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A GENERALIZED DISCRETE FRACTIONAL GRONWALL INEQUALITY AND ITS APPLICATION ON THE UNIQUENESS OF SOLUTIONS FOR NONLINEAR DELAY FRACTIONAL DIFFERENCE SYSTEM

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In this paper, we state and prove a new discrete fractional version of the generalized Gronwall inequality. Based on this, a particular version expressed by means of discrete Mittag–Lefler functions is provided. As an application, we prove the uniqueness and obtain an estimate for the solutions of nonlinear delay Caputo fractional difference system. Numerical example is presented to demonstrate the applicability of the main results.

1. INTRODUCTION

The theory of fractional differential equations has been extensively investigated over the last years due to widespread applications in various fields of science and engineering. For more details, see the the remarkable monographs [1–3] and the references cited therein.

It has been realized that the discrete analogue of ordinary differential equations has tremendous applications in computational analysis and computer simulations [4]. Motivated by this reality, the study of the discrete analogue of fractional differential equations has become pressing and compulsory. In recent years, few mathematicians have taken the lead to develop the theory of fractional difference equations which is the discrete analogue of fractional differential equations. We cite

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here the papers [5–16] whose authors initiated the study of the theory of fractional difference equations. For more comprehensive details about this subject, the reader is recommended to consult the new survey paper [17].

Mathematical inequalities play an essential role in the investigation of many properties of differential and difference equations. Gronwall inequality, which is our main concern herein, has been studied for fractional differential equations [18–22]. Nevertheless, the investigation of Gronwall inequality for fractional difference equations is comparably seldom [23–25]. The existence and uniqueness of solutions, which is a main application of Gronwall inequality, has been the object of many researchers prior to the study of the qualitative properties for different types of differential or difference equations. Recently, there have appeared many results about the existence and uniqueness of solutions for fractional differential equations [26–32]. For fractional difference equations, however, the authors claim that there is few literature on the existence and uniqueness of solutions [33–38].

In parallel to the development of fractional difference equations in the recent years, we state and prove a new discrete fractional version of the generalized Gronwall inequality. A particular version expressed by means of discrete Mittag–Lefler functions is also provided. As an application, we prove the uniqueness and obtain an estimate for the solutions of nonlinear delay Caputo fractional difference system. Numerical example is presented to demonstrate the applicability of the main results. Our result is different and generalize some existing results in the literature.

2. PRELIMINARIES

Let $\mathbb{R}^m$ be the m–dimensional Euclidean space and define $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$, $I_{-\tau} = \{-\tau, -\tau + 1, \ldots, 0\}$ and $\mathbb{N}_{-\tau} = \{-\tau, -\tau + 1, \ldots\}$ where $\tau \in \mathbb{N}_0$. We prove our main results for the system

$$(1) \qquad \begin{cases} {}^c\nabla_0^\alpha x(t) = A_0 x(t) + A_1 x(t-\tau) + f(t, x(t), x(t-\tau)),\ t \in \mathbb{N}_0 \\ x(t) = \varphi(t),\ t \in I_{-\tau}, \end{cases}$$

where ${}^c\nabla_t^\alpha$ denotes the Caputo's fractional difference of order $\alpha \in (0, 1)$, the state vector $x : \mathbb{N}_{-\tau} \to \mathbb{R}^m$, the constant matrices A_0 and A_1 are of appropriate dimensions, the nonlinearity $f : \mathbb{N}_0 \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ and the initial function $\varphi : I_{-\tau} \to \mathbb{R}^m$.

For the convenience of the reader, we recall some definitions of ∇–based fractional operators which will facilitate the analysis of system (1). For any $\alpha, t \in \mathbb{R}$, the α rising function is defined by

$$(2) \qquad t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)},\ t \in \mathbb{R}\backslash\{\ldots, -2, -1, 0\},\ 0^{\overline{\alpha}} = 0,$$

where Γ is the Gamma function satisfying the property $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$.

Definition 1. [11,12] *Let $x : \mathbb{N}_0 \to \mathbb{R}$ and define $\rho(s) = s - 1$. For $\alpha \in (0, 1)$, we have*

1. *The nabla backward difference of x is defined by* $\nabla x(t) = x(t) - x(t-1)$, $t \in \mathbb{N}_1 = \{1, 2, \ldots\}$.

