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ON DISTANCES IN GENERALIZED SIERPIŃSKI GRAPHS

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In this paper we propose formulas for the distance between vertices of a generalized Sierpiński graph $S(G,t)$ in terms of the distance between vertices of the base graph G. In particular, we deduce a recursive formula for the distance between an arbitrary vertex and an extreme vertex of $S(G,t)$, and we obtain a recursive formula for the distance between two arbitrary vertices of $S(G,t)$ when the base graph is triangle-free. From these recursive formulas, we provide algorithms to compute the distance between vertices of $S(G,t)$. In addition, we give an explicit formula for the diameter and radius of $S(G,t)$ when the base graph is a tree.

1. INTRODUCTION

Let $G=(V,E)$ be a non-empty graph, and t a positive integer. We denote by V^t the set of words of length t on alphabet V. The letters of a word u of length t are denoted by $u_1u_2\dots u_t$. The concatenation of two words u and v is denoted by uv. Klavžar and Milutinović introduced in [**10**] the graph $S(K_n,t)$, $t\geq 1$, whose vertex set is V^t, where $\{u,v\}$ is an edge if and only if there exists $i\in\{1,\dots,t\}$ such that:

$$\text{(i) } u_j=v_j, \text{ if } j<i; \text{ (ii) } u_i\neq v_i; \text{ (iii) } u_j=v_i \text{ and } v_j=u_i \text{ if } j>i.$$

As noted in [**8**], in a compact form, the edge set can be described as

$$\{\{wu_iu_j^{d-1},wu_ju_i^{d-1}\}:\ u_i,u_j\in V, i\neq j; d\in[t]; w\in V^{t-d}\}.$$

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The graph $S(K_3,t)$ is isomorphic to the graph of the Tower of Hanoi with t disks [**10**]. Later, those graphs have been called Sierpiński graphs in [**11**] and they were studied by now from numerous points of view. For instance, the authors of [**4**] studied identifying codes, locating-dominating codes, and total-dominating codes in Sierpiński graphs. In [**7**] the authors propose an algorithm, which makes use of three automata and the fact that there are at most two internally vertex-disjoint shortest paths between any two vertices, to determine all shortest paths in Sierpiński graphs. The authors of [**11**] proved that for any $n \geq 1$ and $t \geq 1$, the Sierpiński graph $S(K_n,t)$ has a unique 1-perfect code (or efficient dominating set) if t is even, and $S(K_n,t)$ has exactly n 1-perfect codes if t is odd. The Hamming dimension of a graph G was introduced in [**12**] as the largest dimension of a Hamming graph into which G embeds as an irredundant induced subgraph. That paper gives an upper bound for the Hamming dimension of the Sierpiński graphs $S(K_n,t)$ for $n \geq 3$. It also shows that the Hamming dimension of $S(K_3,t)$ grows as 3^{t-3}. The idea of almost-extreme vertex of $S(K_n,t)$ was introduced in [**13**] as a vertex that is either adjacent to an extreme vertex of $S(K_n,t)$ or is incident to an edge between two subgraphs of $S(K_n,t)$ isomorphic to $S(K_n,t-1)$. The authors of [**13**] deduced explicit formulas for the distance in $S(K_n,t)$ between an arbitrary vertex and an almost-extreme vertex. Also they gave a formula of the metric dimension of a Sierpiński graph, which was independently obtained by Parreau in her Ph.D. thesis. The set $S_u = \{v \in V(S(K_n,t)) :$ there exist two shortest u,v-paths in $S(K_n,t)\}$, where u is any almost-extreme vertex of $S(K_n,t)$, was completely determined in [**17**]. The eccentricity of an arbitrary vertex of Sierpiński graphs was studied in [**6**] where the main result gives an expression for the average eccentricity of $S(K_n,t)$. For a general background on Sierpiński graphs, the reader is invited to read the comprehensive survey [**9**] and references therein.

The construction of $S(K_n,t)$ was generalized in [**5**] for any graph $G=(V,E)$, by defining the t-th *generalized Sierpiński graph* of G, denoted by $S(G,t)$, as the graph with vertex set V^t and edge set $\{\{wu_iu_j^{d-1}, wu_ju_i^{d-1}\} : \{u_i,u_j\} \in E; d \in [t]; w \in V^{t-d}\}$. Figure 1 shows the graph $S(K_4-e,3)$.

Notice that if $\{u,v\}$ is an edge of $S(G,t)$, there is an edge $\{x,y\}$ of G and a word w such that $u = wxyy\ldots y$ and $v = wyxx\ldots x$. In general, $S(G,t)$ can be constructed recursively from G with the following process: $S(G,1) = G$ and, for $t \geq 2$, we copy n times $S(G,t-1)$ and add the letter x at the beginning of each label of the vertices belonging to the copy of $S(G,t-1)$ corresponding to x. Then for every edge $\{x,y\}$ of G, add an edge between vertex $xyy\ldots y$ and vertex $yxx\ldots x$. Vertices of the form $xx\ldots x$ are called *extreme vertices* of $S(G,t)$. Notice that for any graph G of order n and any integer $t \geq 2$, $S(G,t)$ has n extreme vertices and, if x has degree $d(x)$ in G, then the extreme vertex $xx\ldots x$ of $S(G,t)$ also has degree $d(x)$. Moreover, the degrees of two vertices $yxx\ldots x$ and $xyy\ldots y$, which connect two copies of $S(G,t-1)$, are equal to $d(x)+1$ and $d(y)+1$, respectively.

The authors of [**5**] announced some results about generalized Sierpiński graphs concerning their automorphism groups and perfect codes. In our opinion, these results definitely deserve to be published. In the first published article on this

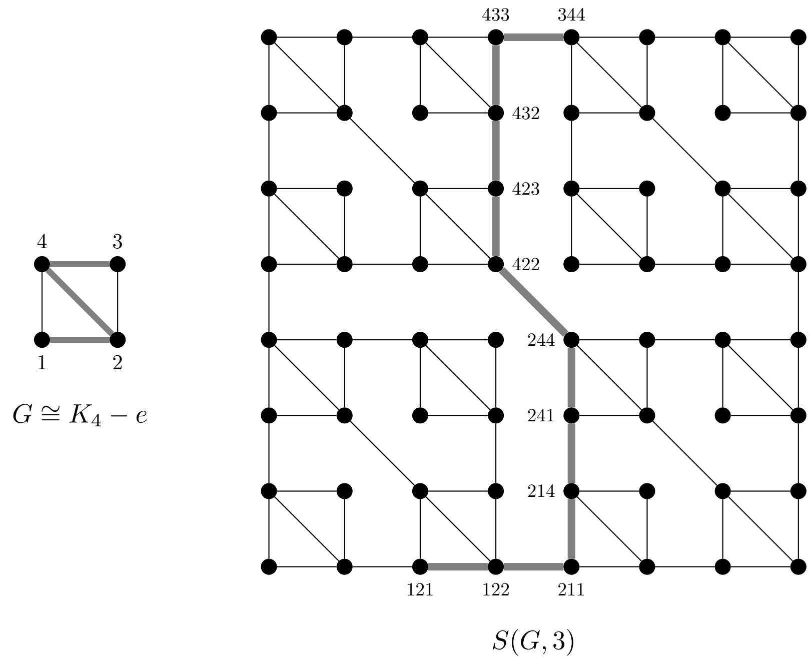

Figure 1: The distance between vertices 121 and 344 in $S(G,3)$ is 10.

subject the authors obtained closed formulae for the Randić index of polymeric networks modelled by generalized Sierpiński graphs [**16**], while in [**2**] this work was extended to the generalized Randić index. Later, the total chromatic number of generalized Sierpiński graphs was studied in [**3**] and the strong metric dimension has recently been studied in [**1**]. The authors of [**15**] obtained closed formulae for the chromatic, vertex cover, clique and domination numbers of generalized Sierpiński graphs $S(G,t)$ in terms of parameters of the base graph G. More recently, a general upper bound on the Roman domination number of $S(G,t)$ was obtained in [**14**]. In particular, it was studied the case in which the base graph G is a path, a cycle, a complete graph or a graph having exactly one universal vertex. In this paper we propose formulas for the distance between vertices of a generalized Sierpiński graph in terms of the distance between vertices of the base graph. In particular, we deduce a recursive formula for the distance between an arbitrary vertex and an extreme vertex of $S(G,t)$, and we obtain a recursive formula for the distance between two arbitrary vertices of $S(G,t)$ when the base graph is triangle-free. From these recursive formulas, we provide algorithms to compute the distance between vertices of $S(G,t)$. In addition, we give an explicit formula for the diameter and radius of $S(G,t)$ when the base graph is a tree.

