

GENERALIZED COSECANT NUMBERS AND TRIGONOMETRIC INVERSE POWER SUMS

*Carlos M. da Fonseca**, *M. Lawrence Glasser*, *Victor Kowalenko*

The generalized cosecant numbers denoted here by $c_{\rho,k}$ represent the coefficients of the power series expansion or generating function of the fundamental function $x^\rho / \sin^\rho x$. In actual fact, these interesting numbers are polynomials in ρ of degree k , whose coefficients are only dependent upon k . In this paper we show how they emerge in the calculation of trigonometric inverse power sums. After introducing the generalized cosecant numbers we present a novel and elegant integral approach for computing the Gardner-Fisher trigonometric inverse power sum, which is given by

$$S_{v,2}(m) = \left(\frac{\pi}{2m}\right)^{2v} \sum_{k=1}^{m-1} \cos^{-2v} \left(\frac{k\pi}{2m}\right),$$

where m and v are positive integers. This method not only confirms the solutions obtained earlier by an empirical method, but it is also much more expedient from a computational point of view. By comparing the formulas from both methods, we derive several new and interesting number-theoretical results involving symmetric polynomials over the set of quadratic powers up to $(v-1)^2$ and the generalized cosecant numbers.

1. INTRODUCTION

Over the last half-century finite sums involving powers of trigonometric functions have fascinated mathematicians and physicists. In 1966 Quoniam [38] posed an open problem in which the following result was conjectured

$$\sum_{k=1}^{\lfloor n/2 \rfloor} 2^{2m} \cos^{2m} \left(\frac{k\pi}{n+1}\right) = (n+1) \binom{2m-1}{m-1} - 2^{2m-1},$$

*Corresponding author. Carlos M. da Fonseca

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where $m < n + 1$ and m and n are positive integers. A solution by Greening et al [21] appeared soon afterwards. Then Gardner [16] found that the finite sum of inverse powers of cosines

$$(1.1) \quad S_{v,2}(m) = \left(\frac{\pi}{2m}\right)^{2v} \sum_{k=1}^{m-1} \cos^{-2v} \left(\frac{k\pi}{2m}\right),$$

where both m and v are positive integers, appeared in the calculation of the v -th cumulant of a certain quadratic form in m independent standardized normal variates. Although he was able to show that

$$\lim_{m \rightarrow \infty} S_{v,2}(m) = \zeta(2v),$$

where ζ is the Riemann zeta function, he asked whether it was possible to obtain a “simpler” closed form expression for $S_{v,2}(m)$, for all m and v .

In solving this problem, Fisher [12, 24] observed that the above sum could also be written as

$$(1.2) \quad S_{v,2}(m) = \left(\frac{\pi}{2m}\right)^{2v} \sum_{k=1}^{m-1} \sin^{-2v} \left(\frac{k\pi}{2m}\right),$$

and then set about devising an ingenious generating function approach for the more general trigonometric power series given by

$$Q_{m,v}(\delta) = \sum_{k=1}^{m-1} \sin^{-2v} \left(\frac{k\pi + \delta}{2m}\right).$$

As a result, he found that

$$S_{1,2}(m) = \frac{\pi^2}{6} \left(1 - \frac{1}{m^2}\right) \quad \text{and} \quad S_{2,2}(2) = \frac{\pi^4}{90} \left(1 + \frac{5}{2m^2} - \frac{7}{2m^4}\right),$$

while for large m , he obtained

$$S_{v,2}(m) = \zeta(2v) + \frac{v}{12m^2} \zeta(2v - 2) + O(m^{-4}).$$

Although Gardner first brought $S_{v,2}(m)$ to the attention of mathematicians, Fisher’s solution is largely responsible for the continuing interest in the sum even though it is not complete. Therefore, we shall refer to this sum from here on as the Gardner-Fisher sum.

With the advent of string theory a surge in interest occurred in trigonometric power sums, probably due to their appearance in important physical applications, e.g., see Dowker [9–11] and Verlinde [44]. The latter article, in particular, resulted in a flurry of proofs by many authors as discussed in [47]. After this period of intense activity came studies of related sums such as Berndt and Yeap [1] on the

explicit evaluation and reciprocity theorems of specific trigonometric power sums and Cvijović and Srivastava [6, 7] on the Dowker and related sums. At the same time others as in [4, 5, 45], which were motivated purely by the intrinsic fascination of these sums, derived formulas where the summand is a power of the secant such as

$$\sum_{k=0}^{n-1} \sec^{2p} \left(\frac{k\pi}{n} \right) = n \sum_{k=1}^{2p-1} (-1)^{p+k} \binom{p-1+kn}{2p-1} \sum_{j=k}^{2p-1} \binom{2p}{j+1}.$$

It should be noted that when both sides of the above result are evaluated separately in Mathematica, it is found that they agree, although for large values of n and p one may encounter problems with complex infinities on the lhs as a result of the summand becoming too small.

More recently, Dowker [8] has concentrated only on sums involving powers of cosecant with a similar argument as the above sum because of their appearance in the Casimir effect, Dedekind sums and, of course, Verlinde's formula. Interestingly, in solving these sums Dowker adopts a method which first appeared in the 1864 paper of Jeffery [23]. Surprisingly, this method begins in a similar manner to the integral approach presented here, but ultimately they progress on different paths so much so that the final solutions are expressed in very different forms from one another. Nevertheless, as a result of [23], it is intriguing to discover that the interest in trigonometric power sums can be traced back to the mid-1800's. For an extensive survey on the vast amount of literature on this topic, the interested reader should consult the introduction of Berndt and Yeap [1] and the references cited therein. More recently, however, these sums have appeared in a study of the entanglement entropy in string theory [43], but there the argument inside the trigonometric powers are the fractions k/n multiplied by 2π rather than π .

It should be emphasized that by far, the most interesting trigonometric power sums are those with inverse powers of the sine or cosine including tangent and cotangent because the evaluation of these sums generally involves the zeta function directly or through related numbers such as the Bernoulli and Euler numbers. That is, whenever the power of the sine or cosine function in the summand is even and negative, the solution is frequently number-theoretical in nature [6, 7, 47], whereas if it is a basic trigonometric power sum, i.e., possessing only even positive powers of sine or cosine, then it tends to be combinatorial [14]. Moreover, the inverse power case can result in the development of new and interesting results in number theory, as will be observed in this work.

There are also new approaches to the analysis and estimation of finite sums involving powers of trigonometric functions that have appeared recently in the literature. See, for example, [32–34, 39].

In concluding this introduction it should be mentioned that such sums are not only affected by the power and type of trigonometric functions in the summand, but also by the different sequences of rationals multiplying π in the arguments of these functions. As a result, it becomes a formidable task to devise a master theorem that can cover all the different kinds of trigonometric sums with inverse

powers. Nevertheless, in Section 6 we shall outline how with the aid of the results for the Gardner-Fisher and untwisted Dowker sums in the earlier sections, one can determine the solutions to very intricate trigonometric inverse power sums. Consequently, the present paper represents a platform for the study of more sophisticated trigonometric inverse power sums in the future.

2. GENERALIZED COSECANT NUMBERS

Before we can study the Gardner-Fisher and untwisted Dowker sums, we need to provide a discussion of the generalized cosecant numbers, which first appeared in [28]. These interesting numbers emerge when it is realized that the Gardner-Fisher sum given here by (1.1) or (1.2) can also be expressed as

$$S_{v,2}(m) = \sum_{j=0}^{\infty} c_{2v,j} \left(\frac{\pi}{2m}\right)^{2j} \sum_{k=1}^{m-1} k^{2j-2v}.$$

In this equation that was first derived in [27], the coefficients $c_{\rho,k}$ represent the coefficients in the generating function of $x^\rho \operatorname{csc}^\rho x$, i.e.,

$$x^\rho \operatorname{csc}^\rho x \equiv \sum_{k=0}^{\infty} c_{\rho,k} x^{2k}.$$

They are also a generalization of the cosecant numbers c_k , which arise in numerous and important applications described in [28]. The latter set of numbers, which correspond to $\rho = 1$ in the above equation, are given by

$$(2.3) \quad c_k = \frac{(-1)^{k+1}}{(2k)!} (2^{2k} - 2) B_{2k} = 2 (1 - 2^{1-2k}) \frac{\zeta(2k)}{\pi^{2k}},$$

where B_{2k} represent the Bernoulli numbers. As a consequence, they yield positive fractions, which converge rapidly to zero as $k \rightarrow \infty$. On the other hand, the generalized cosecant numbers are found to be given by

$$(2.4) \quad c_{\rho,k} = (-1)^k \sum_{\substack{\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_k \\ \sum_{i=1}^k i \lambda_i = k}}^{k, [k/2], [k/3], \dots, 1} (-1)^{N_k(\lambda)} (\rho)_{N_k(\lambda)} \prod_{i=1}^k \left(\frac{1}{(2i+1)!}\right)^{\lambda_i} \frac{1}{\lambda_i!}.$$

In this formula λ denotes a partition summing to k , λ_i denotes the frequency or multiplicity a part i occurs in a partition, while $N_k(\lambda)$ represents the total number of parts in a partition, sometimes called the length of the partition. That is, $N_k(\lambda) = \sum_{i=1}^k \lambda_i$.

The above formula for $c_{\rho,k}$ has been derived by applying the partition method for a power series expansion to $x^\rho \operatorname{csc}^\rho x$. This method first appeared in [31]. Then,

λ	λ_1	λ_2	λ_3	λ_4	λ_5	$N_5(\lambda)$
{5}					1	1
{4, 1}	1			1		2
{3, 2}		1	1			2
{3, 1, 1}	2		1			3
{2, 2, 1}	1	2				3
{2, 1, 1, 1}	3	1				4
{1, 1, 1, 1, 1}	5					5

Table 2.1: Multiplicities of the partitions summing to 5 and the sums of the parts

it was developed further in a series of articles [26, 28–30], culminating in [25]. In particular, [25] not only provides the most comprehensive account of the method by including most of the existing material from the other articles, but also introduces more intricate applications and derivations. For example, general formulas for the three highest and three lowest order coefficients of the generalized cosecant numbers are derived for the first time in Chapter 2 of this book in addition to derivations of the generating functions for generalizations of the elliptic integrals.

To calculate the generalized cosecant numbers or $c_{\rho,k}$ via (2.4), we need to determine the specific contribution made by each integer partition summing to k . For example, if we wish to determine $c_{\rho,5}$, then we require all the contributions made by the seven partitions that sum to 5 or those listed in the first column of Table 2.1. Depending on the function that is being studied, each element or part in a partition is assigned a specific value. Specifically, in the case of $x^\rho / \sin^\rho x$ each part i is assigned a value of $(-1)^{i+1}/(2i+1)!$. Moreover, if the part i occurs λ_i times in a partition or possesses a multiplicity of λ_i , then we need to take the values to the λ_i -th power, i.e., $(-1)^{(i+1)\lambda_i}/((2i+1)!^{\lambda_i})$. Table 2.1 displays the values of all the multiplicities of the parts in all seven partitions summing to 5 with a blank space representing zero.

Associated with each partition is a multinomial factor that is determined by taking the factorial of the total number of parts in the partition $N_k(\lambda)$ and dividing by the factorials of all the multiplicities. E.g., for the partition $\{2, 1, 1, 1\}$ in Table 2.1, we have $N_5(\lambda) = 4$, $\lambda_1 = 3$ and $\lambda_2 = 1$ with all the other multiplicities equal to zero or vanishing. Hence the multinomial factor for this partition becomes $4!/(3! \cdot 1!) = 4$. When the function is accompanied by a power, another modification must be made. Each partition must be multiplied by Pochhammer factor of $\Gamma(N_k(\lambda) + \rho)/\Gamma(\rho)N_k(\lambda)!$ or $(\rho)_{N_k(\lambda)}/N_k(\lambda)!$. For $\rho = 1$, this simply yields unity and thus, the multinomial factor is unaffected. In this case the generalized cosecant numbers reduce to the cosecant numbers given by (2.3). Therefore, there are two methods of calculating the cosecant numbers, either by (2.3) or by placing $\rho = 1$ in (2.4).

In (2.4) the product is concerned with the evaluating the contribution made by each partition based on the values of the multiplicities, while the sum is concerned with all the partitions summing to k . Thus, the sum covers the range of

values for all the multiplicities. For example, λ_1 attains a maximum value of k , which corresponds to the partition with k ones, while λ_2 attains a maximum value of $[k/2]$, which corresponds to the partition with $[k/2]$ twos in it. Here, $[k/n]$ denotes the floor function or the largest integer less than or equal to k/n . Therefore for odd values, the partition with $[k/2]$ twos would also have $\lambda_1 = 1$. Hence we see that λ_i can only attain a maximum value of $[k/i]$, which becomes the upper limit for each multiplicity in (2.4). Furthermore, a valid partition must satisfy the constraint given by $\sum_{i=1}^k i\lambda_i = k$.