2. *The Riemann–Liouville's sum operator of x is defined by*

$$\nabla_0^{-\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t-\rho(s))^{\overline{\alpha-1}} x(s), \quad t \in \mathbb{N}_1. \tag{3}$$

3. *The Riemann–Liouville's difference operator of x is defined by*

$$\nabla_0^{\alpha} x(t) = \nabla\Big(\nabla_0^{-(1-\alpha)} x(t)\Big) = \frac{\nabla}{\Gamma(1-\alpha)} \sum_{s=1}^{t} \big(t-\rho(s)\big)^{\overline{-\alpha}} x(s), \quad t \in \mathbb{N}_1. \tag{4}$$

4. *The Caputo's difference operator of x is defined by*

$${}^{c}\nabla_0^{\alpha} x(t) = \nabla_0^{-(1-\alpha)} \nabla x(t) = \frac{1}{\Gamma(1-\alpha)} \sum_{s=1}^{t} \big(t-\rho(s)\big)^{\overline{-\alpha}} \nabla x(s), \quad t \in \mathbb{N}_1. \tag{5}$$

5. *The relation*

$$\nabla_0^{-\alpha}\, {}^{c}\nabla_0^{\alpha} x(t) = x(t) - x(0). \tag{6}$$

6. *The power rule is defined for* $t \in \mathbb{N}_a = \{a, a+1, a+2, \ldots\}$, $a \in \mathbb{R}$ *by*

$$\nabla_a^{-\alpha} (t-a)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} (t-a)^{\overline{\alpha+\mu}}, \ \mu > -1, \tag{7}$$

and hence

$$\nabla_a^{\alpha} (t-a)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} (t-a)^{\overline{\mu-\alpha}}.$$

7. *The relation between Riemann–Liouville and Caputo's difference operators is defined by*

$$\nabla_0^{\alpha} x(t) = {}^{c}\nabla_0^{\alpha} x(t) + x(0) \frac{t^{-\overline{\alpha}}}{\Gamma(1-\alpha)}. \tag{8}$$

3. A GENERALIZED DISCRETE FRACTIONAL GRONWALL INEQUALITY

We state and prove a new discrete fractional version of the generalized Gronwall's inequality that will be valid for systems involving delay term.

Theorem 1. (Generalized Gronwall Inequality) *Let $\alpha > 0$, $u(t), v(t)$ be nonnegative functions and $w(t)$ be nonnegative and nondecreasing function for $t \in \mathbb{N}_0$ such that $w(t) \le M$ where M is a constant. If*

$$u(t) \le v(t) + w(t)\Gamma(\alpha)\nabla_0^{-\alpha} u(t), \tag{9}$$

then

$$u(t) \le v(t) + \sum_{k=1}^{\infty} \Big(w(t)\Gamma(\alpha)\Big)^k \nabla_0^{-k\alpha} v(t). \tag{10}$$

Proof. Define

$$B\phi(t) = w(t) \sum_{s=1}^{t} (t-\rho(s))^{\overline{\alpha-1}} \phi(s), \ t \in \mathbb{N}_0.$$

It follows that

$$u(t) \le v(t) + Bu(t),$$

which implies that $u(t) \le \sum_{k=0}^{n-1} B^k v(t) + B^n u(t)$. We claim that

$$B^n u(t) \le \sum_{s=1}^{t} \frac{\big(w(t)\Gamma(\alpha)\big)^n}{\Gamma(n\alpha)} (t-\rho(s))^{\overline{n\alpha-1}} u(s) \tag{11}$$

and $B^n u(t) \to 0$ as $n \to \infty$ for $t \in \mathbb{N}_0$. It is easy to see that (11) is valid for $n = 1$. Assume that it is true for $n = k$, that is,

$$B^k u(t) \le \sum_{s=1}^{t} \frac{\big(w(t)\Gamma(\alpha)\big)^k}{\Gamma(k\alpha)} (t-\rho(s))^{\overline{k\alpha-1}} u(s).$$