2. MAIN RESULTS

For any $t \geq 2$ the subgraph $\langle V_x \rangle$ of $S(G,t)$, induced by $V_x = \{xw : x \in V, w \in V^{t-1}\}$, is isomorphic to $S(G,t-1)$. Note that $\langle V_x \rangle$ contains exactly one extreme vertex of $S(G,t)$.

Lemma 1. *Let $G = (V,E)$ be a connected non-trivial graph. For any $x \in V$, $w, w' \in V^{t-1}$ and any integer $t \geq 2$,*

$$d_{S(G,t)}(xw, xw') = d_G(w, w').$$

Proof. For any shortest path $w, w_1, w_2, \ldots, w_l, w'$ between w and w' in $S(G,t-1)$ and $x \in V$, we have a path $xw, xw_1, xw_2, \ldots, xw_l, xw'$ between xw and xw' in $S(G,t)$. Hence, we can conclude that

$$d_{S(G,t)}(xw, xw') \leq d_{S(G,t-1)}(w, w'). \tag{1}$$

Suppose that there exists a shortest path P between xw and xw' of the form

$$xw = v_0 w_0^{(0)}, v_0 w_1^{(0)}, \ldots, v_0 w_{l_0}^{(0)}, v_1 w_0^{(1)}, v_1 w_1^{(1)}, \ldots, v_1 w_{l_1}^{(1)}, \ldots, v_0 w_{l_r}^{(r)} = xw'.$$

In such a case, $v_0 w_0^{(r)} = v_0 (v_{r-1})^{t-1}$, $v_{r-1} w_{l_{r-1}}^{(r-1)} = v_{r-1}(v_0)^{t-1}$, $v_{i+1} w_0^{(i+1)} = v_{i+1}(v_i)^{t-1}$ and $v_i w_{l_i}^{(i)} = v_i (v_{i+1})^{t-1}$ for all $i \in \{0, 1, \ldots, r-2\}$. Also, the trail $x = v_0, v_1, v_2, \ldots, v_{r-1}, v_0 = x$ associated to P has length greater than zero. Since P is a shortest path, we obtain that $v_i w_0^{(i)}, v_i w_1^{(i)}, \ldots, v_i w_{l_i}^{(i)}$ is a shortest path in $\langle V_{v_i} \rangle$, where $i \in \{0, \ldots, r\}$ and $v_r = v_0$, so that

$d_{S(G,t)}\left(v_0 w, v_0 (v_1)^{t-1}\right) = d_{S(G,t-1)}\left(w, (v_1)^{t-1}\right)$,

$d_{S(G,t)}\left(v_0 (v_{r-1})^{t-1}, v_0 w'\right) = d_{S(G,t-1)}\left((v_{r-1})^{t-1}, w'\right)$ and

$d_{S(G,t)}\left(v_{i+1}(v_i)^{t-1}, v_{i+1}(v_{i+2})^{t-1}\right) = d_{S(G,t-1)}\left((v_i)^{t-1}, (v_{i+2})^{t-1}\right)$, $i \in \{0, 1, \ldots, r-2\}$.

Thus,

$$\begin{aligned} d_{S(G,t)}(xw, xw') =& d_{S(G,t)}(v_0 w, v_0 (v_1)^{t-1}) + \sum_{i=0}^{r-2} d_{S(G,t)}(v_{i+1}(v_i)^{t-1}, v_{i+1}(v_{i+2})^{t-1}) + \\ &+ d_{S(G,t)}(v_0 (v_{r-1})^{t-1}, v_0 w') + r \\ =& d_{S(G,t-1)}(w, (v_1)^{t-1}) + \sum_{i=0}^{r-2} d_{S(G,t-1)}((v_i)^{t-1}, (v_{i+2})^{t-1}) + \\ &+ d_{S(G,t-1)}((v_{r-1})^{t-1}, w') + r. \end{aligned}$$

Hence, if r is even, then

$$\begin{aligned} d_{S(G,t)}(xw,xw') >& d_{S(G,t-1)}(w,(v_1)^{t-1}) + \sum_{i=0}^{\frac{r-4}{2}} d_{S(G,t-1)}((v_{2i+1})^{t-1},(v_{2i+3})^{t-1}) \\ &+ d_{S(G,t-1)}((v_{r-1})^{t-1},w') \\ \geq& d_{S(G,t-1)}(w,w') \text{ (by triangle inequality)}, \end{aligned}$$

which contradicts (1). Now, if r is odd, then

$$\begin{aligned} d_{S(G,t)}(xw,xw') >& d_{S(G,t-1)}(w,(v_1)^{t-1}) + \sum_{i=0}^{\frac{r-3}{2}} d_{S(G,t-1)}((v_{2i+1})^{t-1},(v_{2i+3})^{t-1}) \\ &+ \sum_{i=0}^{\frac{r-3}{2}} d_{S(G,t-1)}((v_{2i})^{t-1},(v_{2i+2})^{t-1}) + d_{S(G,t-1)}((v_{r-1})^{t-1},w') \\ \geq& d_{S(G,t-1)}(w,w') \text{ (by triangle inequality)}, \end{aligned}$$

which contradicts (1). Therefore, the result follows. □

From the lemma above, we deduce the following remark.

Remark 2. *Let $G = (V,E)$ be a connected non-trivial graph and let $r \geq 1$ and $t \geq 1$ be two integers. If*
$(*)$
$v_0w_0^{(0)}, v_0w_1^{(0)}, \ldots, v_0w_{l_0}^{(0)}, v_1w_0^{(1)}, v_1w_1^{(1)}, \ldots, v_1w_{l_1}^{(1)}, \ldots, v_rw_0^{(r)}, v_rw_1^{(r)}, \ldots, v_rw_{l_r}^{(r)},$

is a shortest path in $S(G,t)$, where $v_i \in V$, $v_k \neq v_{k+1}$ and $w_j^{(i)} \in V^{t-1}$, for $i \in \{0,1,\ldots,r\}$, $k \in \{0,1,\ldots,r-1\}$ and $j \in \{0,1,\ldots,l_i\}$, then $v_0,v_1,\ldots,v_r$ is a path in G.

From now on, we will refer to the path $v_0,v_1,\ldots,v_r$ in G as the G-path associated to the shortest path $(*)$. We say that a G-path P is *triangle-free* if for any set S composed by three consecutive vertices of P, the subgraph of G induced by S is a path. For instance, the G-path associated to the shortest path $121, 122, 211, 214, 241, 244, 422, 423, 432, 433, 344$ shown in Figure 1 is $1,2,4,3$. Notice that this G-path is not triangle-free.