As an example, let us consider the evaluation of $c_{\rho,5}$. From Table 2.1 there are seven partitions whose contributions must be evaluated. By following the steps given above, we find that according to the order in which the partitions appear in Table 2.1, (2.4) yields

$$(2.5) \quad c_{\rho,5} = (\rho)_1 \frac{1}{1!} - \frac{(\rho)_2}{2!} \frac{2!}{1!1!} \frac{1}{3!9!} - \frac{(\rho)_2}{2!} \frac{2!}{1!1!} \frac{1}{5!7!} + \frac{(\rho)_3}{3!} \frac{3!}{1!2!} \frac{1}{3!^2 7!} \\ + \frac{(\rho)_3}{3!} \frac{3!}{1!2!} \frac{1}{5!^2 3!} - \frac{(\rho)_4}{4!} \frac{4!}{1!3!} \frac{1}{3!^3 5!} + \frac{(\rho)_5}{5!} \frac{5!}{5!} \frac{1}{3!^5} .$$

Consequently, we observe that $c_{\rho,5}$ is a fifth order polynomial in ρ . In fact, the highest order term comes from the partition with k ones in it, which produces the term with $(\rho)_k$. Hence the generalized cosecant numbers are polynomials of order k . Another interesting property is that all the contributions from partitions with the same number of parts possess the same sign, which oscillates in sign or toggles according to whether there is an even or odd number of parts. Moreover, (2.5) can be simplified further, whereupon one eventually obtains the result for $c_{\rho,5}$ in the sixth row below the headings in Table 2.2.

Aside from the trivial $k = 0$ value of unity, Table 2.2 displays the first fifteen generalized cosecant numbers, which have been obtained by determining the multiplicities of all the partitions summing to each order k and then evaluating the sum of their contributions following the steps outlined above. Beyond $k = 10$, the partition method for a power series expansion becomes laborious due to the exponential increase in the number of partitions. To circumvent this problem, a general computing methodology has been developed to determine higher order coefficients via the partition method for a power series expansion. This methodology, which is based on representing all the partitions summing to a specific order as a tree diagram and invoking the bivariate recursive central partition (BRCP) algorithm, is described in great detail in [25] and [26]. The expressions for the coefficients arising from this general theory can then be imported into a mathematical software package such as Mathematica [46] whereupon its symbolic routines can be used to generate the final values displayed in Table 2.2.

A particularly interesting property of the generalized cosecant numbers is that the power of the cosecant or ρ appears as the variable in the polynomials displayed in Table 2.2. This means that the coefficients are invariant irrespective of the value of ρ . Moreover, for the special case, where ρ is an even integer (the case of interest in this work), the generalized cosecant numbers satisfy the following

k	$c_{\rho,k}$
0	1
1	$\frac{1}{3!} \rho$
2	$\frac{2}{6!} (2\rho + 5\rho^2)$
3	$\frac{8}{9!} (16\rho + 42\rho^2 + 35\rho^3)$
4	$\frac{2}{3 \cdot 10!} (144\rho + 404\rho^2 + 420\rho^3 + 175\rho^4)$
5	$\frac{4}{3 \cdot 12!} (768\rho + 2288\rho^2 + 2684\rho^3 + 1540\rho^4 + 385\rho^5)$
6	$\frac{2}{9 \cdot 15!} (1061376\rho + 3327584\rho^2 + 4252248\rho^3 + 2862860\rho^4 + 1051050\rho^5 + 175175\rho^6)$
7	$\frac{1}{27 \cdot 15!} (552960\rho + 1810176\rho^2 + 2471456\rho^3 + 1849848\rho^4 + 820820\rho^5 + 210210\rho^6 + 25025\rho^7)$
8	$\frac{2}{45 \cdot 18!} (200005632\rho + 679395072\rho^2 + 978649472\rho^3 + 792548432\rho^4 + 397517120\rho^5 + 125925800\rho^6 + 23823800\rho^7 + 2127125\rho^8)$
9	$\frac{4}{81 \cdot 21!} (129369047040\rho + 453757851648\rho^2 + 683526873856\rho^3 + 589153364352\rho^4 + 323159810064\rho^5 + 117327450240\rho^6 + 27973905960\rho^7 + 4073869800\rho^8 + 282907625\rho^9)$
10	$\frac{2}{6075 \cdot 22!} (38930128699392\rho + 140441050828800\rho^2 + 219792161825280\rho^3 + 199416835425280\rho^4 + 117302530691808\rho^5 + 47005085727600\rho^6 + 12995644662000\rho^7 + 2422012593000\rho^8 + 280078548750\rho^9 + 15559919375\rho^{10})$
11	$\frac{8}{243 \cdot 25!} (494848416153600\rho + 1830317979303936\rho^2 + 2961137042841600\rho^3 + 2805729689044480\rho^4 + 1747214980192000\rho^5 + 755817391389984\rho^6 + 232489541684400\rho^7 + 50749166067600\rho^8 + 7607466867000\rho^9 + 715756291250\rho^{10} + 32534376875\rho^{11})$
12	$\frac{2}{2835 \cdot 27!} (1505662706987827200\rho + 5695207005856038912\rho^2 + 9487372599204065280\rho^3 + 9332354263294766080\rho^4 + 6096633539052376320\rho^5 + 2806128331871953088\rho^6 + 937291839756592320\rho^7 + 229239926321406000\rho^8 + 40598842049766000\rho^9 + 5005999501002500\rho^{10} + 390802935022500\rho^{11} + 14803141478125\rho^{12})$
13	$\frac{232}{81 \cdot 30!} (844922884529848320\rho + 3261358271400247296\rho^2 + 5576528334428209152\rho^3 + 5668465199488266240\rho^4 + 3858582205451484160\rho^5 + 1870620248833400064\rho^6 + 667822651436228288\rho^7 + 178292330746770240\rho^8 + 35600276746834800\rho^9 + 5225593531158000\rho^{10} + 539680243602500\rho^{11} + 35527539547500\rho^{12} + 1138703190625\rho^{13})$
14	$\frac{2}{1215 \cdot 30!} (138319015041155727360\rho + 543855095595477762048\rho^2 + 952027796641042464768\rho^3 + 996352286992030556160\rho^4 + 703040965960031795200\rho^5 + 356312537387839432192\rho^6 + 134466795172062184832\rho^7 + 38526945410311117760\rho^8 + 8436987713444690400\rho^9 + 1404048942958662000\rho^{10} + 173777038440005000\rho^{11} + 15258232341852500\rho^{12} + 858582205731250\rho^{13} + 23587423234375\rho^{14})$
15	$\frac{1088}{729 \cdot 35!} (562009739464769840087040\rho + 2247511941596311764074496\rho^2 + 4019108379306905439830016\rho^3 + 4317745925208072594259968\rho^4 + 3145163776677939429416960\rho^5 + 1656917203539032341530624\rho^6 + 655643919364420586023424\rho^7 + 199227919419039256217472\rho^8 + 46995751664475880185920\rho^9 + 8614026107092938211680\rho^{10} + 1214778349162323946000\rho^{11} + 128587452922193265000\rho^{12} + 9720180867524627500\rho^{13} + 472946705787806250\rho^{14} + 11260635852090625\rho^{15})$

Table 2.2: Generalized cosecant numbers $c_{\rho,k}$ up to $k = 15$

recurrence relation:

$$(2.6) \quad c_{2n+2,k+1} = \frac{(2k+2-2n)}{2n} \frac{(2k+1-2n)}{(2n+1)} c_{2n,k+1} + \frac{2n}{2n+1} c_{2n,k}.$$

This equation is obtained by introducing the power series expansion for $x^{2n}/\sin^{2n} x$ into (27) of [27], which is

$$(2.7) \quad \frac{d^2}{dx^2} \frac{1}{\sin^{2n} x} = \frac{2n}{\sin^{2n} x} + \frac{2n(2n+1) \cos^2 x}{\sin^{2n+2} x}.$$

Then one equates like powers of x . For the special case of $n=1$, the numbers are related to the cosecant-squared numbers discussed in [25] and [28]. As a consequence, we find that

$$(2.8) \quad c_{2,k} = 2(2k-1) \frac{\zeta(2k)}{\pi^{2k}}.$$

Furthermore, the k -dependence in (2.6) can be removed by deducing that the $c_{2n,k}$ can be expressed generally as

$$(2.9) \quad c_{2n,k} = 4 \sum_{j=0}^{n-1} \frac{\Gamma(k-j)}{\Gamma(k-n+1)} \frac{\Gamma(k-j+1/2)}{\Gamma(k-n+1/2)} \frac{1}{\Gamma(n)} \frac{\Gamma(j+1/2)}{\Gamma(n+1/2)} C(n,j) \frac{\zeta(2k-2j)}{\pi^{2k-2j}},$$

where

$$(2.10) \quad C(n,j) = C(n-1,j) + \frac{(n-1)^2}{j-1/2} C(n-1,j-1),$$

and $C(1,0) = 2$ from equating (2.9) with (2.8). For $n=1$, we find that $C(n,1) = 2 \sum_{l=1}^{n-1} l^2 = 4B_3(n)/3$, where $B_k(x)$ denotes a Bernoulli polynomial. Introducing this result into (2.10) yields $C(n,2)$, which is given by

$$(2.11) \quad C(n,2) = \frac{1}{135} n(n-1)(n-2)(2n-1)(2n-3)(5n+1) = \frac{1}{6} (2n-4)_4 c_{2n,2}.$$

Similarly, if one introduces (2.11) into (2.10), then one obtains

$$C(n,3) = \frac{1}{60} (2n-6)_6 c_{2n,3}.$$

Therefore, we see that the $C(n,j)$ are related to the generalized cosecant numbers. In the next section we shall also see that the generalized cosecant numbers are related to the symmetric polynomials over the set of positive square integers.

3. INTEGRAL APPROACH

With the aid of the material in the previous section we are now in a position to evaluate the Gardner-Fisher sum via a totally different approach from other workers who have studied similar trigonometric power sums, [1–7, 9–15, 17–19, 21, 24, 35]. The main advantage of the integral approach presented here is that it yields a final form for the sum that is far more expedient from a computational point of view than the cited references. Moreover, except for [27] it is both much more informative and compact than the other references as will be seen later. In addition, it verifies the results in [27], which have been determined empirically. As a consequence, comparing the results from the two approaches produces new and interesting number-theoretical results.

Theorem 3.1. *The general solution of the Gardner-Fisher sum, which is represented by either (1.1) or (1.2), can be expressed as*

$$(3.12) \quad S_{v,2}(m) = \frac{1}{(2v-1)!} \sum_{n=0}^{v-1} \left(\frac{\pi}{m}\right)^{2n} s(v,n) \Gamma(2v-2n) \zeta(2v-2n) \left(1 - \frac{1}{m^{2v-2n}}\right),$$

where $n \leq v-1$ and $s(v,n)$ represents the n -th elementary symmetric polynomial obtained by summing over the entire sequence of quadratic powers or squares of integers from 1^2 to $(v-1)^2$. In particular, the polynomials are given by

$$s(v,n) = \sum_{1 \leq i_1 < i_2 < \dots < i_n < v-1} x_{i_1} x_{i_2} \dots x_{i_n},$$

with $x_{i_1} < x_{i_2} < \dots < x_{i_n}$ and each x_{i_j} is equal to at least one value in the set $\{1, 2^2, 3^2, \dots, (v-1)^2\}$.

Proof. From No. 8.365(8) of [20] we have

$$\psi(1-x) = \psi(x) + \pi \cot(\pi x).$$

We should add at this stage that the above equation is the starting point for Dowker's evaluation of finite sums of powers of cosecant [8], which is in turn based on the method originated by Jeffery [23]. Thus, their method follows the first few steps in this proof, but after that we pursue a different path so that in the end our results are in a completely different and far more interesting format. Differentiating the above equation with respect to x yields

$$(3.13) \quad \pi^2 \csc^2(\pi x) = \psi'(x) + \psi'(1-x).$$

By introducing into this equation the series form for the derivative of the digamma function, i.e., $\psi'(x) = \sum_{n=0}^{\infty} 1/(n+x)^2$, which appears as No. 8.363(8) in [20], we can replace the summand in (3.13) by the integral representation for the gamma function. Hence we obtain

$$\pi^2 \csc^2(\pi x) = \sum_{n=0}^{\infty} \int_0^{\infty} t \left(e^{-(n+x)t} + e^{-(n+1-x)t} \right) dt.$$

Since x is less than unity in (3.13), the integral involving the second term is convergent for all values of x and n . Therefore, we can interchange the order of the summation over n and the integral over t , which yields

$$\pi^2 \csc^2(\pi x) = \int_0^\infty t \sum_{n=0}^\infty \left(e^{-(n+x)t} + e^{-(n+1-x)t} \right) dt .$$

Next we treat the two sums over n as the geometric series which are both absolutely convergent since e^{-t} is always less than unity over the entire range of integration. Because $\sum_{k=N}^\infty z^k = z^N/(1-z)$ for $|z| < 1$, we arrive at

$$\pi^2 \csc^2(\pi x) = \int_0^\infty t \left(\frac{e^{-xt}}{1-e^{-t}} + \frac{e^{-(1-x)t}}{1-e^{-t}} \right) dt .$$

Then we make the change of variable $y = e^{-t}$, thereby obtaining

$$(3.14) \quad \pi^2 \csc^2(\pi x) = - \int_0^1 \frac{\ln y}{1-y} \left(y^x + y^{1-x} \right) \frac{dy}{y} .$$