If $n = k+1$ and by virtue of the assumption that w is a nondecreasing function, we have

$$\begin{aligned}
B^{k+1}u(t) &= B(B^k u(t)) \le w^{k+1}(t) \sum_{s=1}^{t} (t-\rho(s))^{\overline{\alpha-1}} \sum_{\nu=1}^{s} \frac{\big(\Gamma(\alpha)\big)^k}{\Gamma(k\alpha)} (s-\rho(\nu))^{\overline{k\alpha-1}} u(\nu) \\
&= w^{k+1}(t) \sum_{\nu=1}^{t} \sum_{s=\nu}^{t} \frac{\big(\Gamma(\alpha)\big)^k}{\Gamma(k\alpha)} (t-\rho(s))^{\overline{\alpha-1}} (s-\rho(\nu))^{\overline{k\alpha-1}} u(\nu) \\
&= \frac{w^{k+1}(t)\big(\Gamma(\alpha)\big)^{k+1}}{\Gamma(k\alpha)} \sum_{\nu=1}^{t} \frac{1}{\Gamma(\alpha)} \sum_{s=\nu}^{t} (t-\rho(s))^{\overline{\alpha-1}} (s-\rho(\nu))^{\overline{k\alpha-1}} u(\nu) \\
&= \frac{\big(w(t)\Gamma(\alpha)\big)^{k+1}}{\Gamma(k\alpha)} \sum_{\nu=1}^{t} \frac{1}{\Gamma(\alpha)} \sum_{s=\nu}^{t} (t-\rho(s))^{\overline{\alpha-1}} (s-\rho(\nu))^{\overline{k\alpha-1}} u(\nu) \\
&= \frac{\big(w(t)\Gamma(\alpha)\big)^{k+1}}{\Gamma(k\alpha)} \sum_{\nu=1}^{t} \nabla_{\rho(\nu)}^{-\alpha} (s-\rho(\nu))^{\overline{k\alpha-1}} u(\nu),
\end{aligned}$$

where $\nabla_{\rho(\nu)}^{-\alpha}u(t)=\frac{1}{\Gamma(\alpha)}\sum_{s=\nu}^{t}(t-\rho(s))^{\overline{\alpha-1}}u(s)$ has been used. It follows from (7) that

$$\begin{aligned} B^{k+1}u(t) &\leq \frac{\big(w(t)\Gamma(\alpha)\big)^{k+1}}{\Gamma(k\alpha)}\sum_{\nu=1}^{t}(s-\rho(\nu))^{\overline{k\alpha+\alpha-1}}\frac{\Gamma(k\alpha)}{\Gamma(k\alpha+\alpha)}u(\nu) \\ &= \sum_{\nu=1}^{t}\frac{\big(w(t)\Gamma(\alpha)\big)^{k+1}}{\Gamma(k\alpha+\alpha)}(s-\rho(\nu))^{\overline{k\alpha+\alpha-1}}u(\nu). \end{aligned}$$

Therefore, relation (11) is obtained. Furthermore, one can figure out that

$$B^{n}u(t)\leq\sum_{s=1}^{t}\frac{\big(M\Gamma(\alpha)\big)^{n}}{\Gamma(n\alpha)}(t-\rho(s))^{\overline{n\alpha-1}}u(s)\to 0 \text{ as } n\to\infty,\ t\in\mathbb{N}_0.$$

To complete the proof, we let $n\to\infty$ in

$$u(t)\leq\sum_{k=0}^{n-1}B^{k}v(t)+B^{n}u(t)=v(t)+\sum_{k=1}^{n-1}B^{k}v(t)+B^{n}u(t)$$

to reach at $u(t)\leq v(t)+\sum_{k=1}^{\infty}B^{k}v(t)$. By the help of the semigroup property

$$(\nabla_0^{-\alpha}\nabla_0^{-\mu}v)(t)=(\nabla_0^{-(\alpha+\mu)}v)(t)$$

and the definition of B we get (10). This completes the proof. □

Let $E_{\overline{\alpha}}(\lambda,z)=\sum_{k=0}^{\infty}\frac{\lambda^{k}z^{\overline{k\alpha}}}{\Gamma(\alpha k+1)}$ be the Mittage–Leffler function in one parameter which was introduced in [12, 39]. Based on Theorem 1, we provide a particular version expressed by means of the discrete Mittag–Lefler functions.