Lemma 3. *Let $G = (V,E)$ be a connected non-trivial graph and let $t \geq 1$ be an integer such that $d_{S(G,t)}(u^t,v^t) = (2^t-1)d_G(u,v)$ for all $u,v \in V$. Then for every $x,y \in V$ and $w,w' \in V^t$ the following assertions hold.*

(i) *The G-path associated to any shortest path between x^{t+1} and yw is triangle-free.*

(ii) *Let $w \neq x^t$, $w' \neq y^t$. If $d_G(x,y) \geq 2$, then the G-path $x,v_1,v_2,\ldots,v_{r-1},y$ associated to any shortest path between xw and yw' is triangle-free whenever $x \notin N(v_2)$.*

Proof. Let P be a shortest path between x^{t+1} and yw and let $x = v_0, v_1, v_2, \ldots, v_{r-1}, v_r = y$ be the G-path P_1 associated to P. If P_1 is triangle-free, then we are done. Now, suppose that j is the minimum subscript such that v_j and v_{j+2} are adjacent and consider the following cases:

(a) $j = 0$. Suppose that $r = 2$. In this case, the G-path is $x = v_0, v_1, v_2 = y$, so that

$$\begin{aligned}
d_{S(G,t+1)}(x^{t+1}, yw) =& d_{S(G,t+1)}((v_0)^{t+1}, v_0(v_1)^t) + d_{S(G,t+1)}(v_1(v_0)^t, v_1(v_2)^t)\\
&+ d_{S(G,t+1)}(v_2(v_1)^t, v_2w) + 2\\
=& d_{S(G,t)}((v_0)^t, (v_1)^t) + d_{S(G,t)}((v_0)^t, (v_2)^t)\\
&+ d_{S(G,t)}((v_1)^t, w) + 2 \text{ (by Lemma 1)}\\
\geq& d_{S(G,t)}((v_0)^t, w) + d_{S(G,t)}((v_0)^t, (v_2)^t) + 2\\
&\text{(by triangle inequality)}\\
>& d_{S(G,t)}((v_0)^t, w) + d_{S(G,t)}((v_0)^t, (v_2)^t) + 1\\
=& d_{S(G,t+1)}((v_0)^{t+1}, v_0(v_2)^t) + 1 + d_{S(G,t+1)}(v_2(v_0)^t, v_2w)\\
&\text{(by Lemma 1)}\\
\geq& d_{S(G,t+1)}(x^{t+1}, yw) \text{ (by triangle inequality)},
\end{aligned}$$

which is a contradiction.

Now, assume that $r \geq 3$ and let

$$\alpha_j = \sum_{i=2}^{r-2} d_{S(G,t+1)}(v_{i+1}v_i^t, v_{i+1}v_{i+2}^t) + d_{S(G,t+1)}(v_r(v_{r-1})^t, v_rw) + r - 2.$$

So,

$$\begin{aligned}
d_{S(G,t+1)}(x^{t+1}, yw) =& d_{S(G,t+1)}((v_0)^{t+1}, v_0(v_1)^t) + d_{S(G,t+1)}(v_1(v_0)^t, v_1(v_2)^t)\\
&+ d_{S(G,t+1)}(v_2(v_1)^t, v_2(v_3)^t) + \alpha_j + 2\\
=& d_{S(G,t)}((v_0)^t, (v_1)^t) + d_{S(G,t)}((v_0)^t, (v_2)^t)\\
&+ d_{S(G,t)}((v_1)^t, (v_3)^t) + \alpha_j + 2 \text{ (by Lemma 1)}\\
\geq& d_{S(G,t)}((v_0)^t, (v_3)^t) + d_{S(G,t)}((v_0)^t, (v_2)^t) + \alpha_j + 2\\
&\text{(by triangle inequality)}\\
=& d_{S(G,t+1)}((v_0)^{t+1}, v_0(v_2)^t) + d_{S(G,t+1)}(v_2(v_0)^t, v_2(v_3)^t)\\
&+ \alpha_j + 2 \text{ (by Lemma 1)}\\
>& d_{S(G,t+1)}((v_0)^{t+1}, v_0(v_2)^t) + 1 + d_{S(G,t+1)}(v_2(v_0)^t, v_2(v_3)^t)\\
&+ \alpha_j\\
\geq& d_{S(G,t+1)}(x^{t+1}, yw) \text{ (by triangle inequality)},
\end{aligned}$$

which is a contradiction.

(b) $1 \le j \le r-3$. Let

$$\alpha_j = d_{S(G,t+1)}((v_0)^{t+1}, v_0(v_1)^t) + \sum_{i=0}^{j-2} d_{S(G,t+1)}(v_{i+1}v_i^t, v_{i+1}(v_{i+2})^t) + \\ + \sum_{i=j+2}^{r-2} d_{S(G,t+1)}(v_{i+1}v_i^t, v_{i+1}(v_{i+2})^t) + d_{S(G,t+1)}(v_r(v_{r-1})^t, v_r w) + r - 2$$

Notice that $d_G(v_{j-1}, v_{j+1}) = 2, d_G(v_j, v_{j+2}) = 1$ and $d_G(v_{j+1}, v_{j+3}), d_G(v_{j-1}, v_{j+2}), d_G(v_j, v_{j+3}) \in \{1, 2\}$. Hence,

$$\begin{aligned} d_{S(G,t+1)}(x^{t+1}, yw) =& \alpha_j + d_{S(G,t+1)}(v_j(v_{j-1})^t, v_j(v_{j+1})^t) \\ &+ d_{S(G,t+1)}(v_{j+1}(v_j)^t, v_{j+1}(v_{j+2})^t) \\ &+ d_{S(G,t+1)}(v_{j+2}(v_{j+1})^t, v_{j+2}(v_{j+3})^t) + 2 \\ =& \alpha_j + d_{S(G,t)}((v_{j-1})^t, (v_{j+1})^t) + d_{S(G,t)}((v_j)^t, (v_{j+2})^t) \\ &+ d_{S(G,t)}((v_{j+1})^t, (v_{j+3})^t) + 2 \text{ (by Lemma 1)} \\ \ge& \alpha_j + 2(2^t - 1) + (2^t - 1) + (2^t - 1) + 2 \text{ (by assumption)} \\ \ge& \alpha_j + d_{S(G,t)}((v_{j-1})^t, (v_{j+2})^t) + d_{S(G,t)}((v_j)^t, (v_{j+3})^t) + 2 \\ &\text{(by assumption)} \\ =& \alpha_j + d_{S(G,t+1)}(v_j(v_{j-1})^t, v_j(v_{j+2})^t) \\ &+ d_{S(G,t+1)}(v_{j+2}(v_j)^t, v_{j+2}(v_{j+3})^t) + 2 \text{ (by Lemma 1)} \\ >& \alpha_j + d_{S(G,t+1)}(v_j(v_{j-1})^t, v_j(v_{j+2})^t) + 1 \\ &+ d_{S(G,t+1)}(v_{j+2}(v_j)^t, v_{j+2}(v_{j+3})^t) \\ \ge& d_{S(G,t+1)}(x^{t+1}, yw) \text{ (by triangle inequality),} \end{aligned}$$

which is a contradiction.

(c) $j = r-2$ and $r \ge 3$. Let

$$\alpha_j = d_{S(G,t+1)}((v_0)^{t+1}, v_0(v_1)^t) + \sum_{i=0}^{r-4} d_{S(G,t+1)}(v_{i+1}(v_i)^t, v_{i+1}(v_{i+2})^t) + r - 2.$$

Notice that $d_G(v_{r-3}, v_{r-1}) = 2, d_G(v_{r-2}, v_{r-1}) = d_G(v_{r-2}, v_r) = 1$ and

$d_G(v_{r-3}, v_r) \in \{1, 2\}$. In this case,

$$\begin{aligned}
d_{S(G,t+1)}(x^{t+1}, yw) =& \alpha_j + d_{S(G,t+1)}(v_{r-2}(v_{r-3})^t, v_{r-2}(v_{r-1})^t) \\
& + d_{S(G,t+1)}(v_{r-1}(v_{r-2})^t, v_{r-1}(v_r)^t) \\
& + d_{S(G,t+1)}(v_r(v_{r-1})^t, v_r w) + 2 \\
=& \alpha_j + d_{S(G,t)}((v_{r-3})^t, (v_{r-1})^t) + d_{S(G,t)}((v_{r-2})^t, (v_r)^t) \\
& + d_{S(G,t)}((v_{r-1})^t, w) + 2 \text{ (by Lemma 1)} \\
=& \alpha_j + 2(2^t - 1) + (2^t - 1) + d_{S(G,t)}((v_{r-1})^t, w) + 2 \\
& \text{(by assumption)} \\
\geq& \alpha_j + d_{S(G,t)}((v_{r-3})^t, (v_r)^t) + d_{S(G,t)}((v_{r-2})^t, (v_{r-1})^t) \\
& + d_{S(G,t)}((v_{r-1})^t, w) + 2 \text{ (by assumption)} \\
>& \alpha_j + d_{S(G,t)}((v_{r-3})^t, (v_r)^t) + d_{S(G,t)}((v_{r-2})^t, (v_{r-1})^t) \\
& + d_{S(G,t)}((v_{r-1})^t, w) + 1 \\
\geq& \alpha_j + d_{S(G,t)}((v_{r-3})^t, (v_r)^t) + d_{S(G,t)}((v_{r-2})^t, w) + 1 \\
& \text{(by triangle inequality)} \\
=& \alpha_j + d_{S(G,t+1)}(v_{r-2}(v_{r-3})^t, v_{r-2}(v_r)^t) + 1 \\
& + d_{S(G,t+1)}(v_r(v_{r-2})^t, v_r w) \text{ (by Lemma 1)} \\
\geq& d_{S(G,t+1)}(x^{t+1}, yw) \text{ (by triangle inequality)},
\end{aligned}$$

which is a contradiction.