We now require the following identity:

$$\prod_{n=1}^{v-1} (\partial_z^2 + 4n^2) \csc^2 z = (2v-1)! \csc^{2v} z, \quad \text{for } v = 1, 2, 3, \dots$$

This result can be readily derived by: (1) replacing $\cos^2 x$ by $1 - \sin^2 x$ in (2.7), which gives

$$\left(\frac{d^2}{dx^2} + 4n^2 \right) \csc^{2n} x = 2n(2n+1) \csc^{2n+2} x ,$$

and (2) multiplying the above result by successive values of n . Moreover, if we make the elementary change of variable, $u = y^{1/2m}$, then (3.14) can be expressed as

$$(3.15) \quad \csc^{2v} \left(\frac{k\pi}{2m} \right) = - \frac{4^v m^2}{\pi^2 \Gamma(2v)} \int_0^1 \frac{\ln u}{1-u^{2m}} \prod_{n=1}^{v-1} \left(\frac{m^2}{\pi^2} \ln^2 u + n^2 \right) (u^k + u^{2m-k}) \frac{du}{u} .$$

Next we sum the lhs from $k = 1$ to $m - 1$, which gives the Gardner-Fisher sum or (1.2). Thus, we arrive at

$$(3.16) \quad \sum_{k=1}^{m-1} \csc^{2v} \left(\frac{k\pi}{2m} \right) = - \frac{4^v m^2}{\pi^2 \Gamma(2v)} \int_0^1 \prod_{n=1}^{v-1} \left(\frac{m^2}{\pi^2} \ln^2 u + n^2 \right) \frac{(1-u^{m-1})}{(1-u^m)(1-u)} du .$$

At this stage we introduce Newton's identities or the Newton-Girard formulas as they should be known, which results in a finite sum in powers of $m^2 \ln^2 u / \pi^2$. Consequently, (3.16) becomes

$$(3.17) \quad S_{v,2}(m) = - \frac{4^v}{\Gamma(2v)} \sum_{n=0}^{v-1} s(v,n) \left(\frac{\pi}{m} \right)^{2n-2v} I(m,v) ,$$

where

$$I(m, v) = \int_0^1 \frac{(1 - u^{m-1})}{(1 - u^m)(1 - u)} \ln^{2v-2n-1} u \, du,$$

and $s(v, n)$ represents the n -th elementary symmetric polynomial derived by summing quadratic powers of integers, viz. $1^2, 2^2, \dots, (v-1)^2$. That is,

$$s(v, n) = \sum_{1 \leq i_1 < i_2 < \dots < i_n < v-1} x_{i_1} x_{i_2} \cdots x_{i_n},$$

where $x_{i_1} < x_{i_2} < \dots < x_{i_n}$ and each x_{i_j} is equal to at least one value in the set $\{1, 2^2, 3^2, \dots, (v-1)^2\}$. Specifically, for the three lowest values of n , they are given by

$$s(v, 0) = 1, \quad s(v, 1) = (v-1)v(2v-1)/6,$$

and

$$s(v, 2) = \frac{(5v+1)}{4 \cdot 6!} (2v-4)_5,$$

while for the three highest values of n , they are given by

$$s(v, v-1) = (v-1)!^2, \quad s(v, v-2) = (v-1)!^2 (\zeta(2) - \zeta(2, v)),$$

and

$$s(v, v-3) = \frac{(v-1)!^2}{2} \left((\zeta(2) - \zeta(2, v))^2 + \zeta(4, v) - \zeta(4) \right).$$

Since these important polynomials have not appeared in the literature before to the best of our knowledge, we shall discuss the above results in more detail once the proof has been completed.

The integral $I(m, v)$ can be evaluated by first decomposing the denominator of the integrand, i.e.,

$$\frac{1 - u^{m-1}}{(1-u)(1-u^{2m})} = \frac{1}{1-u} - \frac{u^{m-1}}{1-u^m}.$$

Then we apply No. 4.271(4) from [20], which is

$$\int_0^1 \frac{\ln^{p-1} x}{1-x} dx = e^{i(p-1)\pi} \Gamma(p) \zeta(p), \quad p > 1.$$

Hence we find that

$$I(m, v) = \Gamma(2v-2n) \zeta(2v-2n) (m^{-2(v-n)} - 1).$$

Finally, by introducing the result for $I(m, v)$ into (3.17), we arrive at (3.12). This completes the proof of Theorem 3.1. \square

Before studying (3.12) in detail, let us turn our attention to the evaluation of the interesting symmetric polynomials $s(v, n)$. As stated in the proof, these polynomials arise by expanding the product in (3.16) in powers of $\ln u$ with the aid of Newton's identities. This means that the product is expressed as

$$(3.18) \quad \prod_{n=1}^{v-1} \left(\frac{m^2}{\pi^2} \ln^2 u + n^2 \right) = \prod_{i=1}^{v-1} i^2 + \frac{m^2}{\pi^2} \ln^2 u \sum_{j=1}^{v-1} \prod_{i=1}^{v-1} \frac{i^2}{j^2} + \dots + \left(\frac{m^2}{\pi^2} \ln^2 u \right)^{v-1} .$$

Consequently, a finite sum in powers of $m^2 \ln^2 u / \pi^2$ emerges with the powers ranging from zero to $v - 1$. The coefficients of $m^2 \ln^2 u / \pi^2$ raised to the k -th power represent the symmetric polynomials $s(v, v - 1 - k)$. Note also that the highest powers in the expansion of the product yield the lowest n values in $s(v, n)$ and vice versa. Furthermore, from (3.18) we observe that

$$s(v, v - 1) = \prod_{i=1}^{v-1} i^2 = ((v - 1)!)^2 = \Gamma(v)^2 ,$$

while $s(v, 0) = 1$. The situation, however, becomes more complicated as we move to the inner powers from these extreme results. For example, to determine the symmetric polynomial over the quadratic powers for the next highest power, we need to sum over squares of the positive integers ranging from unity to $v - 1$. That is, we find that

$$(3.19) \quad s(v, 1) = \sum_{n=1}^{v-1} n^2 = v(v - 1)(2v - 1)/6 ,$$

where we have used No. 4.1.1.8 in [37].

The coefficient of the next highest order term in the expansion of the product is determined by summing over all possible pairs of $\sum i^2 j^2$, where i runs over all the integers between unity and $v - 2$ and j runs over all the integers from $i + 1$ to $v - 1$. These limits ensure that not only do i and j not equal one another, there is also no repetition of any pairs of their values. Hence we arrive at

$$(3.20) \quad s(v, 2) = \sum_{i=1}^{v-2} \sum_{j=i+1}^{v-1} i^2 j^2 = \sum_{i=1}^{v-2} i^2 \left(\sum_{j=1}^{v-1} j^2 - \sum_{j=1}^i j^2 \right) .$$

The first sum on the rhs of (3.20) is virtually the square of the rhs of (3.19), while the second sum can be obtained from No. 4.1.1.12 in [37]. To evaluate $s(v, 2)$, we require the general result for sums of powers of integers, which also appears in Section 4.1.1 of [37] and is given by

$$(3.21) \quad \sum_{k=1}^n k^m = \frac{1}{m + 1} (B_{m+1}(n + 1) - B_{m+1}) ,$$

where $B_{m+1}(x)$ and B_{m+1} are respectively the Bernoulli polynomials and numbers of index $m + 1$. Since Bernoulli numbers with odd index vanish, (3.20) reduces to

$$(3.22) \quad s(v, 2) = \sum_{i=1}^{v-2} \sum_{j=i+1}^{v-1} i^2 j^2 = \frac{1}{9} B_3(v-1) B_3(v) - \frac{1}{3} \sum_{i=1}^{v-2} i^2 B_3(i+1) .$$

Now we introduce $B_3(x) = x^3 - 3x^2/2 + x/2$ into the above result. After some algebraic manipulation, (3.22) becomes

$$(3.23) \quad s(v, 2) = \frac{1}{4 \cdot 6!} 2v(2v-1)(2v-2)(2v-3)(2v-4)(5v+1) .$$

Introducing the Pochhammer notation into the above result yields the result given in the proof of Theorem 3.1.

The coefficient of the next highest order term in powers of $m^2 \ln^2 u/\pi^2$ arising from the product of (3.18) can be expressed as

$$(3.24) \quad s(v, 3) = \sum_{i_1=1}^{v-3} \sum_{i_2=i_1+1}^{v-2} \sum_{i_3=i_2+1}^{v-1} i_1^2 i_2^2 i_3^2 .$$

We can introduce (3.21) into the above result, but a better method of evaluating this nested sum is to implement it in Mathematica [46] by the following line of code:

```
Sv3[v_] := Sum[i^2 Sum[j^2 Sum[k^2, {k, j + 1, v - 1}], {j, i + 1, v - 2}], {i, 1, v - 3}]
```

Then with the aid of the Simplify and Factor routines, (3.24) yields

$$s(v, 3) = \frac{1}{2 \cdot 9!} (35v^2 + 21v + 4)(2v - 6)_7 .$$

Moreover, we see a pattern developing for the symmetric polynomials $s(v, n)$. The first sum ranges from unity to $v - n$, while for the other sums the bottom limits are the previous summation variable plus one and the upper limits increment from $v - n$ until the upper limit is $v - 1$ in the last sum. As a consequence, the symmetric polynomials $s(v, n)$ can be written generally as

$$(3.25) \quad s(v, n) = \sum_{i_1=1}^{v-n} \sum_{i_2=i_1+1}^{v-n+1} \sum_{i_3=i_2+1}^{v-n+2} \cdots \sum_{i_n=i_{n-1}+1}^{v-1} \prod_{j=1}^n i_j^2 .$$

By studying $s(v, 1)$, $s(v, 2)$ and $s(v, 3)$ more closely, we observe that the symmetric polynomials $s(v, n)$ are related to the generalized cosecant numbers in Table 2.2. For example, $s(v, 1) = 2^{-2} (2v-2)_2 c_{2v,1}$, while $s(v, 3) = 2^{-6} (2v-6)_6 c_{2v,3}$. Indeed, by encoding (3.25) it is found that for all the values in Table 2.2 the $s(v, n)$ are given by

$$(3.26) \quad s(v, n) = 2^{-2n} (2v - 2n)_{2n} c_{2v,n} .$$

The above result will be derived by another method shortly. Therefore, our result for the Gardner-Fisher sum or (3.12) reduces to

$$(3.27) \quad S_{v,2}(m) = \sum_{n=0}^{v-1} \left(\frac{\pi}{2m}\right)^{2n} c_{2v,n} \zeta(2v-2n) \left(1 - \frac{1}{m^{2v-2n}}\right).$$

Moreover, if we use No. 9.542 from [20], which states that

$$\zeta(2m) = \frac{2^{2m-1}}{(2m)!} \pi^{2m} |B_{2m}|,$$

then the above result can be expressed as

$$(3.28) \quad S_{v,2}(m) = \frac{(2\pi)^{2v}}{2} \sum_{n=0}^{v-1} \left(\frac{1}{4m}\right)^{2n} \frac{|B_{2v-2n}|}{(2v-2n)!} \left(1 - \frac{1}{m^{2v-2n}}\right) c_{2v,n}.$$

Let us now turn our attention to the symmetric polynomials obtained from the lower powers of $m^2 \ln^2 u / \pi^2$ in the product of (3.18). From the discussion below (3.18) the symmetric polynomial $s(v, v-2)$ represents the coefficient of $m^2 \ln^2 u / \pi^2$ and is given by

$$s(v, v-2) = \sum_{j=1}^{v-1} \prod_{i=1}^{v-1} i^2 / j^2 = \prod_{i=1}^{v-1} i^2 \sum_{j=1}^{v-1} j^{-2} = \Gamma(v)^2 \left(\frac{\pi^2}{6} - \zeta(2, v)\right),$$

where $\zeta(n, x)$ represents the Hurwitz zeta function. The symmetric polynomial $s(v, v-3)$ is the coefficient of the square of $m^2 \ln^2 u / \pi^2$ in the expanded product in (3.18). It possesses two summation indices in the denominator with one index always less than the other. Therefore, $s(v, v-3)$ becomes

$$s(v, v-3) = \sum_{\substack{j,k=1 \\ j < k}}^{v-1} \prod_{i=1}^{v-1} \frac{i^2}{j^2 k^2}.$$

The product is easily evaluated as it is simply the square of $\Gamma(v)$. The double summation is evaluated by adding the $j > k$ terms. Since this amounts to doubling the number of terms in the double summation, we must halve the double sum. Hence we obtain

$$s(v, v-3) = \frac{1}{2} \Gamma(v)^2 \sum_{\substack{j,k=1 \\ j \neq k}}^{v-1} \frac{1}{j^2 k^2}.$$

Next we remove the $j \neq k$ constraint by subtracting the $j = k$ terms in the double summation. Then we find that

$$\begin{aligned} s(v, v-3) &= \frac{1}{2} \Gamma(v)^2 \left(\left(\sum_{j=1}^{v-1} \frac{1}{j^2} \right)^2 - \sum_{j=1}^{v-1} \frac{1}{j^4} \right) \\ &= \frac{1}{2} \Gamma(v)^2 \left((\zeta(2) - \zeta(2, v))^2 - (\zeta(4) - \zeta(4, v)) \right), \end{aligned}$$

which is the result given in the proof of Theorem 3.1.