Corollary 1. *Under the hypothesis of Theorem 1, assume further that $v(t)$ is a nondecreasing function for $t\in\mathbb{N}_0$, then*

$$u(t)\leq v(t)E_{\overline{\alpha}}\big(w(t)\Gamma(\alpha),t\big),\quad t\in\mathbb{N}_0. \tag{12}$$

Proof. From (10) and the assumption that $v(t)$ is a nondecreasing function for $t\in\mathbb{N}_0$, we may write

$$u(t)\leq v(t)\Big[1+\sum_{k=1}^{\infty}\sum_{s=1}^{t}\frac{\big(w(t)\Gamma(\alpha)\big)^{k}}{\Gamma(k\alpha)}(t-\rho(s))^{\overline{k\alpha-1}}\Big]$$

or

$$u(t)\leq v(t)\Big[1+\sum_{k=1}^{\infty}\nabla_0^{-k\alpha}\Big(w(t)\Gamma(\alpha)\Big)^{k}\Big].$$

It follows that

$$\begin{aligned} u(t) &\leq v(t)\Big[1+\sum_{k=1}^{\infty}\Big(w(t)\Gamma(\alpha)\Big)^{k}\nabla_0^{-k\alpha}1\Big] \\ &= v(t)\Big[1+\sum_{k=1}^{\infty}\Big(w(t)\Gamma(\alpha)\Big)^{k}\frac{t^{\overline{k\alpha}}}{\Gamma(k\alpha+1)}\Big] \\ &= v(t)\sum_{k=0}^{\infty}\frac{\big(w(t)\Gamma(\alpha)\big)^{k}t^{\overline{k\alpha}}}{\Gamma(k\alpha+1)} = v(t)E_{\overline{\alpha}}\big(w(t)\Gamma(\alpha),t\big). \end{aligned}$$

The proof is complete. □

For more comprehensive details on the properties of Mittage–Leffler function in one parameter, one can consult the papers [40, 41].

4. APPLICATIONS AND AN EXAMPLE

Based on the results obtained in the previous section, we prove the uniqueness and obtain an estimate for the solutions of system (1). Moreover, numerical example is presented to demonstrate the applicability of the main results.

Let $|\cdot|$ *be any Euclidean norm and* $\|\cdot\|$ *be the matrix norm induced with this vector. Let* $D = D(\mathbb{N}_0\times\mathbb{R}^m\times\mathbb{R}^m,\mathbb{R}^m)$ *be the set of all bounded functions (sequences). Clearly, the space* D *is a Banach space induced with the norm* $\|z\|_D := \sup_{t\in I_{-\tau}}|z(t)|$.

We make use of the following assumptions:

(H.1) $f\in D(\mathbb{N}_0\times\mathbb{R}^m\times\mathbb{R}^m,\mathbb{R}^m)$ *is a Lipschitz–type function. That is, there exists a positive constant* $L_1>0$ *such that*

$$\big\|f(t,x(t),x(t-\tau))-f(t,y(t),y(t-\tau))\big\| \leq L_1\Big(\|x(t)-y(t)\|+\|x(t-\tau)-y(t-\tau)\|\Big),$$

for $t\in\mathbb{N}_0$.

(H.2) There exists a positive constant L_2 *such that* $\|f(t,x(t),x(t-\tau))\|\leq L_2$.

The first result in this section provides a representation for the solutions of system (1) that will be useful in the subsequent analysis.

Theorem 2. $x:\mathbb{N}_{-\tau}\to\mathbb{R}^m$ *is a solution of system (1) if and only if*

$$\begin{cases} x(t)=\varphi(0)+\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}\big(t-\rho(s)\big)^{\overline{\alpha-1}}\Big[A_0x(s)+A_1x(s-\tau) \\ +f(s,x(s),x(s-\tau))\Big],\ t\in\mathbb{N}_0 \\ x(t)=\varphi(t),\ t\in I_{-\tau}, \end{cases} \tag{13}$$