Considering the previous cases, we deduce that the G-path associated to any shortest path between x^{t+1} and yw is triangle-free. Therefore, (i) holds.

Now, assume that $d_G(x, y) \geq 2$. If $x \notin N(v_2)$ and $x = v_0, v_1, v_2, \ldots, v_{r-1}, v_r = y$ is the G-path associated to a shortest path between xw and yw', then by analogy to the proof of (i), cases (b) and (c), we deduce that the above mentioned G-path is triangle-free. Therefore, (ii) holds. □

Theorem 4. *Let $G = (V, E)$ be a connected non-trivial graph. For any $x, y \in V$ and any integer $t \geq 1$,*

$$d_{S(G,t)}(x^t, y^t) = (2^t - 1)d_G(x, y).$$

Proof. We will proceed by induction on t. For $t = 1$, we have that $d_{S(G,1)}(x^1, y^1) = d_G(x, y) = (2^1 - 1)d_G(x, y)$. Suppose that $d_{S(G,t)}(x^t, y^t) = (2^t - 1)d_G(x, y)$ holds true for an integer $t \geq 1$ and any pair of vertices of G. We will show that $d_{S(G,t+1)}(x^{t+1}, y^{t+1}) = (2^{t+1} - 1)d_G(x, y)$.

Let P be a shortest path between x^{t+1}, y^{t+1} and let $x = v_0, v_1, v_2, \ldots, v_{r-1}, v_r =$

y be the G-path associated to P. So,

$$d_{S(G,t+1)}(x^{t+1},y^{t+1}) = d_{S(G,t+1)}(v_0^{t+1},v_0v_1^t)+\sum_{i=0}^{r-2} d_{S(G,t+1)}(v_{i+1}v_i^t,v_{i+1}v_{i+2}^t)+ \\ + d_{S(G,t+1)}(v_rv_{r-1}^t,v_r^{t+1})+r.$$

By hypothesis and Lemmas 1 and 3, we obtain

$$d_{S(G,t+1)}(x^{t+1},y^{t+1}) = 2^t-1+2(2^t-1)(r-1)+2^t-1+r.$$

Also, since $r \geq d_G(x,y)$ we have

$$d_{S(G,t+1)}(x^{t+1},y^{t+1}) \geq (2^{t+1}-1)d_G(x,y).$$

Now, let $x=u_0,u_1,\ldots,u_s=y$ be a shortest path between x and y. By Lemma 1 and induction hypothesis we have that

$d_{S(G,t+1)}(x^{t+1},x(u_1)^t)=2^t-1$, $d_{S(G,t+1)}(y(u_{s-1})^t,y^{t+1})=2^t-1$ and

$d_{S(G,t+1)}(u_{i+1}(u_i)^t,u_{i+1}(u_{i+2})^t)=2(2^t-1)$ for $i\in\{0,1,\ldots,s-2\}$.

Thus, since $u_i(u_{i+1})^t$ is adjacent to $u_{i+1}(u_i)^t$ for all $i\in\{0,1,\ldots,s-1\}$, we have

$$\begin{aligned} d_{S(G,t+1)}(x^{t+1},y^{t+1}) \leq & d_{S(G,t+1)}(x^{t+1},x(u_1)^t)+\sum_{i=0}^{s-2} d_{S(G,t+1)}(u_{i+1}(u_i)^t,u_{i+1}(u_{i+2})^t)+ \\ & + d_{S(G,t+1)}(y(u_{s-1})^t,y^{t+1})+s \\ = & 2^t-1+\sum_{i=0}^{s-2} 2\left(2^t-1\right)+2^t-1+s \\ = & 2^t-1+2\left(2^t-1\right)(d_G(x,y)-1)+2^t-1+d_G(x,y) \\ = & (2^{t+1}-1)d_G(x,y). \end{aligned}$$

Therefore, $d_{S(G,t+1)}(x^{t+1},y^{t+1})=(2^{t+1}-1)d_G(x,y)$. □

For any $w\in V^{t-1}$, the subgraph induced by $\{wx:\, x\in V\}$ is isomorphic to G and so $d_{S(G,t)}(x^t,x^{t-1}y)=d_G(x,y)$. Hence, as we will see in Theorem 5, we only study $d_{S(G,t)}(x^t,w)$ for the cases in which w is not of the form $x^{t-1}y$.

Given two vertices $x,y\in V$, we define $\mathcal{P}(x,y)$ as the set of all shortest paths between x and y. For any $P_i\in\mathcal{P}(x,y)$, the neighbour of y lying on P_i will be denoted by $y^{(i)}$. With this notation in mind we can state the following result.

Theorem 5. *Let $G=(V,E)$ be a connected non-trivial graph. For any integer $t\geq 2$ and any $x\in V$ and $w=x^{j-1}z_jz_{j+1}\cdots z_t\in V^t$ such that $1\leq j\leq t-1$ and $x\neq z_j$,*

$$d_{S(G,t)}(x^t,w)=\min_{P_i\in\mathcal{P}(x,z_j)}\left\{d_{S(G,t-j)}\left(\left(z_j^{(i)}\right)^{t-j},z_{j+1}\cdots z_t\right)\right\}+ \\ +(2^{t-j+1}-1)d_G(x,z_j)-(2^{t-j}-1).$$

Proof. By Lemma 1, we have that $d_{S(G,t)}(x^t,w) = d_{S(G,t-j+1)}(x^{t-j+1}, z_j \cdots z_t)$. Let $x = v_0, v_1, v_2, \ldots, v_{r-1}, v_r = z_j$ be a shortest path P between x and z_j. Let P_1 be a path of minimum length among all the paths from x^{t-j+1} to $z_j \cdots z_t$ having P as its associated path. So, the length of P_1 is given by

$$\begin{aligned} l(P_1) =& d_{S(G,t-j+1)}(x^{t-j+1}, x(v_1)^{t-j}) + \sum_{i=0}^{r-2} d_{S(G,t-j+1)}(v_{i+1}(v_i)^{t-j}, v_{i+1}(v_{i+2})^{t-j}) + \\ &+ r + d_{S(G,t-j+1)}(v_r(v_{r-1})^{t-j}, z_j \cdots z_t) \\ =& d_{S(G,t-j)}(x^{t-j}, v_1^{t-j}) + \sum_{i=0}^{r-2} d_{S(G,t-j)}((v_i)^{t-j}, (v_{i+2})^{t-j}) + r \\ &+ d_{S(G,t-j)}((v_{r-1})^{t-j}, z_{j+1} \cdots z_t) \text{ (by Lemma 1)} \\ =& (2^{t-j}-1) + 2(2^{t-j}-1)(r-1) + r + d_{S(G,t-j)}((v_{r-1})^{t-j}, z_{j+1} \cdots z_t) \\ & \text{(by Theorem 4).} \end{aligned} \tag{2}$$

Now, let P_2 be a shortest path between x^{t-j+1} and $z_j \cdots z_t$ and let P_2' be the G-path associated to P_2. By Lemma 3 and Theorem 4, we learned that P_2' is triangle-free. Suppose that P_2' given by $x = u_0, u_1, u_2, \ldots, u_{s-1}, u_s = z_j$ has length $s > d_G(x, z_j)$. Analogously to the way in which we obtained the length of P_1 in (2), we deduce that the length of P_2 is given by

$$l(P_2) = (2^{t-j}-1) + 2(2^{t-j}-1)(s-1) + s + d_{S(G,t-j)}((u_{s-1})^{t-j}, z_{j+1} \cdots z_t).$$

