pi}{2}-x\Big)^5.
\end{aligned}$$

One proof of this statement is based on equivalent mixed trigonometric polynomial inequalities:

$$f(x) = \left(\pi^2 - 4x^2\right)\sin x - x\,T^{\psi,0}_4\Big(\frac{\pi}{2}-x\Big)\cos x > 0$$

and

$$g(x)=\left(\pi^2-4x^2\right)\sin x - x\,T_5^{\psi,0}\left(\frac{\pi}{2}-x\right)\cos x<0,$$

for $x\in(0,\pi/2)$. References [**15, 16**] show that problem of proving mixed trigonometric polynomial inequalities is a decidable problem. In these two references are presented appropriate algorithms that follow mentioned inequalities. Some interesting applications of the algorithmic approach in proving mixed trigonometric inequalities can be found in [**21, 31**]; see also [**17, 18**]. G. Bercu in [**33, 34**] presented some interesting approximations of trigonometric functions using Pade approximant.

Remark 4 *It is obvious that Statement 5 is a consequence of Theorem 5.*

Further, let us observe the array $(\alpha_k)_{k\in N}$ defined by:

$$\alpha_1=1,\ \alpha_{2j}=0,\ \alpha_{2j+1}=-\frac{2^{2j}|B_{2j}|}{(2j)!}$$

for $j\in N$. Then based on [**6**], we have the following series representations:

$$\begin{aligned}\psi(t) &= \frac{8}{\pi}\,t\,(\pi-t)\,\frac{1}{1-\left(\frac{2\,t}{\pi}\right)}\cot t\\ &= \frac{8}{\pi}\,t\,(\pi-t)\left(\sum_{i=0}^{\infty}\left(\frac{2\,t}{\pi}\right)^i\right)\left(\sum_{j=0}^{\infty}\alpha_{2j+1}t^{2j-1}\right)\end{aligned}$$

for $t\in(0,\pi/2)$. Let $r_2(m)$ be the remainder after division of the natural number m by 2. We are posing the following conjecture:

Conjecture 1

1. *For the function $\psi(t)$ on $t\in(0,\pi/2)$, the following equality holds:*

$$\psi(t)=\sum_{m=0}^{\infty}\left(\frac{8\,\alpha_{m+1-r_2(m)}}{\pi^{r_2(m)}}+\sum_{i=1}^{[m/2]}\frac{2^{2i+2+r_2(m)}\alpha_{m+1-2i-r_2(m)}}{\pi^{2i+r_2(m)}}\right)t^m. \tag{29}$$

2. *For the function $\psi(t)$ on $t\in(0,\pi/2)$ and $\ell\in N$, the following inequalities are true:*

$$\underline{T}_{2\ell}^{\psi,0}(t)<\psi(t)<\overline{T}_{2\ell+1}^{\psi,0}(t)\quad(\,t\in(0,\pi/2)\ \wedge\ \ell\in N\,). \tag{30}$$

3.6 IMPROVEMENTS OF INEQUALITY IN STATEMENT 6

Let us denote the following function:

$$f(x)=\left(\pi^2-4x^2\right)\frac{\tan x}{x}$$

for $x\in(0,\pi/2)$.

According to [**6**] and (14), we have:

$$f(x) = \sum_{k=1}^{\infty} C_k x^{2k-2} \tag{31}$$

where

$$C_k = \frac{\pi^2 \cdot 2^{2k}(2^{2k}-1)\,|B_{2k}|}{(2k)!} - \frac{4 \cdot 2^{2k-2}(2^{2k-2}-1)\,|B_{2k-2}|}{(2k-2)!}, \tag{32}$$

and $x \in (0, c)$ and $0 < c < \pi/2$. It is not hard to check $C_k < 0$ for $k \in N$.

Finally, based on *Theorem WD* we get the following theorem:

Theorem 6 *For every $x \in (0, c)$, where $0 < c < \pi/2$, the following inequalities hold:*

$$\underline{T}_{m_1}^{f;0,c}(x) = \sum_{k=1}^{m_1-1} C_k x^{2k-2} + \left(\frac{1}{c}\right)^{2m_1-2}\left(f(c) - \left(\sum_{k=1}^{m_1-1} C_k c^{2k-2}\right)\right) x^{2m_1-2}$$
$$< f(x) < \sum_{k=1}^{m_2} C_k x^{2k-2} = \overline{T}_{m_2}^{f,0}(x),$$

for $m_1, m_2 \in N$.

Remark 5 *It is obvious that Statement 6 is a consequence of Theorem 6 for $m_2 =$* 3.

The approximations discussed in this paper can be of great significance for potential application of analytic inequalities in engineering. Some specific inequalities of a similar type are considered in [**11, 12, 13**].

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