Proof. For $t \in I_{-\tau}$, it is clear that $x(t) = \varphi(t)$ is the solution of (1). For $t \in \mathbb{N}_0$, we apply ∇_0^α on both sides of equation (**??**) to obtain

$$\nabla_0^\alpha x(t) = \varphi(0)\frac{t^{\overline{-\alpha}}}{\Gamma(1-\alpha)} + A_0 x(t) + A_1 x(t-\tau) + f(t, x(t), x(t-\tau)),$$

where $(\nabla_0^\alpha \nabla_0^{-\alpha} u)(t) = u(t)$ have been used. By using relation (8), we end up with the desired form

$${}^c\nabla_0^\alpha x(t) = A_0 x(t) + A_1 x(t-\tau) + f(t, x(t), x(t-\tau)),\ t \in \mathbb{N}_0.$$

From system (1), we can see that $x(t) = \varphi(t)$ for $t \in I_{-\tau}$. For $t \in \mathbb{N}_0$, we apply $\nabla_0^{-\alpha}$ on both sides of equation (1) to get

$$\nabla_0^{-\alpha}[{}^c\nabla_0^\alpha x(t)] = \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}\big(t-\rho(s)\big)^{\overline{\alpha-1}}\Big[A_0 x(s) + A_1 x(s-\tau) + f(s, x(s), x(s-\tau))\Big].$$

In view of relation (6), one can easily see that

$$x(t) = \varphi(0) + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}\big(t-\rho(s)\big)^{\overline{\alpha-1}}\Big[A_0 x(s) + A_1 x(s-\tau) + f(s, x(s), x(s-\tau))\Big].$$

□

The first main application in this paper is provided in the following theorem.

Theorem 3. *Let condition (H.1) hold. If $x(t)$ and $y(t)$ are two solutions for the system (1), then $x(t) = y(t)$.*

Proof. Let x and y be two solutions of system (1). Denote z by $z(t) = x(t) - y(t)$. Then, one can easily figure out that $z(t) = 0$ for $t \in I_{-\tau}$. This implies that system (1) has a unique solution for $t \in I_{-\tau}$.

For $t \in \mathbb{N}_0$, however, we have

$$z(t) = \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}\big(t-\rho(s)\big)^{\overline{\alpha-1}}\big[A_0 z(s) + A_1 z(s-\tau) + f(s, x(s), x(s-\tau)) - f(s, y(s), y(s-\tau))\big].$$

If $t \in I_\tau = \{0, 1, \ldots, \tau\}$, then $z(t-\tau) = 0$. Therefore,

$$z(t) = \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}\big(t-\rho(s)\big)^{\overline{\alpha-1}}\Big[A_0 z(s) + f(s, x(s), x(s-\tau)) - f(s, y(s), y(s-\tau))\Big]. \tag{14}$$

This implies

$$\begin{aligned}
\|z(t)\| &\leq \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}\big(t-\rho(s)\big)^{\overline{\alpha-1}}\big[\|A_0\|\|z(s)\| \\
&+ \|f(s,x(s),x(s-\tau))-f(s,y(s),y(s-\tau))\|\big] \\
&\leq \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}\big(t-\rho(s)\big)^{\overline{\alpha-1}}\big[\|A_0\|\|z(s)\| \\
&+ L_1\Big(\|x(s)-y(s)\|+\|x(s-\tau)-y(s-\tau)\|\Big)\big] \\
&= \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}\big(t-\rho(s)\big)^{\overline{\alpha-1}}\big[(\|A_0\|+L_1)\|z(s)\|+L_1\|z(s-\tau)\|\big] \\
&= \frac{\|A_0\|+L_1}{\Gamma(\alpha)}\sum_{s=1}^{t}\big(t-\rho(s)\big)^{\overline{\alpha-1}}\|z(s)\|. \qquad (15)
\end{aligned}$$

By applying the result of Corollary 1, we have

$$\|z(t)\| \leq 0\cdot E_{\overline{\alpha}}\big[\|A_0\|+L_1,t\big], \qquad (16)$$

which implies that $x(t)=y(t)$ for $t\in I_\tau$.