The next lowest order term in the expansion of the product given by (3.18) is the cubic power of $m^2 \ln^2 u / \pi^2$, whose coefficient is given by

$$s(v, v-4) = \sum_{\substack{j,k,l=1 \\ j < k < l}}^{v-1} \prod_{i=1}^{v-1} \frac{i^2}{j^2 k^2 l^2} .$$

As before the product over i yields $\Gamma(v)^2$. In this case, however, there are six possible combinations of the constraint, viz. $k < j < l$, $k < l < j$, $j < k < l$, $j < l < k$, $l < j < k$ and $l < k < j$. All these are equivalent to one another, which means that we can divide by 6 or 3! and replace the constraint by the condition that none of the indices is equal to one another, i.e., $i \neq j \neq k$. Thus, (3.29) can be expressed as

$$(3.29) \quad s(v, v-4) = \frac{1}{6} \Gamma(v)^2 \sum_{\substack{j,k,l=1 \\ j \neq k \neq l}}^{v-1} \frac{1}{j^2 k^2 l^2} .$$

In order to dispense with the condition that none of the summation indices is equal to each other, we need to remove all those cases where at least one index is equal to another. This is complicated by the fact that we need to consider all the cases where only one index is equal to another and where they are all equal to each other. Hence we obtain

$$(3.30) \quad \begin{aligned} \sum_{j \neq k \neq l} \frac{1}{j^2 k^2 l^2} &= \sum_{j,k,l} \frac{1}{j^2 k^2 l^2} - 3 \sum_{\substack{j,k \\ j \neq k}} \frac{1}{j^4 k^2} - \sum_j \frac{1}{j^6} \\ &= \sum_{j,k,l} \frac{1}{j^2 k^2 l^2} - 3 \sum_{j,k} \frac{1}{j^4 k^2} + 2 \sum_j \frac{1}{j^6} . \end{aligned}$$

Introducing (3.30) into (3.29) yields

$$s(v, v-4) = \frac{1}{6} \Gamma(v)^2 \left((\zeta(2) - \zeta(2, v))^3 - 3(\zeta(4) - \zeta(4, v)) (\zeta(2) - \zeta(2, v)) + 2(\zeta(6) - \zeta(6, v)) \right) .$$

Similarly, we can evaluate $s(v, v-5)$ by modifying (3.29), which becomes

$$s(v, v-5) = \frac{1}{4!} \Gamma(v)^2 \sum_{\substack{j,k,l,m=1 \\ j \neq k \neq l \neq m}}^{v-1} \frac{1}{j^2 k^2 l^2 m^2} .$$

Again, in order to dispense with the constraint that none of the indices is equal to one another, we need to remove all the cases where at least one index is equal to

another. Therefore, the equivalent of (3.30) is

$$(3.31) \quad \sum_{j \neq k \neq l \neq m} \frac{1}{j^2 k^2 l^2 m^2} = \sum_{j,k,l,m} \frac{1}{j^2 k^2 l^2 m^2} - 6 \sum_{j,k,l} \frac{1}{j^4 k^2 l^2} + 8 \sum_{j,k} \frac{1}{j^6 k^2} + 3 \sum_{j,k} \frac{1}{j^4 k^4} - 6 \sum_j \frac{1}{j^8}.$$

Then $s(v, v - 5)$ becomes

$$s(v, v - 5) = \frac{1}{24} \Gamma(v)^2 \left((\zeta(2) - \zeta(2, v))^4 - 6(\zeta(4) - \zeta(4, v))(\zeta(2) - \zeta(2, v))^2 + 8(\zeta(6) - \zeta(6, v))(\zeta(2) - \zeta(2, v)) + 3(\zeta(4) - \zeta(4, v))^2 - 6(\zeta(8) - \zeta(8, v)) \right).$$

By continuing with the above analysis we can derive a general formula for $s(v, v - n)$, which is

$$s(v, v - n) = \sum_{\substack{j_1, \dots, j_{n-1}=1 \\ j_1 < j_2, \dots < j_{n-1}}}^{v-1} \prod_{i=1}^{v-1} \prod_{k=1}^{n-1} \frac{i^2}{j_k^2} = \frac{\Gamma(v)^2}{(n-1)!} \sum_{\substack{j_1, \dots, j_{n-1}=1 \\ j_1 \neq j_2 \dots j_{n-2} \neq j_{n-1}}}^{v-1} \prod_{k=1}^{n-1} \frac{1}{j_k^2}.$$

In removing the condition that none of the summation indices is equal to one another, one ends up with a number of distinct multiple sums as exemplified by the rhs of (3.30). The number of these sums is dependent upon the total number of partitions summing to $n - 1$, which in the case of (3.30) is three since the number of partitions summing to 3 is 3. Moreover, the powers of the indices can be related to the parts in each partition. In fact, all one needs to do is divide the power of each index by 2 to determine the part in a partition. Thus, the first and second terms on the rhs of (3.30) correspond respectively to the partitions $\{1,1,1\}$ and $\{2,1\}$, while the final term corresponds to the single part partition $\{3\}$. Consequently, we have a means of determining the number of sums and the powers on the summation indices. The problem is now determining the factors multiplying each of these sums.

In order to determine the coefficients appearing in the sums such as (3.31), let us consider $s(v, v - n)$ where $n = 6$. Based on the preceding examples we now have a sum over five indices. We also know that there will be seven distinct sums on the rhs since the number of partitions summing to $n - 1$, in this case 5, is 7 as indicated in Section 2. In addition, the coefficients are negative or positive according to the number of parts in each partition. If n is even, then the sums corresponding to partitions with an even number of parts will be negative, while those corresponding to an odd number of parts will be positive. The reverse applies when n is odd. Hence we can treat the coefficients as signless. Thus, the sum in $s(v, v - 6)$ can be expressed as

$$(3.32) \quad \sum_{i \neq j \neq k \neq l \neq m} \frac{1}{i^2 j^2 k^2 l^2 m^2} = \sum_{i,j,k,l,m} \frac{1}{i^2 j^2 k^2 l^2 m^2} - A_1 \sum_{i,j,k,l} \frac{1}{i^4 j^2 k^2 l^2} + A_2 \sum_{i,j,k} \frac{1}{i^4 j^4 k^2} + A_3 \sum_{i,j,k} \frac{1}{i^6 j^2 k^2} - A_4 \sum_{i,j} \frac{1}{i^6 j^4} - A_5 \sum_{i,j} \frac{1}{i^8 j^2} + A_5 \sum_i \frac{1}{i^{10}}.$$

If we sum the signless forms of the coefficients in (3.30) and (3.31), then we obtain values of 6 and 24, respectively. These correspond to $(n-1)!$ since $(n-1)!$ is the number of permutations of $n-1$ objects relative to their natural order [2, 40]. Therefore, in (3.32), $1 + \sum_{i=1}^5 A_i = 5!$, while it is found that $\sum_{i=1}^5 (-1)^i A_i = 0$. More generally, we shall see that if the upper limit is replaced by k , then the sums will equal $k!$ and 0, respectively. Moreover, if one examines the last or the one-dimensional sum in the previous results, then one finds that its coefficient is equal to $(n-2)!$. Therefore, in (3.32), $A_5 = 4!$.

In actual fact, if we type the sequence

$$1, 1, 1, 1, 3, 2, 1, 6, 8, 3, 6, \dots$$

into the on-line encyclopedia of integer sequences, then it is found that the sequence yields the triangle of refined rencontres numbers where $T(n, k)$ is the number of permutations of n elements with cycle type/class k [36]. According to Corollary 12.1 in [2], the number of permutations that can be decomposed into k cycles equals the signless Stirling numbers of the first kind or $|s_1(n, k)|$. This means that if there is only one partition with k parts, then the coefficient of that sum will be $|s_1(n, k)|$. Unfortunately, there are only a few values where there is a single partition that has a fixed number of parts. These are: (1) the partitions possessing only ones, represented by the first sum of the rhs of (3.32), (2) the partitions with a single part, represented by the last sum on the rhs and (3) the partitions with one two and all ones, represented by the second sum on the rhs. We already know the coefficients of the sums corresponding to the first two partitions, viz. $|s_1(5, 5)|$ or unity and $|s_1(5, 1)|$, which equals $4!$. The coefficient of the sum for the remaining or third type of partition has a signless coefficient of $|s_1(5, 4)|$. Hence we find that $A_1 = 10$.

The remaining coefficients A_2, A_3, A_4 and A_5 are affected by the refinement process. This is because the sums with coefficients A_2 and A_3 correspond to partitions with three parts, while the sums with coefficients A_4 and A_5 correspond to partitions with two parts. In these cases $|s_1(5, 3)|$ yields the sum of A_2 and A_3 , while $|s_1(5, 2)|$ is equal to the sum of A_4 and A_5 . To evaluate them separately, we require cycle classes as described in [40]. A permutation with a cycle class (k_1, k_2, \dots, k_n) possesses k_1 unit cycles, k_2 2-cycles, etc. This is equivalent to partitions with k_1 ones, k_2 twos and so on up to k_n n 's. From Riordan [40] the number of permutations of n elements of class (k) denoted by $N_p(k_1, k_2, \dots, k_n)$ is given by

$$(3.33) \quad N_p(k_1, k_2, \dots, k_n) = \frac{n!}{\prod_{i=1}^n i^{k_i} k_i!}.$$

In terms of partitions, i corresponds to the part i in a partition, while k_i corresponds to its multiplicity, which was denoted by λ_i in the previous section. The above result means for the sum corresponding to the partition $\{2, 2, 1\}$, we have $n = 2 + 2 + 1 = 5$, $k_1 = 1$ and $k_2 = 2$. Hence from (3.33), $A_2 = 5!/2! \cdot 2^2 = 15$. For the sum corresponding to the partition $\{1, 1, 3\}$ we have $A_3 = 5!/3! = 20$. In addition, $A_2 + A_3 = 35$, which is equal to $|s_1(5, 3)|$. Similarly, for the sum corresponding to the

partition $\{2,3\}$, we obtain $A_4 = 5!/2 \cdot 3 = 20$, while for the sum corresponding to the partition $\{1,4\}$, we arrive at $A_5 = 5!/4 = 30$. Then we see that $A_4 + A_5 = |s_1(5, 2)|$. As a consequence, (3.32) becomes

$$(3.34) \quad \sum_{i \neq j \neq k \neq l \neq m} \frac{1}{i^2 j^2 k^2 l^2 m^2} = \sum_{i,j,k,l,m} \frac{1}{i^2 j^2 k^2 l^2 m^2} - 10 \sum_{i,j,k,l} \frac{1}{i^4 j^2 k^2 l^2} + 15 \sum_{i,j,k} \frac{1}{i^4 j^4 k^2} \\ + 20 \sum_{i,j,k} \frac{1}{i^6 j^2 k^2} - 20 \sum_{i,j} \frac{1}{i^6 j^4} - 30 \sum_{i,j} \frac{1}{i^8 j^2} + 24 \sum_i \frac{1}{i^{10}}.$$

Similarly, for the $n = 7$ case one obtains

$$(3.35) \quad \sum_{i \neq j \neq k \neq l \neq m \neq p} \frac{1}{i^2 j^2 k^2 l^2 m^2 p^2} = \sum_{i,j,k,l,m,p} \frac{1}{i^2 j^2 k^2 l^2 m^2 p^2} - 15 \sum_{i,j,k,l,m} \frac{1}{i^4 j^2 k^2 l^2 m^2} \\ + 45 \sum_{i,j,k,m} \frac{1}{i^4 j^4 k^2 m^2} + 40 \sum_{i,j,k,m} \frac{1}{i^6 j^2 k^2 m^2} - 120 \sum_{i,j,m} \frac{1}{i^6 j^4 m^2} - 90 \sum_{i,j,m} \frac{1}{i^8 j^2 m^2} \\ - 15 \sum_{i,j,m} \frac{1}{i^4 j^4 m^4} + 144 \sum_{i,j} \frac{1}{i^{10} j^2} + 90 \sum_{i,j} \frac{1}{i^8 j^4} + 40 \sum_{i,j} \frac{1}{i^6 j^6} - 120 \sum_i \frac{1}{i^{12}}.$$

Finally, multiplying (3.34) and (3.35) by $\Gamma(v)^2/5!$ and $\Gamma(v)^2/6!$ yields $s(v, v - 6)$ and $s(v, v - 7)$, respectively.

We can devise a general formula for these results, but before doing so, we need to represent each partition in descending order. This means that each partition λ summing to k is now written as $\{\sigma_1, \sigma_2, \dots, \sigma_{N_k(\lambda)}\}$, where each part σ_i is ordered according to $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \dots \geq \sigma_{N_k(\lambda)}$ as in Table 2.1. We can also replace the quadratic power or square on the lhs by an arbitrary power α . Therefore, (3.35) can be expressed more generally as

$$(3.36) \quad \sum_{k_1 \neq k_2 \neq \dots \neq k_{n-1}} \prod_{i=1}^{n-1} \frac{1}{k_i^\alpha} = (-1)^{\delta(n)} (n-1)! \sum_{\lambda} (-1)^{N_{n-1}(\lambda)} \\ \sum_{k_1, \dots, k_{N_{n-1}(\lambda)}} \prod_{i=1}^{n-1} \frac{1}{\lambda_i! i^{\lambda_i} k_i^{\alpha \sigma_i}},$$

where $\delta(n) = (1 + (-1)^n)/2$. The phase factor involving $\delta(n)$ has been introduced in order to ensure that the correct sign is obtained for each sum when n is either odd or even. Note that the sum over λ is carried out over all partitions summing to $n - 1$ or $p(n - 1)$, where $p(k)$ is the partition function in number theory. For $n = 6$ this means the sum is over all seven partitions summing to 5. The critical point is the σ_i appear as the power of k_i in the product on the rhs. Even though i ranges from 1 to $n - 1$, the σ_i vanish when $i > N_{n-1}(\lambda)$.