Hence,

$$\begin{aligned} l(P_2) >& (2^{t-j}-1) + 2(2^{t-j}-1)r + r + d_{S(G,t-j)}((u_{s-1})^{t-j}, z_{j+1} \cdots z_t) \text{ (as } s \geq r+1) \\ =& (2^{t-j}-1) + 2(2^{t-j}-1)(r-1) + 2(2^{t-j}-1) + r + \\ &+ d_{S(G,t-j)}((u_{s-1})^{t-j}, z_{j+1} \cdots z_t) \\ \geq& (2^{t-j}-1) + 2(2^{t-j}-1)(r-1) + r + d_{S(G,t-j)}((v_{r-1})^{t-j}, (u_{s-1})^{t-j}) \\ &+ d_{S(G,t-j)}((u_{s-1})^{t-j}, z_{j+1} \cdots z_t) \text{ (by Theorem 4 and } u_{s-1}, v_{r-1} \in N(z_j)) \\ \geq& (2^{t-j}-1) + 2(2^{t-j}-1)(r-1) + r + d_{S(G,t-j)}((v_{r-1})^{t-j}, z_{j+1} \cdots z_t) \\ & \text{(by triangle inequality)} \\ =& l(P_1), \end{aligned}$$

which is a contradiction. Therefore, the G-path associated to any shortest path between x^{t-j+1} and $z_j \cdots z_t$ is a shortest path between x and z_j, so that (2) leads to the result. □

We can use Theorem 5 as a tool to prove the following known result.

Corollary 6. [**10**] *Let $t \geq 1$ and $n \geq 2$ be integers, let $K_n = (V,E)$ be a complete graph, $x \in V$ and and $w = z_1 z_2 \cdots z_t \in V^t$. Then*

$$d_{S(K_n,t)}(x^t, w) = \sum_{z_i \neq x} 2^{t-i}.$$

Theorem 5 leads to Algorithm 1 which allows us to compute the distance between an extreme vertex and any vertex of $S(G,t)$. In this algorithm we are using two functions, $dist(x,y)$ and $dist^*(x,y)$. The first one gives the distance between x and y and the second one gives the same distance and stores in $\nu(x,y)$ the set of neighbours of y lying on the corresponding shortest paths. For instance, if we compute the distances by using Dijkstra's algorithm, then Algorithm 1 has time complexity of $O(t|V|^2)$, while Dijkstra's algorithm for $S(G,t)$ has time complexity of $O(|V|^{2t})$.

Algorithm 1

Input: A connected graph G, a vertex x and a word $w = z_1 z_2 \cdots z_t$.
Output: $d_{S(G,t)}(x^t, w)$
function RecursiveExtreme(x, w)
  $j \leftarrow 1$
  while $j < t$ **and** $x = z_j$ **do**
    $j \leftarrow j + 1$
  end while
  if $j = t$ **then**
    $d_{S(G,t)}(x^t, w) \leftarrow dist(x, z_t)$
  else
    $m \leftarrow +\infty$
    $(d_G(x, z_j), \nu(x, z_j)) \leftarrow dist^*(x, z_j)$
    for each $v \in \nu(x, z_j)$ **do**
      $m \leftarrow \min\{m, \text{RecursiveExtreme}(v, z_{j+1} \cdots z_t)\}$
    end for
    $d_{S(G,t)}(x^t, w) \leftarrow m + (2^{t-j+1} - 1)d_G(x, z_j) - (2^{t-j} - 1)$
  end if
end function

The result exposed in Lemma 3 cannot be generalized to the case of two non-extreme vertices. For instance, for the graph $S(G,3)$ shown in Figure 1 the G-path $1, 2, 4, 3$ associated to the shortest path between 121 and 344 contains two triangles. Our next result concerns triangle-free G-paths.

Theorem 7. *Let $G = (V, E)$ be a connected non-trivial graph, $t \geq 2$ an integer, $x, y, z \in V$ and $w, w' \in V^t$ such that $w = z^{j-1} x x_{j+1} \cdots x_t$, $w' = z^{j-1} y y_{j+1} \cdots y_t$, $1 \leq j \leq t - 1$ and $x \neq y$. If (a) x does not belong to any cycle or (b) any path $x, u_1, \ldots, u_{s-1}, y$ of length $s \geq d_G(x,y) + 2$ satisfies that $x \notin N_G(u_2)$ (or $y \notin N_G(u_{s-2})$), then*

$$d_{S(G,t)}(w, w') = \lambda(x,y) + (2^{t-j+1} - 1)d_G(x,y) - 2(2^{t-j} - 1),$$

where

$$\lambda(x,y) = \min_{P_i\in\mathcal{P}(x,y)}\left\{d_{S(G,t-j)}\left(\left(x^{(i)}\right)^{t-j}, x_{j+1}\cdots x_t\right) + \right.$$
$$\left. + \, d_{S(G,t-j)}\left(\left(y^{(i)}\right)^{t-j}, y_{j+1}\cdots y_t\right)\right\}$$

Proof. By Lemma 1, we have that $d_{S(G,t)}(w,w') = d_{S(G,t-j+1)}(xx_{j+1}\cdots x_t, yy_{j+1}\cdots y_t)$. Let $x = v_0, v_1, v_2, \ldots, v_{r-1}, v_r = y$ be a shortest path P between x and y in G. Let P_1 be a path of minimum length among all the paths from $xx_{j+1}\cdots x_t$ to $yy_{j+1}\cdots y_t$ having P as its associated path. So, the length of P_1 is given by

$$\begin{aligned}
l(P_1) =& d_{S(G,t-j+1)}(xx_{j+1}\cdots x_t, x(v_1)^{t-j}) + \sum_{i=0}^{r-2} d_{S(G,t-j+1)}(v_{i+1}(v_i)^{t-j}, v_{i+1}(v_{i+2})^{t-j}) \\
&+ r + d_{S(G,t-j+1)}(v_r(v_{r-1})^{t-j}, yy_{j+1}\cdots y_t) \\
=& d_{S(G,t-j)}(x_{j+1}\cdots x_t, (v_1)^{t-j}) + \sum_{i=0}^{r-2} d_{S(G,t-j)}((v_i)^{t-j}, (v_{i+2})^{t-j}) + r + \\
&+ d_{S(G,t-j)}((v_{r-1})^{t-j}, y_{j+1}\cdots y_t) \text{ (by Lemma 1)} \\
=& d_{S(G,t-j)}(x_{j+1}\cdots x_t, (v_1)^{t-j}) + 2(2^{t-j}-1)(r-1) + r \\
&+ d_{S(G,t-j)}((v_{r-1})^{t-j}, y_{j+1}\cdots y_t) \text{ (by Theorem 4).}
\end{aligned} \tag{3}$$

Let P_2 be a shortest path between $xx_{j+1}\cdots x_t$ and $yy_{j+1}\cdots y_t$, and let $x = u_0, u_1, u_2, \ldots, u_{s-1}, u_s = y$ be the G-path P_2' associated to P_2. Suppose that $s > d_G(x,y)$.