For $t\in\mathbb{N}_\tau=\{\tau,\tau+1,\ldots\}$, we get

$$\begin{aligned}
z(t) &= \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}\big(t-\rho(s)\big)^{\overline{\alpha-1}}\big[A_0z(s)+f(s,x(s),x(s-\tau))-f(s,y(s),y(s-\tau))\big] \\
&+ \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}\big(t-\rho(s)\big)^{\overline{\alpha-1}}A_1z(s-\tau). \qquad (17)
\end{aligned}$$

It follows that

$$\begin{aligned}
\|z(t)\| &\leq \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}\big(t-\rho(s)\big)^{\overline{\alpha-1}}\big[\|A_0\|\|z(s)\| \\
&+ \|f(s,x(s),x(s-\tau))-f(s,y(s),y(s-\tau))\|\big] \\
&+ \frac{\|A_1\|}{\Gamma(\alpha)}\sum_{s=1}^{t}\big(t-\rho(s)\big)^{\overline{\alpha-1}}\|z(s-\tau)\| \\
&\leq \frac{\|A_0\|+L_1}{\Gamma(\alpha)}\sum_{s=1}^{t}\big(t-\rho(s)\big)^{\overline{\alpha-1}}\|z(s)\| \qquad (18) \\
&+ \frac{\|A_1\|+L_1}{\Gamma(\alpha)}\sum_{s=1}^{t}\big(t-\rho(s)\big)^{\overline{\alpha-1}}\|z(s-\tau)\|.
\end{aligned}$$

Let $\bar{z}(t) = \sup_{\theta \in I_{-\tau}} \|z(t+\theta)\|$, then we get

$$\begin{aligned} \bar{z}(t) &\le \frac{\|A_0\| + L_1}{\Gamma(\alpha)} \sum_{s=1}^{t} \big(t-\rho(s)\big)^{\overline{\alpha-1}} \bar{z}(s) + \frac{\|A_1\| + L_1}{\Gamma(\alpha)} \sum_{s=1}^{t} \big(t-\rho(s)\big)^{\overline{\alpha-1}} \bar{z}(s) \\ &\le \frac{\|A_0\| + \|A_1\| + 2L_1}{\Gamma(\alpha)} \sum_{s=1}^{t} \big(t-\rho(s)\big)^{\overline{\alpha-1}} \bar{z}(s). \end{aligned} \tag{19}$$

By applying the result of Corollary 1, we obtain

$$\|z(t)\| \le \bar{z}(t) \le 0 \cdot E_{\overline{\alpha}}\big[\big(\|A_0\| + \|A_0\| + 2L_1\big)\Gamma(\alpha), t\big], \tag{20}$$

Hence, we end up with $x(t) = y(t)$ for $t \in \mathbb{N}_{-\tau}$.

□

In the following theorem, we provide an estimate for the solution of system (1).

Theorem 4. *Let condition (H.2) hold. Then, the following estimate for the solution $x(t)$ of system (1) is valid:*

$$\|x(t)\| \le \Big[\|\varphi\| + \frac{L_2 + \|\varphi\|\big(\|A_0\| + \|A_1\|\big)}{\Gamma(\alpha+1)} t^{\overline{\alpha}}\Big] E_{\overline{\alpha}}\big[\big(\|A_0\| + \|A_1\|\big), t\big]. \tag{21}$$

Proof. For $t \in \mathbb{N}_0$, the solution of system (1) has the form

$$x(t) = \varphi(0) + \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} \big(t-\rho(s)\big)^{\overline{\alpha-1}} \Big[A_0 x(s) + A_1 x(s-\tau) + f(s, x(s), x(s-\tau))\Big]. \tag{22}$$

It follows that

$$\begin{aligned} \|x(t)\| &\le \|\varphi(0)\| + \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} \big(t-\rho(s)\big)^{\overline{\alpha-1}} \big\|A_0 x(s) + A_1 x(s-\tau) \\ &+ f(s, x(s), x(s-\tau))\big\| \\ &\le \|\varphi\| + \frac{\|A_0\|}{\Gamma(\alpha)} \sum_{s=1}^{t} \big(t-\rho(s)\big)^{\overline{\alpha-1}} \|x(s)\| \\ &+ \frac{\|A_1\|}{\Gamma(\alpha)} \sum_{s=1}^{t} \big(t-\rho(s)\big)^{\overline{\alpha-1}} \|x(s-\tau)\| \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} \big(t-\rho(s)\big)^{\overline{\alpha-1}} \|f(s, x(s), x(s-\tau))\|. \end{aligned}$$