The preceding analysis leads to fascinating results concerning the Hurwitz zeta function and the generalized cosecant numbers, which can be obtained by introducing the results for $s(v, v - n)$ into (3.26), bearing in mind that the latter result will be derived more formally later as (3.44). For $n = 1$ and setting $\alpha = 2$, we find that

$$\frac{\Gamma(v)}{\Gamma(v + 1/2)} = \frac{2}{\sqrt{\pi}} c_{2v, v-1}.$$

This result is only valid for integer values of v greater than unity. Alternatively, it can be expressed as $B(v, 1/2) = 2 c_{2v, v-1}$, where $B(x, y)$ represents the beta function. In fact, by (1) introducing (3.36) into (3.26), (2) using the above result and (3) carrying out some algebra, one obtains

$$(-1)^{\delta(n)} \sum_{\lambda} (-1)^{N_{n-1}(\lambda)} \sum_{k_1, \dots, k_{N_{n-1}(\lambda)}=1}^{v-1} \prod_{i=1}^{n-1} \frac{1}{\lambda_i! i^{\lambda_i}} \frac{1}{k_i^{2\sigma_i}} = \frac{2^{2n-2}}{\Gamma(2n)} \frac{c_{2v, v-n}}{c_{2v, v-1}}.$$

For $n = 2$ this result yields

$$(3.37) \quad \sum_{k=1}^{v-1} \frac{1}{k^2} = \frac{\pi^2}{6} - \zeta(2, v) = \frac{2}{3} \frac{c_{2v, v-2}}{c_{2v, v-1}},$$

where (the integer) $v > 2$. Similarly, for $n = 3$ to 6, it is found that

$$\sum_{k=1}^{v-1} \frac{1}{k^4} = \frac{\pi^4}{90} - \zeta(4, v) = \frac{4}{9} \left(\frac{c_{2v, v-2}}{c_{2v, v-1}} \right)^2 - \frac{4}{15} \frac{c_{2v, v-3}}{c_{2v, v-1}},$$

$$\sum_{k=1}^{v-1} \frac{1}{k^6} = \frac{\pi^6}{945} - \zeta(6, v) = \frac{4}{105} \frac{c_{2v, v-4}}{c_{2v, v-1}} - \frac{4}{15} \frac{c_{2v, v-3}}{c_{2v, v-1}} \frac{c_{2v, v-2}}{c_{2v, v-1}} + \frac{8}{27} \left(\frac{c_{2v, v-2}}{c_{2v, v-1}} \right)^3,$$

$$\begin{aligned} \sum_{k=1}^{v-1} \frac{1}{k^8} &= \frac{\pi^8}{9450} - \zeta(8, v) = \frac{8}{14175} \left(350 \left(\frac{c_{2v, v-2}}{c_{2v, v-1}} \right)^4 - 420 \frac{c_{2v, v-3}}{c_{2v, v-1}} \left(\frac{c_{2v, v-2}}{c_{2v, v-1}} \right)^2 \right. \\ &\quad \left. + 63 \left(\frac{c_{2v, v-3}}{c_{2v, v-1}} \right)^2 + 60 \frac{c_{2v, v-4}}{c_{2v, v-1}} \frac{c_{2v, v-2}}{c_{2v, v-1}} - 5 \frac{c_{2v, v-5}}{c_{2v, v-1}} \right), \end{aligned}$$

and

$$(3.38) \quad \begin{aligned} \sum_{k=1}^{v-1} \frac{1}{k^{10}} &= \frac{\pi^{10}}{93555} - \zeta(10, v) = \frac{4}{93555} \left(3080 \left(\frac{c_{2v, v-2}}{c_{2v, v-1}} \right)^5 - 4620 \frac{c_{2v, v-3}}{c_{2v, v-1}} \left(\frac{c_{2v, v-2}}{c_{2v, v-1}} \right)^3 \right. \\ &\quad \left. + 1386 \left(\frac{c_{2v, v-3}}{c_{2v, v-1}} \right)^2 \frac{c_{2v, v-2}}{c_{2v, v-1}} + 660 \frac{c_{2v, v-4}}{c_{2v, v-1}} \left(\frac{c_{2v, v-2}}{c_{2v, v-1}} \right)^2 - 198 \frac{c_{2v, v-4}}{c_{2v, v-1}} \frac{c_{2v, v-3}}{c_{2v, v-1}} \right. \\ &\quad \left. - 55 \frac{c_{2v, v-5}}{c_{2v, v-1}} \frac{c_{2v, v-2}}{c_{2v, v-1}} + 3 \frac{c_{2v, v-6}}{c_{2v, v-1}} \right), \end{aligned}$$

where $v > 2$ for the first result, $v > 3$ in the second, $v > 4$ for the third result and $v > 5$ for the final result. In principle, this process can be continued for higher powers of the sum on the lhs by determining higher values of n in the symmetric polynomials, $s(v, v-n)$. Thus, we see that integer values of the Hurwitz zeta function for even powers can be expressed in terms of the ratios of the generalized

cosecant numbers. In addition, the number of ratios appearing on the rhs is equal to the number of partitions summing to half the power in the zeta functions.

It should also be mentioned that the series on the lhs of (3.37) to (3.38) are also specific values of the generalized harmonic numbers, which are defined as $H_{n,r} \doteq \sum_{k=1}^n 1/k^r$. In particular, for the case of $r = 2$ given by (3.37) the numbers are known as Wolstenholme numbers, which appear as Sequences A007406, A007408, A11354 and A123751 in the Online Encyclopedia of Integer Sequences [42].

Our results for the Gardner-Fisher sum given by either (3.12), (3.27) or (3.28) are computationally expedient because they can all be implemented as a one-line instruction in Mathematica [46]. The last two forms, however, rely on importing the generalized cosecant numbers into a notebook before they can be evaluated, whereas (3.12) relies on the intrinsic routines of the package and therefore, does not require additional information. Therefore, for the time being we shall concentrate on evaluating the solutions of the Gardner-Fisher sum via (3.12).

By using the SymmetricPolynomial, Gamma and Zeta routines we can express (3.12) as

$$\begin{aligned} S[m_-, v_-] &:= (1/((2 v - 1)!)) \text{Sum}[(\text{Pi}/m)^{(2 n)} \\ &\text{SymmetricPolynomial}[n, \text{Table}[k^2, k, 1, v - 1]] \\ &\text{Gamma}[2 v - 2 n] \text{Zeta}[2 v - 2 n] (1 - 1/(m^{(2 v - 2 n)})), \{n, 0, v - 1\} \end{aligned} .$$

If we type in the instruction

$$F[m, n] := \text{Table}[\text{Simplify}[S[m, v]], \{v, 1, n\}] \quad ,$$

then we can construct a table of solutions of the Gardner-Fisher sum. In fact, by typing

$$\text{Timing}[\text{Grid}[\text{Partition}[F[m, 15], 1]]] \quad ,$$

one obtains the the first 15 solutions of the Gardner-Fisher sum, i.e., for $v = 1$ to 15. For a Venom Blackbook 17S Pro High Performance laptop with 8 Gb RAM and equipped with Mathematica 10.4 it takes less than 0.16 seconds for the values of the sum to appear on the screen. Table 3.3 displays the output from the above instruction. Besides presenting a greater number of results than in Table 2 of [27], we have corrected the typographical error occurring in the last term of the second result, which should be $-7/2m^4$, not $-7/2m^2$. In addition, the results have been formatted in terms of the zeta function in order to maintain consistency with Gardner's limit mentioned in the introduction. Moreover, the results can be further simplified by expressing them as

$$S_{v,2}(m) = (2m^2)^{-v} (m^2 - 1) p_{v-1}(m^2) ,$$

where $p_v(x)$ is a polynomial of degree v with the coefficients $p_{v,j}$ satisfying

$$(3.39) \quad p_{v,v} = 2^{v+1} \zeta(2v+2); \quad , \quad p_{v,v-1} = 2^{v+1} \zeta(2v+2) + 2^{v-1} \pi^2 \zeta(2v)/3 ,$$

$$p_{v,j-1} = p_{v,j} + \frac{2^{v+1}}{(2v+1)!} \pi^{2v+2-2j} s(v+1, v+1-j) \Gamma(2j) \zeta(2j),$$

and

$$(3.40) \quad p_{v,0} = \frac{2^{v+1}}{(2v+1)!} \sum_{j=0}^v \pi^{2j} s(v+1, j) \Gamma(2v+2-2j) \zeta(2v+2-2j).$$

As mentioned previously, solutions of the Gardner-Fisher sum have been obtained in [27] by using an empirical method. According to this approach, the coefficients of m^{-2i} in $S_{v,2}(m)$ for $i < v$ were determined to be

$$(3.41) \quad C_i^v = c_{2v,i} \zeta(2v-2i) \left(\frac{\pi}{2}\right)^{2i},$$

while for $i = v$, they were given by

$$(3.42) \quad C_v^v = 2^v \left(\frac{\pi}{2}\right)^{2v} - \sum_{i=0}^{v-1} 2^{2v-2i} c_{2v,i} \left(\frac{\pi}{2}\right)^{2i} \zeta(2v-2i).$$

In these results we have dropped dividing by $\zeta(2v)$, which occurs when this factor is taken outside the results for $S_v(m)$ in order to confirm Gardner's limit. In addition, (3.42) has an extra factor of 2^v in the first term on the rhs, which is missing in (44) of [27]. Moreover, in the future work mentioned previously it will be shown that the final coefficient of $S_v(m)$ can also be expressed as

$$(3.43) \quad C_v^v = -\frac{1}{2} (c_{2v,v} + 1) \left(\frac{\pi}{2}\right)^{2v}.$$

By equating like powers of m^2 in the above results with (3.12), we find that

$$(3.44) \quad c_{2v,i} = 2^{2i} \frac{\Gamma(2v-2i)}{\Gamma(2v)} s(v, i), \quad i < v,$$

$$\sum_{i=0}^{v-1} \left(2^{2v-2i} c_{2v,i} - \frac{2^{2i}}{(2v-1)!} s(v, i) \Gamma(2v-2i) \right) \left(\frac{\pi}{2}\right)^{2i-2v} \zeta(2v-2i) = 2^v,$$

and

$$(3.45) \quad \sum_{n=0}^{v-1} \pi^{2n} s(v, n) \Gamma(2v-2n) \zeta(2v-2n) = \frac{1}{2} \Gamma(2v) (c_{2v,v} + 1) \left(\frac{\pi}{2}\right)^{2v}.$$

The first of the above results has already been derived as (3.26), while the other results have been verified by programming the generalized cosecant numbers into Mathematica [46]. Hence the solutions of the Gardner-Fisher sum can be expressed in terms of the generalized cosecant numbers as

$$(3.46) \quad S_{v,2}(m) = \sum_{i=0}^{v-1} c_{2v,i} \zeta(2v-2i) \left(\frac{\pi}{2m}\right)^{2i} - \frac{1}{2} (c_{2v,v} + 1) \left(\frac{\pi}{2m}\right)^{2v}.$$