We first assume premiss (a). In this case $u_1 = v_1$ and $s \geq r+1$, and by Lemma 3 and Theorem 4 we can conclude that P_2' is triangle-free. Following a procedure analogous to that described in (3), we deduce that the length of P_2 is given by

$$\begin{aligned}
l(P_2) =& d_{S(G,t-j)}(x_{j+1}\cdots x_t, (v_1)^{t-j}) + 2(2^{t-j}-1)(s-1) + s \\
&+ d_{S(G,t-j)}((u_{s-1})^{t-j}, y_{j+1}\cdots y_t).
\end{aligned}$$

Hence,

$$\begin{aligned}
l(P_2) >& d_{S(G,t-j)}(x_{j+1}\cdots x_t,(v_1)^{t-j})+2(2^{t-j}-1)r+r\\
&+d_{S(G,t-j)}((u_{s-1})^{t-j},y_{j+1}\cdots y_t)\ (\text{as } s\geq r+1)\\
=& d_{S(G,t-j)}(x_{j+1}\cdots x_t,(v_1)^{t-j})+2(2^{t-j}-1)(r-1)+r\\
&+2(2^{t-j}-1)+d_{S(G,t-j)}((u_{s-1})^{t-j},y_{j+1}\cdots y_t)\\
\geq& d_{S(G,t-j)}(x_{j+1}\cdots x_t,(v_1)^{t-j})+2(2^{t-j}-1)(r-1)+r\\
&+d_{S(G,t-j)}((v_{r-1})^{t-j},(u_{s-1})^{t-j})+d_{S(G,t-j)}((u_{s-1})^{t-j},y_{j+1}\cdots y_t)\\
&(\text{by Theorem 4 and the fact that } u_{s-1},v_{r-1}\in N(y))\\
\geq& d_{S(G,t-j)}((x_{j+1}\cdots x_t,(v_1)^{t-j})+2(2^{t-j}-1)(r-1)+r\\
&+d_{S(G,t-j)}((v_{r-1})^{t-j},y_{j+1}\cdots y_t)\ (\text{by triangle inequality})\\
=& l(P_1),
\end{aligned}$$

which is a contradiction.

Finally, assume premiss (b). In this case, $s\geq r+2$ and by Lemma 3 and Theorem 4 we can conclude that P_2' is triangle-free. Following a procedure analogous to that described in (3), we deduce that the length of P_2 is given by

$$\begin{aligned}
l(P_2) =& d_{S(G,t-j)}(x_{j+1}\cdots x_t,(u_1)^{t-j})+2(2^{t-j}-1)(s-1)+s\\
&+d_{S(G,t-j)}((u_{s-1})^{t-j},y_{j+1}\cdots y_t).
\end{aligned}$$

Hence,

$$\begin{aligned}
l(P_2) >& d_{S(G,t-j)}(x_{j+1}\cdots x_t,(u_1)^{t-j})+2(2^{t-j}-1)(r+1)+r\\
&+d_{S(G,t-j)}((u_{s-1})^{t-j},y_{j+1}\cdots y_t)\ (\text{as } s\geq r+2)\\
=& d_{S(G,t-j)}(x_{j+1}\cdots x_t,(u_1)^{t-j})+2(2^{t-j}-1)+2(2^{t-j}-1)(r-1)+r\\
&+2(2^{t-j}-1)+d_{S(G,t-j)}((u_{s-1})^{t-j},y_{j+1}\cdots y_t)\\
\geq& d_{S(G,t-j)}(x_{j+1}\cdots x_t,(u_1)^{t-j})+d_{S(G,t-j)}((u_1)^{t-j},(v_1)^{t-j})+2(2^{t-j}-1)(r-1)\\
&+r+d_{S(G,t-j)}((v_{r-1})^{t-j},(u_{s-1})^{t-j})+d_{S(G,t-j)}((u_{s-1})^{t-j},y_{j+1}\cdots y_t)\\
&(\text{by Theorem 4 and the fact that } u_1,v_1\in N(x) \text{ and } u_{s-1},v_{r-1}\in N(y))\\
\geq& d_{S(G,t-j)}((x_{j+1}\cdots x_t,(v_1)^{t-j})+2(2^{t-j}-1)(r-1)+r\\
&+d_{S(G,t-j)}((v_{r-1})^{t-j},y_{j+1}\cdots y_t)\ (\text{by triangle inequality})\\
=& l(P_1),
\end{aligned}$$

which is a contradiction.

Therefore, the G-path associated to any shortest path between $xx_{j+1}\cdots x_t$ and $yy_{j+1}\cdots y_t$ is a shortest path between x and y, so that (3) leads to the result. □

From Theorem 7 we can state the formula for the distance between vertices in $S(G,t)$ for any bipartite graph G.

Corollary 8. *Let $G=(V,E)$ be a connected bipartite graph, $t\geq 2$ an integer, $x,y,z\in V$ and $w,w'\in V^t$ such that $w=z^{j-1}xx_{j+1}\cdots x_t$, $w'=z^{j-1}yy_{j+1}\cdots y_t$, $1\leq j\leq t-1$ and $x\neq y$. Then*

$$d_{S(G,t)}(w,w')=\lambda(x,y)+(2^{t-j+1}-1)d_G(x,y)-2(2^{t-j}-1),$$

where

$$\lambda(x,y)=\min_{P_i\in\mathcal{P}(x,y)}\left\{d_{S(G,t-j)}\left(\left(x^{(i)}\right)^{t-j},x_{j+1}\cdots x_t\right)+\right.$$
$$\left.+\,d_{S(G,t-j)}\left(\left(y^{(i)}\right)^{t-j},y_{j+1}\cdots y_t\right)\right\}$$

For instance, for the cycle graph $C_4=(V,E)$ whose vertex set is $V=\{a,b,c,d\}$ and edge set is $E=\{\{a,b\},\{b,c\},\{c,d\},\{d,a\}\}$, Corollary 8 leads to $d_{S(C_4,3)}(dab,bdc)=13$. In this case, $t=3$, $j=1$, $x=d$, $y=b$ and

$$\begin{aligned}\lambda(d,b)&=\min\{d_{S(C_4,2)}(aa,ab)+d_{S(C_4,2)}(aa,dc),d_{S(C_4,2)}(cc,ab)+d_{S(C_4,2)}(cc,dc)\}\\&=\min\{1+4,5+2\}\\&=5.\end{aligned}$$

Analogously, $d_{S(C_4,3)}(dab,cad)=8$, as $t=3$, $j=1$, $x=d$, $y=c$ and $\lambda(d,c)=d_{S(C_4,2)}(cc,ab)+d_{S(C_4,2)}(dd,ad)=5+2=7$.

Given two vertices $x,y\in V$, we define $\mathcal{P}'(x,y)$ as the set of all paths between x and y of length $d_G(x,y)+1$. For any $P_k\in\mathcal{P}'(x,y)$ between x and y, the neighbour of y lying on P_k will be denoted by $y^{(k)}$. With this notation in mind we proceed to state the following result which can be deduced by analogy to the proof of Theorem 7.

Theorem 9. *Let $G=(V,E)$ be a connected non-trivial triangle-free graph, $t\geq 2$ an integer, $x,y,z\in V$ and $w,w'\in V^t$ such that $w=z^{j-1}xx_{j+1}\cdots x_t$, $w'=z^{j-1}yy_{j+1}\cdots y_t$, $1\leq j\leq t-1$ and $x\neq y$. Then*

$$d_{S(G,t)}(w,w')=\min\{\vartheta(x,y),\vartheta'(x,y)\},$$

where

$$\begin{aligned}\vartheta(x,y)=&\lambda(x,y)+\left(2^{t-j+1}-1\right)d_G(x,y)-2(2^{t-j}-1),\\ \vartheta'(x,y)=&\lambda'(x,y)+\left(2^{t-j+1}-1\right)d_G(x,y)+1,\\ \lambda(x,y)=&\min_{P_i\in\mathcal{P}(x,y)}\left\{d_{S(G,t-j)}\left(\left(x^{(i)}\right)^{t-j},x_{j+1}\cdots x_t\right)+\right.\\&\left.+\,d_{S(G,t-j)}\left(\left(y^{(i)}\right)^{t-j},y_{j+1}\cdots y_t\right)\right\},\\ \lambda'(x,y)=&\min_{P_k\in\mathcal{P}'(x,y)}\left\{d_{S(G,t-j)}\left(\left(x^{(k)}\right)^{t-j},x_{j+1}\cdots x_t\right)+\right.\\&\left.+\,d_{S(G,t-j)}\left(\left(y^{(k)}\right)^{t-j},y_{j+1}\cdots y_t\right)\right\}.\end{aligned}$$