By the assumption (H.2), the above inequality can be rewritten as

$$\begin{aligned}\|x(t)\| &\leq \|\varphi\| + \frac{\|A_0\|+\|A_1\|}{\Gamma(\alpha)}\sum_{s=1}^{t}\big(t-\rho(s)\big)^{\overline{\alpha-1}}\Big[\sup_{\theta\in I_{-\tau}}\|x(s+\theta)\|+\|\varphi\|\Big]\\ &+ \frac{L_2}{\Gamma(\alpha)}\sum_{s=1}^{t}\big(t-\rho(s)\big)^{\overline{\alpha-1}} \qquad (23)\\ &= \|\varphi\| + \frac{L_2+\|\varphi\|\big(\|A_0\|+\|A_1\|\big)}{\Gamma(\alpha+1)}t^{\overline{\alpha}}\\ &+ \frac{\|A_0\|+\|A_1\|}{\Gamma(\alpha)}\sum_{s=1}^{t}\big(t-\rho(s)\big)^{\overline{\alpha-1}}\sup_{\theta\in I_{-\tau}}\|x(s+\theta)\|, \qquad (24)\end{aligned}$$

where the power rule (7) has been used. Let $v(t) = \|\varphi\| + \frac{L_2+\|\varphi\|\big(\|A_0\|+\|A_1\|\big)}{\Gamma(\alpha+1)}t^{\overline{\alpha}}$, then v is nondecreasing function. Therefore, Corollary 1 with $w(t) = \|A_0\| + \|A_1\|$ implies that

$$\|x(t)\| \leq \sup_{\theta\in I_{-\tau}}\|x(t+\theta)\| \leq v(t)E_{\overline{\alpha}}\big[\big(\|A_0\|+\|A_1\|\big)\Gamma(\alpha),t\big]. \qquad (25)$$

Hence, the solution x of (1) satisfies the estimate

$$\|x(t)\| \leq \Big[\|\varphi\| + \frac{L_2+\|\varphi\|\big(\|A_0\|+\|A_1\|\big)}{\Gamma(\alpha+1)}t^{\overline{\alpha}}\Big]E_{\overline{\alpha}}\big[\big(\|A_0\|+\|A_1\|\big)\Gamma(\alpha),t\big]. \qquad (26)$$

The proof is complete. □

Example 1. *Consider the nonlinear delay fractional difference equation of the form*

$$^{c}\nabla_t^{\frac{1}{2}}x(t) = 2x(t) + 3x(t-3) - \sin x(t) + 3\sin x(t-3),\ t\in\mathbb{N}_0 \qquad (27)$$

with the initial function $x(t) = \cos 2t,\ t\in I_{-3}$. *Clearly, equation (27) is a scalar equation and* $A_0 = 2$ *and* $A_1 = 3$. *The nonlinearity has the form* $f(t,x(t),x(t-\tau)) = -\sin x(t) + 3\sin x(t-3)$. *Therefore, we have*

$$\begin{aligned}\big\|f(t,x(t),x(t-\tau)) &- f(t,y(t),y(t-\tau))\big\|\\ &= \big\|-\sin x(t) + 3\sin x(t-3) + \sin y(t) - 3\sin y(t-3)\big\|\\ &\leq 3\Big(\big\|\sin x(t)-\sin y(t)\big\| + \big\|\sin x(t-3)-\sin y(t-3)\big\|\Big).\end{aligned}$$

Thus, condition (H.1) holds with $L_1 = 3$. *By the consequence of Theorem 2, equation (27) has a unique solution. Moreover,*

$$\big\|f(t,x(t),x(t-\tau))\big\| = \big\|-\sin x(t) + 3\sin x(t-3)\big\| \leq 4,$$

which implies that condition (H.2) is satisfied with $L_2 = 4$. *By Theorem 4, the solution has the estimate*

$$\big\|x(t)\big\| \leq \Big[1+\frac{9}{\Gamma(\frac{3}{2})}t^{\overline{\frac{1}{2}}}\Big]\sum_{k=0}^{\infty}\frac{(5\sqrt{\pi})^k t^{\overline{\frac{k}{2}}}}{\Gamma(\frac{k}{2}+1)}.$$

Remark 1. *Solving equation (27) is not an easy task. However, getting a bound for the solution could be considered as a substantial step.*

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