v	$(2m^2)^v S_{v,2}(m)/(m^2 - 1)$
1	$2\zeta(2)$
2	$2\zeta(4)(7 + 2m^2)$
3	$\zeta(6)(71 + 29m^2 + 8m^4)$
4	$\frac{2}{3}\zeta(8)(521 + 251m^2 + 104m^4 + 24m^6)$
5	$\zeta(10)(1693 + 901m^2 + 450m^4 + 164m^6 + 32m^8)$
6	$\frac{1}{691}\zeta(12)(5710469 + 3253469m^2 + 1815032m^4 + 821182m^6 + 262624m^8 + 44224m^{10})$
7	$\frac{1}{30}\zeta(14)(1212457 + 726457m^2 + 436531m^4 + 224891m^6 + 91472m^8 + 25952m^{10} + 3840m^{12})$
8	$\frac{2}{10851}\zeta(16)(1074010337 + 669172337m^2 + 424359179m^4 + 238674979m^6 + 112425856m^8 + 41352256m^{10} + 10528128m^{12} + 1388928m^{14})$
9	$\frac{1}{219335}\zeta(18)(212920335247 + 136906047247m^2 + 90463687339m^4 + 54230674609m^6 + 28237533526m^8 + 12198655216m^{10} + 4088714368m^{12} + 943572928m^{14} + 112299520m^{16})$
10	$\frac{1}{3666831}\zeta(20)(17471801743019 + 11530685023019m^2 + 7871509097999m^4 + 4954255382249m^6 + 2775847146014m^8 + 1339874175764m^{10} + 533660885024m^{12} + 164177309024m^{14} + 34637202944m^{16} + 3754834944m^{18})$
11	$\frac{22}{51270780}\zeta(22)(54543344015461 + 36797151215461m^2 + 25796160434461m^4 + 16874095108441m^6 + 10001178676906m^8 + 5238677224486m^{10} + 2352811821856m^{12} + 868592118736m^{14} + 246835895296m^{16} + 47953445376m^{18} + 4772843520m^{20})$
12	$\frac{1}{3545461365}\zeta(24)(407813841938063843 + 280376888294063843m^2 + 200956537702208843m^4 + 135642820502091743m^6 + 84063888200850398m^8 + 46897459147087298m^{10} + 23026002351214568m^{12} + 9661082872248968m^{14} + 3322054926238208m^{16} + 876937263364608m^{18} + 157850700656640m^{20} + 14522209751040m^{22})$
13	$\frac{1}{82899306}\zeta(26)(46885126608855949 + 32769033282135949m^2 + 23932490769301549m^4 + 16581373742311501m^6 + 10657001539527466m^8 + 6251551315383406m^{10} + 3288926574739816m^{12} + 1518316248517036m^{14} + 597409832798656m^{16} + 192183289614336m^{18} + 47355550989312m^{20} + 7940215990272m^{22} + 679111114752m^{24})$
14	$\frac{1}{6785560294}\zeta(28)(18876870638218467389 + 13386158464071267389m^2 + 9936028183278303389m^4 + 7037937839930870909m^6 + 4662484330428335720m^8 + 2849737068204025620m^{10} + 1584446206207320450m^{12} + 787977205890303150m^{14} + 343067669320476480m^{16} + 127034728547601280m^{18} + 38382891063494656m^{20} + 8866606540512256m^{22} + 1391299071164416m^{24} + 111174619856896m^{26})$
15	$\frac{1}{6892673020804}\zeta(30)(94347422281763617652599 + 67773208432462419572599m^2 + 51020868251978707090999m^4 + 36832275607229172708727m^6 + 25034877087339771576013m^8 + 15834856690259096336845m^{10} + 9212522702209442756325m^{12} + 4863979571266245490245m^{14} + 2292234370375168188000m^{16} + 943898600252928069440m^{18} + 329986597479890425856m^{20} + 93975868433656203776m^{22} + 20429529466374004736m^{24} + 3012284194329018368m^{26} + 225859109545705472m^{28})$

Table 3.3: The first 15 $(2m^2)^v S_{v,2}(m)/(m^2 - 1)$ for v , a positive integer

To conclude our analysis of the symmetric polynomials over quadratic numbers, it should be mentioned that by introducing (3.44) into (2.6), one can obtain a recurrence relation for them. For $k < n$, this is given by

$$s(n+1, k+1) = s(n, k+1) + n^2 s(n, k).$$

As a check on this result, let us consider $k = 1$. We already know $s(n, 1)$ from (3.19), which is a third order polynomial with no constant. Because of the n^2 factor in the above equation, we immediately observe that the second term on the rhs will be fifth order in n . However, the order of $s(n, 2)$ must be at least one order higher because the highest order terms will be subtracted out. Thus, we conjecture that $s(2, k)$ is a sixth order polynomial with no constant term, i.e., $s(n, 2) = \sum_{i=1}^6 a_i n^i$. Introducing this form for $s(n, 2)$ into the above equation yields

$$\begin{aligned} \sum_{i=1}^6 a_i + n \sum_{i=2}^6 \binom{i}{1} a_i + n^2 \sum_{i=3}^6 \binom{i}{2} a_i + n^3 \sum_{i=4}^6 \binom{i}{3} a_i + n^4 \sum_{i=5}^6 \binom{i}{4} a_i + 6n^5 a_6 \\ = \frac{n^3}{6} - \frac{n^2}{2} + \frac{n^5}{3}. \end{aligned}$$

By equating the coefficients on both sides, we find that $a_6 = 1/18$, $a_5 = -4/15$, $a_4 = 31/72$, $a_3 = -1/4$, $a_2 = 1/72$ and $a_1 = 1/60$. If this polynomial is introduced into Mathematica and the Factor routine is applied to it, then one obtains (3.23) except v is replaced by n .

4. UNTWISTED DOWKER SUM

As stated in [6], in a series of papers [9–11] Dowker encountered the following general family of cosecant sums:

$$(4.47) \quad S_v(m, r) := \sum_{k=1}^{m-1} \cos\left(\frac{2rk\pi}{m}\right) \csc^{2v}\left(\frac{k\pi}{m}\right),$$

where m is a positive integer or natural number greater than unity and r is an integer ranging from zero to $m - 1$. These are now known as Dowker sums where the untwisted form is represented by $r = 0$ and the twisted forms by $r \neq 0$. In this section we shall concentrate on the untwisted case, while the twisted case will be studied in the future. Suffice it to say, to obtain the latter, one needs to study the untwisted case first. Moreover, we see that the major difference between the untwisted Dowker and Gardner-Fisher sums is the argument of the cosecant, which is now composed of fractions multiplying π as opposed to $\pi/2$. There is also no normalization constant in the above definition. Because of the change in the argument, the empirical method in [27] is no longer applicable, which will be explained in the future work mentioned previously. Consequently, we can only rely on the integral approach of the previous section to evaluate this sum.

Theorem 4.2. *The untwisted Dowker sum given by the $r = 0$ form of (4.47) possesses the following solution:*

$$(4.48) \quad S_{v,1}(m, 0) = \sum_{k=1}^{m-1} \csc^{2v} \left(\frac{k\pi}{m} \right) = 2^{2v+1} \sum_{n=0}^{v-1} \left(\frac{m}{2\pi} \right)^{2v-2n} \frac{\Gamma(2v-2n)}{\Gamma(2v)} \\ \times s(v, n) \left(1 - \frac{1}{m^{2v-2n}} \right) \zeta(2v-2n).$$

Proof. Before considering the untwisted Dowker sum specifically, we shall begin the proof by studying a more general form of the sum where the $\pi/2$ argument inside the cosecant power is replaced by π/ℓ , and ℓ is any integer except 0. For negative values of ℓ , ℓ should be replaced by $|\ell|$ in what follows. Hence we define the series $S_{v,\ell}(m, r)$ as

$$S_{v,\ell}(m, r) := \sum_{k=1}^{m-1} \cos \left(\frac{2rk\pi}{m} \right) \csc^{2v} \left(\frac{k\pi}{\ell m} \right).$$

Consequently, we need to modify (3.15) so that the sum over the new argument in the cosecant power becomes

$$\sum_{k=1}^{m-1} \csc^{2v} \left(\frac{k\pi}{\ell m} \right) = -\frac{4^{v-1} m^2 \ell^2}{\pi^2 \Gamma(2v)} \int_0^1 \frac{\ln u}{1-u^{\ell m}} \prod_{n=1}^{v-1} \left(\frac{\ell^2 m^2}{4\pi^2} \ln^2 u + n^2 \right) \sum_{k=1}^{m-1} (u^k + u^{\ell m - k}) \frac{du}{u}.$$

Next we re-arrange the sum over k from one to $m-1$ as follows:

$$(4.49) \quad S_{v,\ell}(m, 0) = -\frac{4^{v-1} m^2 \ell^2}{\pi^2 \Gamma(2v)} \int_0^1 \frac{(1+u^{(\ell-1)m})}{1-u^{\ell m}} \ln u \prod_{n=1}^{v-1} \left(\frac{\ell^2 m^2}{4\pi^2} \ln^2 u + n^2 \right) \sum_{k=0}^{m-2} u^k du.$$

From the discussion below (3.18), we can write the product in (4.49) as

$$\prod_{n=1}^{v-1} \left(\frac{\ell^2 m^2}{4\pi^2} \ln^2 u + n^2 \right) = \sum_{n=0}^{v-1} \left(\frac{\ell^2 m^2}{4\pi^2} \ln^2 u \right)^n s(v, v-1-n).$$

Introducing the above result into (4.49) yields

$$(4.50) \quad S_{v,\ell}(m, 0) = -\frac{4^{v-1} \ell^2 m^2}{\pi^2 \Gamma(2v)} \sum_{n=0}^{v-1} \left(\frac{\ell^2 m^2}{4\pi^2} \right)^n s(v, v-1-n) \\ \sum_{k=0}^{m-2} \int_0^1 \frac{(u^k + u^{k+(\ell-1)m})}{1-u^{\ell m}} \ln^{2n+1} u du.$$

The integrals in (4.50) are of the same generic form as No. 4.271(16) in [20], which is

$$\int_0^1 \frac{x^{p-1}}{1-x^q} \ln^n x dx = -\frac{1}{q^{n+1}} \psi^{(n)} \left(\frac{p}{q} \right).$$

where $p, q > 0$ and $\psi^{(n)}(x)$ denotes the n -th derivative of the psi function. Hence n and q in the above integral correspond to $2n+1$ and ℓm in both integrals of (4.50). The only difference is that p corresponds to $k+1$ for the first integral, while for

the second integral it corresponds to $k + 1 + (\ell - 1)m$. As a consequence, (4.50) becomes

$$(4.51) \quad S_{v,\ell}(m, 0) = \frac{4^{v-1}}{\pi^2 \Gamma(2v)} \sum_{n=0}^{v-1} \left(\frac{1}{4\pi^2} \right)^n s(v, v-1-n) \sum_{k=1}^{m-1} \left(\psi^{(2n+1)} \left(\frac{k}{\ell m} \right) + \psi^{(2n+1)} \left(1 - \frac{1}{\ell} + \frac{k}{\ell m} \right) \right).$$

By putting $n = v - 1 - n$ in (4.51), we obtain

$$S_{v,\ell}(m, 0) = \frac{2^{2v}}{\Gamma(2v)} \sum_{n=1}^{v-1} (2\pi)^{2n-2v} s(v, n) \sum_{k=1}^{m-1} \left(\psi^{(2v-2n-1)} \left(\frac{k}{\ell m} \right) + \psi^{(2v-2n-1)} \left(1 - \frac{1}{\ell} + \frac{k}{\ell m} \right) \right).$$

Note that when $\ell = 1$ or 2 , the summation over k reduces to much simpler forms. For $\ell = 1$ there is a doubling of the first digamma function, while for $\ell = 2$ the sums can be combined into one sum. In the second case the union of the sums is the reason why the empirical method in [27] can be applied. For other values of ℓ this union does not occur. Hence the empirical method will not yield correct values of the sums for $\ell \neq 2$. On the other hand, it means that the empirical method can be applied to the alternating version of the Gardner-Fisher sum when we examine this case in the future work mentioned previously.

With the aid of No. 8.363(8) in [20] we can replace the derivative of digamma function by the Hurwitz zeta function. Then $S_{v,\ell}(m, 0)$ can be expressed as

$$(4.52) \quad S_{v,\ell}(m, 0) = \frac{2^{2v}}{\Gamma(2v)} \sum_{n=0}^{v-1} (2\pi)^{2n-2v} s(v, n) \Gamma(2v - 2n) \times \sum_{k=1}^{m-1} \left(\zeta(2v - 2n, k/\ell m) + \zeta(2v - 2n, 1 - 1/\ell + k/\ell m) \right).$$

For $|\ell| \neq 1$ or 2 , the Hurwitz zeta function is intractable and consequently, this generalization of the Gardner-Fisher sum will not yield polynomials as in Table 3.3. However, for the specific case of $\ell = 1$ or the untwisted Dowker sum (4.52) reduces to

$$(4.53) \quad S_{v,1}(m, 0) = 2^{2v+1} \sum_{n=0}^{v-1} (2\pi)^{2n-2v} \frac{\Gamma(2v - 2n)}{\Gamma(2v)} s(v, n) \sum_{k=1}^{m-1} \zeta(2v - 2n, k/m).$$

Finally, by using the fact that

$$\sum_{k=1}^{m-1} \zeta(2v - 2n, k/m) = m^{2v-2n} \zeta(2v - 2n) - \zeta(2v - 2n),$$

we arrive at (4.48). This completes the proof of the Theorem 4.2. \square

Because there is no normalization factor outside the sum as in the Gardner-Fisher sum, we obtain polynomials in powers of m^2 of degree v . Although these polynomials are denoted by $C_{2v}(m)$ in [7], we shall denote them by $q_v(m^2)$ with the coefficients of m^{2i} represented by $q_{v,i}$. From (4.53) we find that

$$(4.54) \quad q_{v,0} = -2^{2v+1} \sum_{n=0}^{v-1} (2\pi)^{2n-2v} \frac{\Gamma(2v-2n)}{\Gamma(2v)} s(v,n) \zeta(2v-2n), \quad q_{v,1} = \frac{1}{6} \frac{\Gamma(v)\Gamma(1/2)}{\Gamma(v+1/2)},$$

$$(4.55) \quad q_{v,i} = \frac{2^{2v-2i+1}}{\pi^{2i}} \frac{\Gamma(2i)}{\Gamma(2v)} s(v, v-i) \zeta(2i), \quad i < v, \quad \text{and} \quad q_{v,v} = \frac{2}{\pi^{2v}} \zeta(2v).$$