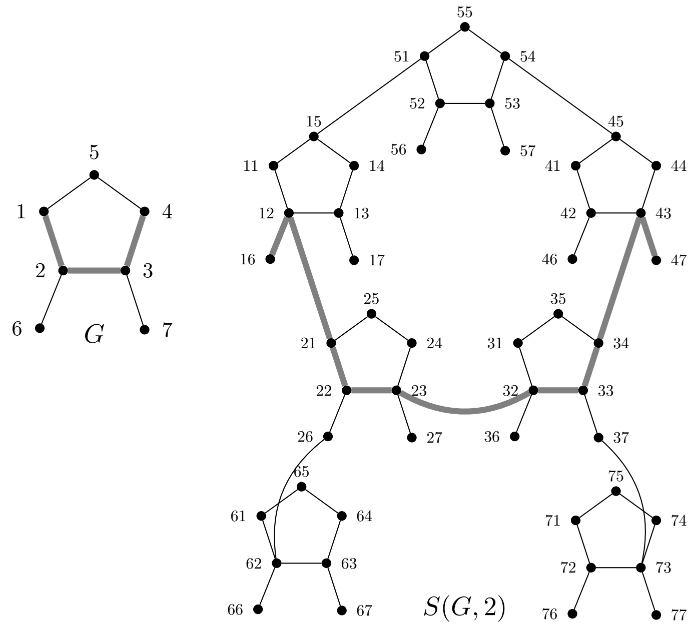

Figure 2: The G-path $1, 2, 3, 4$ associated to the shortest path $16, 12, 21, 22, 23, 32, 33, 34, 43, 47$ between 16 and 47, is not a shortest path between the vertices 1 and 4.

Proof. By Lemma 1, we have that $d_{S(G,t)}(w, w') = d_{S(G,t-j+1)}(xx_{j+1}\cdots x_t, yy_{j+1}\cdots y_t)$. Let $x = v_0, v_1, v_2, \ldots, v_{r-1}, v_r = y$ be a shortest path P between x and y in G. Let P_1 be a path of minimum length among all the paths from $xx_{j+1}\cdots x_t$ to $yy_{j+1}\cdots y_t$ having P as its associated path. Following a procedure analogous to that described in the proof of Theorem 7 we deduce that the length of P_1 is given by

$$\begin{aligned}l(P_1) =& d_{S(G,t-j)}(x_{j+1}\cdots x_t, (v_1)^{t-j}) + 2(2^{t-j} - 1)(r - 1) + r+ \\ &+ d_{S(G,t-j)}((v_{r-1})^{t-j}, y_{j+1}\cdots y_t).\end{aligned}$$

Let P_2 be a shortest path between $xx_{j+1}\cdots x_t$ and $yy_{j+1}\cdots y_t$, and let $x = u_0, u_1, u_2, \ldots, u_{s-1}, u_s = y$ be the G-path P_2' associated to P_2. Following a procedure analogous to that described in the proof of Theorem 7 we deduce that the length of P_2 is given by

$$\begin{aligned}l(P_2) =& d_{S(G,t-j)}(x_{j+1}\cdots x_t, (u_1)^{t-j}) + 2(2^{t-j} - 1)(s - 1) + s+ \\ &+ d_{S(G,t-j)}((u_{s-1})^{t-j}, y_{j+1}\cdots y_t).\end{aligned}$$

Now, since G is a triangle-free graph, if $s \geq d_G(x, y) + 2$, then we can follow a procedure analogous to that described in the proof of Theorem 7 (when we used premiss (b)) to conclude that $l(P_2) > l(P_1)$, which is a contradiction. Hence, the

G-path associated to any shortest path between $xx_{j+1}\cdots x_t$ and $yy_{j+1}\cdots y_t$ is a shortest path between x and y or it has length $d(x,y)+1$. Therefore, the result follows. □

Figure 2 shows an example where the path associated to the shortest path between the vertices 16 and 47 of $S(G,2)$ is not a shortest path between 1 and 4. According to Theorem 9 we have that $d_{S(G,2)}(16,47)=9=\vartheta'(1,4)$, as $\lambda'(1,4)=2$ while $\lambda(1,4)=6$.

Theorem 9 leads to Algorithm 2 which allows us to compute the distance between two arbitrary vertices of $S(G,t)$ for any connected triangle-free graph G. In this algorithm we are using a function, $dist^{**}(x,y)$ which gives the distance between x and y and stores in $\varphi(x,y)$ the set of pairs (x',y') such that x' and y' are, respectively, neighbours of x and y lying on the shortest paths between them, and stores in $\varphi'(x,y)$ the set of pairs (x'',y'') such that x'' and y'' are, respectively, neighbours of x and y lying on the paths of length $d_G(x,y)+1$.

Algorithm 2

Input: A connected triangle-free graph G and two words $w=x_1x_2\cdots x_t$ and $w=y_1y_2\cdots y_t$.
Output: $d_{S(G,t)}(w,w')$
$j \leftarrow 1$
while $j<t$ **and** $x_j=y_j$ **do**
  $j \leftarrow j+1$
end while
if $j=t$ **then**
  $d_{S(G,t)}(w,w') \leftarrow dist(x_t,y_t)$
else
  $(d_G(x_j,y_j),\varphi(x_j,y_j),\varphi'(x_j,y_j)) \leftarrow dist^{**}(x_j,y_j)$
  $\lambda \leftarrow +\infty$
  for each $(u,v)\in\varphi(x_j,y_j)$ **do**
    $d_x \leftarrow$ RecursiveExtreme$(u,x_{j+1}\cdots x_t)$
    $d_y \leftarrow$ RecursiveExtreme$(v,y_{j+1}\cdots y_t)$
    $\lambda \leftarrow \min\{\lambda, d_x+d_y\}$
  end for
  $\lambda' \leftarrow +\infty$
  for each $(u,v)\in\varphi'(x_j,y_j)$ **do**
    $d_x \leftarrow$ RecursiveExtreme$(u,x_{j+1}\cdots x_t)$
    $d_y \leftarrow$ RecursiveExtreme$(v,y_{j+1}\cdots y_t)$
    $\lambda' \leftarrow \min\{\lambda', d_x+d_y\}$
  end for
  $\vartheta \leftarrow \lambda+(2^{t-j+1}-1)d_G(x,z_j)-2(2^{t-j}-1)$
  $\vartheta' \leftarrow \lambda'+(2^{t-j+1}-1)d_G(x,z_j)+1$
  $d_{S(G,t)}(w,w') \leftarrow \min\{\vartheta,\vartheta'\}$
end if

3. DISTANCES IN TREES

As shown in [**15**], for any tree T and any positive integer t the Sierpiński graph $S(T,t)$ is a tree. Thus, there exists only one path between two vertices of $S(T,t)$. Notice that Corollary 8 leads to the next remark.

Remark 10. *Let $T=(V,E)$ be a tree and $t\geq 2$ an integer. For any $w = z^{j-1}xx_{j+1}\cdots x_t$ and $w' = z^{j-1}yy_{j+1}\cdots y_t$, where $1\leq j\leq t-1$, $x,y,z\in V$, $x\neq y$, and $w,w'\in V^t$,*

$$\begin{aligned} d_{S(T,t)}(w,w') =& d_{S(T,t-j)}\left((x')^{t-j}, x_{j+1}\cdots x_t\right) + d_{S(T,t-j)}\left((y')^{t-j}, y_{j+1}\cdots y_t\right) \\ &+ (2^{t-j+1}-1)d_T(x,y) - 2(2^{t-j}-1), \end{aligned}$$

where x' and y' are the neighbours of x and y lying on the path between x and y, respectively.

The *eccentricity* $\epsilon(v)$ of a vertex v in a connected graph G is the maximum distance between v and any other vertex u of G. The *diameter* of G is defined as

$$D(G) = \max_{v\in V(G)}\{\epsilon(v)\},$$

and the *radius* of G is defined as

$$r(G) = \min_{v\in V(G)}\{\epsilon(v)\}.$$

For a vertex v, each vertex at distance $\epsilon(v)$ from v is an *eccentric vertex* for v. A *leaf* in a tree is a vertex of degree one, while a *support vertex* is a vertex adjacent to a leaf.