Moreover, we can use (3.44) to express $q_{v,0}$ and $q_{v,i}$ in terms of the generalized cosecant numbers. Hence we obtain

$$q_{v,0} = -2 \sum_{n=0}^{v-1} \pi^{2n-2v} c_{2v,n} \zeta(2v-2n),$$

and

$$q_{v,i} = \frac{2}{\pi^{2i}} c_{2v,v-i} \zeta(2i), \quad i < v.$$

As in the case of the Gardner-Fisher sum the polynomials obtained from (4.48) possess a common factor of (m^2-1) . Consequently, we can simplify the presentation of the polynomials for $S_{v,1}(m)$ by removing this factor. Thus, Table 4.4 presents the first 15 values of $S_{v,1}(m)/(m^2-1)$, which were obtained by writing (4.48) as a one-line instruction in Mathematica [46] as in the case of the Gardner-Fisher sum. In this instance the instruction becomes

$$\begin{aligned} \text{CS}[m_-, v_-] &:= (2^{2v+1}/(2v-1)!) \text{Sum}[(2\text{Pi}/m)^{(2n-2v)} \\ &\text{SymmetricPolynomial}[n, \text{Table}[k^{2v-1}, \{k, 1, v-1\}]] \\ &\text{Gamma}[2(v-n)] \text{Zeta}[2(v-n)] (1 - 1/(m^{2(v-n)})), \{n, 0, v-1\}] \end{aligned}$$

Then we can use the same instructions below the previous instructions for the Gardner-Fisher sum to tabulate the polynomials and time the calculation except that $S[m,v]$ is now replaced by $\text{CS}[m,v]$. It is found that the results in Table 4.4 took 0.12 seconds to compute on the same Venom laptop mentioned before. It should also be noted that the first five results in the table are identical to those given in [7]. In fact, these authors prove that

$$(4.56) \quad q_v(m^2) = (-1)^{v-1} \frac{2^{2v}}{(2v)!} \sum_{n=0}^v \binom{2v}{2n} B_{2v-2n} B_{2n}^{(2v)}(v) m^{2v-2n},$$

v	$S_{v,1}(m)/(m^2 - 1)$
1	$\frac{2\zeta(2)}{\pi^2}$
2	$\frac{2\zeta(4)}{\pi^4}(m^2 + 11)$
3	$\frac{\zeta(6)}{\pi^6}(2m^4 + 23m^2 + 191)$
4	$\frac{2\zeta(8)}{3\pi^8}(m^2 + 11)(3m^4 + 10m^2 + 227)$
5	$\frac{\zeta(10)}{\pi^{10}}(2m^8 + 35m^6 + 321m^4 + 2125m^2 + 14797)$
6	$\frac{2\zeta(12)}{1382\pi^{12}}(1382m^{10} + 28682m^8 + 307961m^6 + 2295661m^4 + 13803157m^2 + 92427157)$
7	$\frac{2\zeta(14)}{60\pi^{14}}(60m^{12} + 1442m^{10} + 17822m^8 + 151241m^6 + 997801m^4 + 5636617m^2 + 36740617)$
8	$\frac{2\zeta(16)}{10851\pi^{16}}(10851m^{14} + 296451m^{12} + 4149467m^{10} + 39686267m^8 + 292184513m^6 + 1777658113m^4 + 9611679169m^2 + 61430943169)$
9	$\frac{2\zeta(18)}{438670\pi^{18}}(438670m^{16} + 13427317m^{14} + 209998657m^{12} + 2237483869m^{10} + 18276362179m^8 + 122248926511m^6 + 701977130191m^4 + 3674288164303m^2 + 23133945892303)$
10	$\frac{2\zeta(20)}{7333662\pi^{20}}(7333662m^{18} + 248602162m^{16} + 4296730477m^{14} + 50482177477m^{12} + 453588822847m^{10} + 3325534763347m^8 + 20752800653227m^6 + 114104919557227m^4 + 582479437959787m^2 + 3624331198599787)$
11	$\frac{2\zeta(22)}{4660980\pi^{22}}(4660980m^{20} + 173335206m^{18} + 3280873486m^{16} + 42140637451m^{14} + 413195563231m^{12} + 3299060965861m^{10} + 22349066775541m^8 + 132315729680101m^6 + 703327910545381m^4 + 3519581550481381m^2 + 21691682977681381)$
12	$\frac{2\zeta(24)}{7090922730\pi^{24}}(7090922730m^{22} + 287029381530m^{20} + 5904893152686m^{18} + 82314820117486m^{16} + 874693313368831m^{14} + 7557153052851631m^{12} + 55300066644597091m^{10} + 352631499074701891m^8 + 2003157332714424931m^6 + 10363313134329413731m^4 + 51026532637359173731m^2 + 312017413700271173731)$
13	$\frac{2\zeta(26)}{165798612\pi^{26}}(165798612m^{24} + 7256721342m^{22} + 161222873682m^{20} + 2424156289698m^{18} + 27750815238718m^{16} + 257977419168313m^{14} + 2028587745391093m^{12} + 13879086707965453m^{10} + 84366290294270413m^8 + 463526111272448653m^6 + 2345412070181900941m^4 + 11394031603324326541m^2 + 69213549869569446541)$
14	$\frac{2\zeta(28)}{13571120588\pi^{28}}(13571120588m^{26} + 638631887828m^{24} + 15238841788898m^{22} + 245834814624698m^{20} + 3016204736003030m^{18} + 30020322332612430m^{16} + 252475090617525765m^{14} + 1845413091251560365m^{12} + 11967739987225201725m^{10} + 69975652378403124925m^8 + 374033701594727629117m^6 + 1857855957388613058877m^4 + 8923722772452603330877m^2 + 53903636903066465730877)$
15	$\frac{2\zeta(30)}{13785346041608\pi^{30}}(13785346041608m^{28} + 694064907756284m^{26} + 17703093493737716m^{24} + 304993480084683806m^{22} + 3992661121432093526m^{20} + 42362161294746946250m^{18} + 379446103825306975890m^{16} + 2951191304716384278135m^{14} + 20345363828489173342455m^{12} + 126302707637283630630775m^{10} + 715104013050446845937527m^8 + 3735238034142133535912311m^6 + 18264356902365656743358839m^4 + 86881942281626943067992439m^2 + 522273861988577772410712439)$

Table 4.4: The first 15 $S_{v,1}(m)/(m^2 - 1)$ for v , a positive integer

where B_{2v-2n} are the ordinary Bernoulli numbers and $B_k^{(m)}(x)$ are the Bernoulli polynomials of order m and degree k , sometimes referred to as Nörlund polynomials. By equating like powers of m between (4.53) and (4.56), we arrive at the following results:

$$(4.57) \quad B_{2n}^{(2v)}(v) = (-1)^n (2n)! \frac{\Gamma(2v-2n)}{\Gamma(2v)} s(v, n), \quad n < v,$$

and

$$(4.58) \quad B_{2v}^{(2v)}(v) = (-1)^v 4v \sum_{n=0}^{v-1} (2\pi)^{2n-2v} \Gamma(2v-2n) s(v, n) \zeta(2v-2n).$$

In obtaining the second result we have used No. 9.616 in [20], which expresses the Bernoulli numbers in terms of the Riemann zeta function. Furthermore, introducing (3.44) into (4.57) and (4.58) yields

$$B_{2n}^{(2v)}(v) = (-1)^n 2^{-2n} (2n)! c_{2v,n}, \quad n < v,$$

and

$$B_{2v}^{(2v)}(v) = (-1)^v 2^{1-2v} \Gamma(2v+1) \sum_{n=0}^{v-1} \pi^{2n-2v} c_{2v,n} \zeta(2v-2n).$$

The above results can be verified in Mathematica [46], where the Bernoulli polynomials of order m and degree k are determined via the `NorlundB` routine. Thus, we observe that the Nörlund polynomials are related to generalized cosecant numbers.

5. OTHER SUMS

In this section we study more sophisticated trigonometric power sums than the Gardner-Fisher and untwisted Dowker sums. However, in order to carry out the proposed study, we shall require the results presented from the previous sections. For example, let us consider the trigonometric power sum defined by

$$(5.59) \quad S_{v,w,l}^{CC}(m) := \sum_{k=1}^{m-1} \csc^{2v} \left(\frac{k\pi}{\ell m} \right) \cot^{2w} \left(\frac{k\pi}{\ell m} \right),$$

where the superscript CC is an abbreviation for $\csc() \cot()$, $v \geq 0$ and $w \geq 0$ excluding $v+w=0$, while $\ell=1$ and $\ell=2$ correspond respectively to the untwisted Dowker and Gardner-Fisher cases when $w=0$. A factor of $\cos(2pk\pi/\ell m)$, where p is an integer less than $\ell m-1$, could also have been introduced into the summand. In that case the resulting sum when $\ell=1$ becomes a generalization of the twisted Dowker sum [9]. However, that sum will be studied in the future work mentioned previously since we need to determine if the simpler $S_{v,w,l}^{CC}(m)$ given above can be solved first.

Theorem 5.3. For $\ell = 2$, i.e., the analogue of the Gardner-Fisher sum, the trigonometric power sum as defined by (5.59) possesses the following solution:

$$(5.60) \quad S_{v,w,2}^{CC}(m) = (m^2 - 1) \sum_{j=0}^w (-1)^{w-j} \binom{w}{j} \frac{2^{j+v}}{\pi^{2j+2v}} R_{j+v}(m) \quad ,$$

while for $\ell = 1$ or the analogue of the untwisted Dowker sum, its solution is

$$(5.61) \quad S_{v,w,1}^{CC}(m) = (m^2 - 1) \sum_{j=0}^w (-1)^{w-j} \binom{w}{j} T_{j+v}(m) \quad .$$

In the above results $R_v(m) = (2m^2)^v S_{v,2}(m)/(m^2 - 1)$ and $T_v(m) = S_{v,1}(m)/(m^2 - 1)$ with $R_0(m) = T_0(m) = 1/(m + 1)$.

Proof. We begin by expressing the general trigonometric power sum given by (5.59) as

$$S_{v,w,\ell}^{CC}(m) = \sum_{k=1}^{m-1} \frac{\cos^{2w}(k\pi/\ell m)}{\sin^{2v+2w}(k\pi/\ell m)} \quad .$$

Now replacing the cosine power in the numerator by $(1 - \sin^2(k\pi/\ell m))^w$ and applying the binomial theorem, we find that

$$(5.62) \quad S_{v,w,\ell}^{CC}(m) = \sum_{j=0}^w \sum_{k=1}^{m-1} (-1)^{w-j} \binom{w}{j} \operatorname{csc}^{2j+2v} \left(\frac{k\pi}{\ell m} \right) \quad .$$

Therefore, $S_{v,w,\ell}^{CC}(m)$ has become a finite sum of cosecant power sums, which reduce to the Gardner-Fisher and untwisted Dowker sums when $\ell = 2$ and 1, respectively.

For $\ell = 1$ or 2 and $v + w \leq 15$, we can use the results in Tables 3.3 and 4.4 to obtain the solutions of $S_{v,w,\ell}^{CC}(m)$. In the case of $\ell = 2$, we denote the results in Table 3.3 by $R_v(m)$, i.e., $R_v(m) = (2m^2)^v S_{v,2}(m)/(m^2 - 1)$. For the special case of $v = 0$, $R_0(m) = 1/(m + 1)$ according to this formula. By introducing $R_{v+j}(m)$ into (5.62), we find that $S_{v,w,2}^{CC}(m)$ is given by (5.60). On the other hand, for $\ell = 1$ or the untwisted Dowker case, we denote the quantities listed in the Table 4.4 by $T_v(m)$, i.e., $T_v(m) = S_{v,1}(m)/(m^2 - 1)$. For $v = 0$ again, we find that $T_0(m) = 1/(m + 1)$. By introducing $T_{v+j}(m)$ into (5.62), we obtain (5.61), which completes the proof. \square

As an example, if we set v and w equal to 4 and 5 respectively with $\ell = 2$, then (5.59) becomes

$$S_{4,5,2}^{CC}(m) = (m^2 - 1) \left(\frac{2}{\pi^2} \right)^4 \sum_{j=0}^5 (-1)^{j+1} \binom{5}{j} \left(\frac{2}{\pi^2} \right)^j R_{j+4}(m) \quad .$$

Consequently, we require $R_4(m)$ to $R_9(m)$ from Table 3.3 to determine $S_{4,5,2}^{CC}(m)$. To evaluate the sum, we use the Sum routine in Mathematica [46] and then apply

the Simplify and FactorList routines, which not only simplify the result, but also determine any factors in it. Hence we find that

$$(5.63) \quad \sum_{k=1}^{m-1} \csc^8\left(\frac{k\pi}{2m}\right) \cot^{10}\left(\frac{k\pi}{2m}\right) = \frac{16}{194896477400625} (m^2 - 1) (4m^2 - 1) (2280413161 + 712556555m^2 - 2906805048m^4 - 2535353600m^6 + 2920623488m^8 + 2565749760m^{10} - 3310462976m^{12} + 898396160m^{14}) .$$

This result can be checked by introducing the sum on the lhs directly into Mathematica via the NSum routine and then comparing the results of both sides for specific values of m . For example, both sides yield a value of $1.604289608 \dots \times 10^{27}$ when $m = 51$. The advantage that the rhs possesses is that it yields rational values, whereas evaluating the lhs only yields decimal values, whose rationality is difficult to ascertain. In addition, the results determined from the lhs will be subject to the machine precision of the computing system.