Remark 11. *Let u and v be two different vertices in a tree T. If v is an eccentric vertex for u, then v is a leaf and $\epsilon(v)=D(T)$.*

From now on we will assume that T has order $n\geq 3$, as $S(K_2,t)\cong P_{2^t}$.

Lemma 12. *Let $T=(V,E)$ be a tree of order greater than or equal to three, $u,v\in V$ and $t\geq 2$ an integer. Then the following statements hold.*

(i) *If $\epsilon(u)\geq\epsilon(v)$, then $\epsilon(u^t)\geq\epsilon(v^t)$.*

(ii) *$\epsilon(u^t)=\epsilon\left(z^{t-1}\right)+(2^t-1)\epsilon(u)-(2^{t-1}-1)$, where z is the support vertex of an eccentric vertex for u.*

Proof. Let $w=u^{i-1}xx_{i+1}\cdots x_t\in V^t$ and $w'=v^{j-1}yy_{j+1}\cdots y_t\in V^t$ such that $\epsilon(u^t)=d(u^t,w)$ and $\epsilon(v^t)=d(v^t,w')$. If $i>1$, then for any $u'\in N_T(u)$ we have that

$$\begin{aligned} d_{S(T,t)}(u^t, u'u^{i-2}xx_{i+1}\cdots x_t) >& d_{S(T,t)}(u^t, u(u')^{t-1}) + \\ &+ d_{S(T,t)}(u'u^{t-1}, u'u^{i-2}xx_{i+1}\cdots x_t) \\ >& d_{S(T,t)}(u'u^{t-1}, u'u^{i-2}xx_{i+1}\cdots x_t) \\ =& d_{S(T,t)}(u^t, w), \end{aligned}$$

which is a contradiction. Hence, $i = j = 1$ and by Remark 10 we have

$$d_{S(T,t)}(u^t, w) = d_{S(T,t-1)}(x' \cdots x', x_2 \cdots x_t) + (2^t - 1)d_T(u, x) - (2^{t-1} - 1) \tag{4}$$

and

$$d_{S(T,t)}(v^t, w') = d_{S(T,t-1)}(y' \cdots y', y_2 \cdots y_t) + (2^t - 1)d_T(v, y) - (2^{t-1} - 1), \tag{5}$$

where x' is the neighbour of x lying on the path between x and u and y' is the neighbour of y lying on the path between y and v. From now on we assume that $\epsilon(u) \geq \epsilon(v)$ and then we will show that $\epsilon(u^t) \geq \epsilon(v^t)$ by induction. Let $t = 2$. By (4) $d_{S(T,2)}(u^2, w) = d_T(x', x_2) + 3d_T(u, x) - 1 \leq (D(T) - 1) + 3\epsilon(u) - 1$ and the equality holds for x, x_2 satisfying $d_T(x, u) = \epsilon(u)$ and $d_T(x_2, x) = D(T)$. Hence,

$$\epsilon(u^2) = D(T) + 3\epsilon(u) - 2 \geq D(T) + 3\epsilon(v) - 2 = \epsilon(v^2).$$

Our hypothesis is that $\epsilon(u^t) \geq \epsilon(v^t)$. For $x \in V$ such that $d_T(x, u) = \epsilon(u)$, and taking x' as the neighbour of x lying on the path between x and u, we have $\epsilon(x') = D(T) - 1$ (by Remark 11). Hence, by hypothesis we have that $\epsilon((x')^t) \geq \epsilon((y')^t)$, for every internal vertex y'. Thus, (4) leads to

$$\epsilon(u^{t+1}) = \epsilon((x')^t) + (2^{t+1} - 1)\epsilon(u) - (2^t - 1) \tag{6}$$

and, analogously,

$$\epsilon(v^{t+1}) = \epsilon((y')^t) + (2^{t+1} - 1)\epsilon(v) - (2^t - 1), \tag{7}$$

which implies that $\epsilon(u^{t+1}) \geq \epsilon(v^{t+1})$. Therefore, (i) follows by induction and (ii) by (6). □

Theorem 13. *Let $T = (V, E)$ be a tree of order greater than or equal to three. Then for any $u \in V$ and any integer $t \geq 1$,*

$$\epsilon(u^t) = (2^t - 1)\epsilon(u) + (2^t - t - 1)(D(T) - 2).$$

Proof. We proceed by induction on t. The equality holds for $t = 1$. Suppose that $\epsilon(u^t) = (2^t - 1)\epsilon(u) + (2^t - t - 1)(D(T) - 2)$. Then we have

$$\begin{aligned}
\epsilon(u^{t+1}) =& \epsilon(v^t) + (2^{t+1} - 1)\epsilon(u) - (2^t - 1) \text{ (by Lemma 12 (ii))} \\
=& (2^t - 1)\epsilon(v) + (2^t - t - 1)(D(T) - 2) + (2^{t+1} - 1)\epsilon(u) - (2^t - 1) \\
& \text{(by hipothesis)} \\
=& (2^t - 1)(D(T) - 1) + (2^t - t - 1)(D(T) - 2) + (2^{t+1} - 1)\epsilon(u) - (2^t - 1) \\
& \text{(as } v \text{ is the support of a diametral vertex)} \\
=& (2^{t+1} - 1)\epsilon(u) + (2^{t+1} - (t + 1) - 1)(D(T) - 2).
\end{aligned}$$

Therefore, the result follows □

Theorem 14. *For any tree T of order greater than or equal to three and any positive integer t,*

$$D(S(T,t)) = (3 \cdot 2^t - 2t - 3)D(T) - 4(2^t - t - 1).$$

Proof. Let $u, u', v, v' \in V$ and $w_1, w_2 \in V^{t-1}$ such that $d_T(u,v) = D(T)$, u' and v' are the support vertices of u and v, respectively, $d_{S(T,t-1)}((u')^{t-1}, w_1) = \epsilon((u')^{t-1})$ and $d_{S(T,t-1)}((v')^{t-1}, w_2) = \epsilon((v')^{t-1})$. By Remark 10 and Lemma 12 (i) we have that for any $x, y \in V$ and $w, w' \in V^{t-1}$,

$$\begin{aligned} d_{S(T,t)}(uw_1, vw_2) &= \epsilon((u')^{t-1}) + \epsilon((v')^{t-1}) + (2^t - 1)D(T) - 2(2^{t-1} - 1) \\ &\geq d_{S(T,t)}(xw, yw'). \end{aligned}$$

Therefore, uw_1 and vw_2 are diametral vertices and so Theorem 13 leads to

$$D(S(T,t)) = d_{S(T,t)}(uw_1, vw_2) = (3 \cdot 2^t - 2t - 3)D(T) - 4(2^t - t - 1).$$

□

It is well known that if $D(T)$ is even, then $r(T) = \frac{1}{2}D(T)$, otherwise, $r(T) = \frac{1}{2}(D(T) + 1)$. Now, by Theorem 14 we have that $D(T)$ is even if and only if $D(S(T,t))$ is even, so that we deduce the following result.

Theorem 15. *For any tree T of order greater than or equal to three and any positive integer t,*

$$r(S(T,t)) = \begin{cases} \dfrac{1}{2}(3 \cdot 2^t - 2t - 3)D(T) - 2(2^t - t - 1), & \text{for } D(T) \text{ even;} \\ \\ \dfrac{1}{2}\left((3 \cdot 2^t - 2t - 3)D(T) - 2^{t+2} + 4t + 5\right), & \text{otherwise.} \end{cases}$$

4. CONCLUDING REMARKS

In this paper we have discussed the problem of finding formulas for the distance between two vertices of a generalized Sierpiński graph $S(G,t)$ in terms of the distance between vertices of the base graph G. We have solved the problem for the case of an arbitrary vertex and an extreme vertex of $S(G,t)$, and also for two arbitrary vertices of $S(G,t)$ when G is a triangle-free graph. From the recursive formulas proposed, we have devised algorithms to compute the distances. In addition, we have derived explicit formulas for the diameter and radius of $S(T,t)$, for every tree T.

The general problem of computing the distance between two non-extreme vertices of $S(G,t)$ remains open as long as the graph G has triangles.

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