Table 5.5 presents the solutions of $S_{v,w,2}^{CC}(m)$ for v ranging from 0 to 3 and w from 1 to 4. There it can be seen that for v and w greater than zero the solutions are polynomials in even powers of m as we found for the Gardner-Fisher and untwisted Dowker sums. However, the degree of the polynomials is now $2v + 2w$ in m . For $v = 0$, when the sum is only in even powers of $\cot(k\pi/2m)$, the solutions are polynomials of degree $2w$ with odd powers of m . In addition, there is a common factor of $(m - 1)(2m - 1)$ in the solutions, but when $v > 0$, the common factor is $(m^2 - 1)(4m^2 - 1)$. In addition, the value of the denominator is invariant for fixed values of $v + w$. For example, when $v + w = 4$, the value in the denominator is 14175, which is present when either $v = 0$ and $w = 4$, $v = 1$ and $w = 3$, $v = 2$ and $w = 2$ or $v = 3$ and $w = 1$. On the other hand, the value in the numerator is dependent upon 2^v .

A general formula for $S_{v,w,2}^{CC}(m)$ can be derived by introducing (3.46) into (5.62) after it has been divided by $(\pi/2m)^{2v}$ so that the normalization factor can be removed. Thus, we arrive at

$$(5.64) \quad S_{v,w,2}^{CC}(m) = \sum_{j=0}^w (-1)^{w-j} \binom{w}{j} \frac{2^{2j+2v}}{\pi^{2j+2v}} \sum_{i=0}^{j+v} C_i^{j+v} m^{2j+2v-2i} ,$$

where the coefficients C_i^{j+v} in terms of the generalized cosecant numbers are given by (3.41) and (3.43). Moreover, by using (3.41), (3.43), (3.44), and (3.45), we can express (5.64) in terms of the symmetric polynomials over quadratic powers of the integers up to $v - 1$ by replacing the coefficients with the following results:

$$(5.65) \quad C_i^v = (2v)_{-2i} \zeta(2v - 2i) s(v, i) , \quad i < v ,$$

and

$$C_v^v = - \sum_{n=0}^{v-1} \pi^{2n} (2v)_{-2n} \zeta(2v - 2n) s(v, n) .$$

If we expand (5.64), then we find that

$$(5.66) \quad S_{v,w,2}^{CC}(m) = \sum_{j=0}^w (-1)^{w+j} \binom{w}{j} \left(\frac{2}{\pi}\right)^{2j+2v} \left(C_0^{j+v} m^{2j+2v} + C_1^{j+v} m^{2j+2v-2} \right. \\ \left. + C_2^{j+v} m^{2j+2v-4} + \dots + C_{j+v-1}^{j+v} m^2 + C_{j+v}^{j+v} \right).$$

Since we have seen from (5.64) and Table 5.5 that the solutions of $S_{v,w,2}^{CC}(m)$ are polynomials in even powers of m of degree $2v+2w$, we can represent the solutions by $S_{v,w,2}^{CC}(m) = \sum_{l=0}^{v+w} D_{l,v,w} m^{2l}$. Moreover, we observe that the highest power in $S_{v,w,2}^{CC}(m)$ is due to the highest power in the first term on the rhs of (5.66). Hence the leading coefficient becomes

$$D_{v+w,v,w} = \left(\frac{2}{\pi}\right)^{2v+2w} C_0^{w+v} = \left(\frac{2}{\pi}\right)^{2v+2w} \zeta(2w+2v),$$

where the second form has been obtained from (5.65) and by noting that $s(v,0) = 1$. On the other hand, the lowest power or the constant term in $S_{v,w,2}^{CC}(m)$ is represented by the last term on the rhs of (5.66). Therefore, D_0 is given by

$$D_{0,v,w} = -\frac{1}{2} \sum_{j=0}^w (-1)^{j+w} \binom{w}{j} (c_{2j+2v,j+v} + 1) \\ = -\frac{1}{2} \delta_w - \frac{1}{2} \sum_{j=0}^w (-1)^{j+w} \binom{w}{j} c_{2j+2v,j+v}.$$

In obtaining this result we have used (3.43) and No. 4.2.1.3 from Prudnikov et al [37]. In fact, it is only the constant term in the solutions that require (3.43). All the other terms in (5.66) utilize (3.41). The coefficient of the second highest power in $S_{v,w,2}^{CC}(m)$ is more formidable to evaluate than the highest order term because it is composed of two distinct terms, namely the $j = w-1$ and $j = w$ values respectively from the first and second terms on the rhs of (5.66). That is, we find that for $v+w > 1$,

$$D_{v+w-1,v,w} = -\left(\frac{2}{\pi}\right)^{2v+2w-2} C_0^{v+w-1} w + C_1^{w+v} \left(\frac{2}{\pi}\right)^{2v+2w}.$$

In terms of the generalized cosecant numbers, the above result reduces to

$$D_{v+w-1,v,w} = (c_{2v+2w,1} - w c_{2v+2w-2,0}) \left(\frac{2}{\pi}\right)^{2v+2w-2} \zeta(2v+2w-2).$$

Furthermore, $D_{v+w-1,v,w}$ can be simplified even further with the aid of the results for the generalized cosecant numbers in Table 2.2. Hence the above result becomes

$$D_{v+w-1,v,w} = \left(\frac{v-2w}{3}\right) \left(\frac{2}{\pi}\right)^{2v+2w-2} \zeta(2v+2w-2).$$

The coefficient of the next highest power in $S_{v,w,2}^{CC}(m)$ is composed of $j = w-2$ value in the first term on the rhs of (5.66), the $j = w-1$ value in the second term

v	w	$S_{v,w,2}^{CC}(m)$
0	1	$\frac{1}{3}(2m-1)(m-1)$
0	2	$\frac{1}{45}(2m-1)(m-1)(4m^2+6m-13)$
0	3	$\frac{1}{945}(2m-1)(m-1)(32m^4+48m^3-112m^2-192m+251)$
0	4	$\frac{1}{14175}(2m-1)(m-1)(192m^6+288m^5-944m^4-1560m^3+1828m^2+3522m-3551)$
1	1	$\frac{2}{45}(m^2-1)(4m^2-1)$
1	2	$\frac{2}{945}(m^2-1)(4m^2-1)(8m^2-11)$
1	3	$\frac{2}{14175}(m^2-1)(4m^2-1)(48m^4-140m^2+107)$
1	4	$\frac{2}{467775}(m^2-1)(4m^2-1)(640m^6-2896m^4+4580m^2-2549)$
2	1	$\frac{4}{945}(m^2-1)(4m^2-1)(4m^2+5)$
2	2	$\frac{4}{14175}(m^2-1)(4m^2-1)(24m^4-10m^2-29)$
2	3	$\frac{4}{467775}(m^2-1)(4m^2-1)(320m^6-656m^4-20m^2+491)$
2	4	$\frac{4}{638512875}(m^2-1)(4m^2-1)(176896m^8-652480m^6+653688m^4+150830m^2-399809)$
3	1	$\frac{8}{14175}(m^2-1)(4m^2-1)(12m^4+25m^2+23)$
3	2	$\frac{8}{467775}(m^2-1)(4m^2-1)(160m^6+68m^4-175m^2-233)$
3	3	$\frac{8}{638512875}(m^2-1)(4m^2-1)(88448m^8-107840m^6-120876m^4+61765m^2+135203)$
3	4	$\frac{8}{383107725}(m^2-1)(4m^2-1)(21504m^{10}-61568m^8+22496m^6+55524m^4-8005m^2-41291)$

Table 5.5: $S_{v,w,2}^{CC}(m)$ for various values of v and w

of the same equation and the $j = w$ value of the third term. In terms of the generalized cosecant numbers, this coefficient can be written as

$$D_{v+w-2,v,w} = \left(\frac{(w^2-w)}{2}c_{2v+2w-4,0} - wc_{2v+2w-2,1} + c_{2v+2w,2}\right) \left(\frac{2}{\pi}\right)^{2v+2w-4} \zeta(2v+2w-4),$$

where $v + w > 3$. Again, introducing the results from Table 2.2 yields

$$D_{v+w-2,v,w} = \left(\frac{v^2}{18} + \frac{v}{90} + \frac{2w^2}{9} - \frac{7w}{45} - \frac{2vw}{9}\right) \left(\frac{2}{\pi}\right)^{2v+2w-4} \zeta(2v+2w-4).$$

More generally, we find that the coefficient $D_{l,v,w}$ for $v \leq l \leq v + w$ is composed of: (1) the $j = l - v$ value of the first sum on the rhs of (5.66), (2) the $j = l - v + 1$ value of the second sum on the rhs of the same equation, (3) the $j = l - v + 2$ value of the next sum in the equation and so on up till the $l - v + w$ value of the $w + 1$ -th sum on the rhs of (5.66). This means that

$$D_{l,v,w} = \sum_{j=0}^w (-1)^{j+w+l-v} \binom{w}{l-v+j} \left(\frac{2}{\pi}\right)^{2l+2j} C_j^{j+l}.$$

Introducing (3.44) into the above result yields

$$D_{l,v,w} = \frac{2^{2l} \zeta(2l)}{\pi^{2l}} \sum_{j=0}^w (-1)^{j+w+l-v} \binom{w}{l-v+j} c_{2l+2j,j} .$$

For $l \leq v-1$, the situation becomes intriguing. For example, when $l = v-1$, the first sum on the rhs of (5.66) no longer contributes to $D_{v-1,v,w}$. Then the first contribution comes from the second sum or the term involving $C_1^{j+v} m^{2j+2v-2}$ with $j = 0$. When $l = v-2$, this term does not contribute to $D_{v-2,v,w}$. Instead the $j = 0$ value of the next sum becomes the first contribution. By introducing (3.41) into (5.66), we find that for $l < v$,

$$(5.67) \quad D_{v-l,v,w} = \frac{2^{2v-2l}}{\pi^{2v-2l}} \zeta(2v-2l) \sum_{j=0}^w (-1)^{j+w} \binom{w}{j} c_{2j+2v,j+l} .$$

The coefficients $D_{l,v,w}$ form a right-angled triangle as w is incremented. Thus, with w increasing there is an increasing number of coefficients. So far, we have been concentrating on the coefficients lying diagonally in the triangle. Now we turn our attention to those lying vertically. As will be seen, there will be some overlap with the results derived above. Already, we have derived a general formula for the constant term in the solutions. So, now we consider the next lowest order term, which is the m^2 or penultimate term on the rhs of (5.66). This is given by

$$D_{1,v,w} = \sum_{j=0}^w (-1)^{w+j} \binom{w}{j} \left(\frac{2}{\pi}\right)^{2j+2v} C_{j+v-1}^{j+v} .$$

With the aid of (3.41) $D_{1,v,w}$ reduces to

$$D_{1,v,w} = \frac{2}{3} \sum_{j=0}^w (-1)^{w+j} \binom{w}{j} c_{2j+2v,j+v-1} ,$$

where $v > 0$ and we have used the fact that $\zeta(2) = \pi^2/6$. Similarly, to obtain the coefficient of m^4 in $S_{v,w,2}^{CC}$, we need to consider the m^6 -term in (5.66). This is given by

$$D_{2,v,w} = \frac{8}{45} \sum_{j=0}^w (-1)^{j+w} \binom{w}{j} c_{2j+2v,j+v-2} ,$$

where on this occasion, we have used the fact that $2^4 \zeta(4)/\pi^4 = 8/45$. In more general form we have

$$D_{l,v,w} = \frac{2^{2l}}{\pi^{2l}} \zeta(2l) \sum_{j=0}^w (-1)^{j+w} \binom{w}{j} c_{2j+2v,j+v-l} .$$

Ideally, this result is valid for $l < v$. In fact, it is identical to (5.67) if l is replaced by $v-l$. However, it also yields the correct coefficients of the solutions if we make sure

that $c_{\rho,-k} = 0$ for $k > 0$. This can be accomplished by introducing the Heaviside step-function in the following form:

$$\Theta(k) = \begin{cases} 1, & k \geq 0, \\ 0, & k < 0. \end{cases}$$

Then the coefficients can be expressed as

$$(5.68) \quad D_{l,v,w} = \frac{2^{2l}}{\pi^{2l}} \zeta(2l) \sum_{j=0}^w (-1)^{j+w} \binom{w}{j} c_{2j+2v,j+v-l} \Theta(j+v-l).$$

The above result can be programmed in Mathematica [46] as follows:

```
D[l_, v_, w_] := (2/Pi)^(2 l) Zeta[2 l] Sum[(-1)^(j + w) Binomial[w, j] c[2 j + 2 v, j + v - l] HeavisideTheta[j + v - l + 1/2], {j, 0, w}]
```

Note that the argument of the Heaviside step-function has been bolstered by $1/2$ to overcome the fact that `HeavisideTheta[0]` is undefined in Mathematica, whereas we require it to yield unity according to the above definition. If we should enter `D[9,4,5]` and `D[8,4,5]` into Mathematica, then we obtain respectively the values of $11499470848/38979295480125$ and $-237043712/162820783125$. These are identical to the coefficients of the m^{18} - and m^{16} -terms when (5.63) is expanded out via the `Expand` routine in Mathematica. Hence we have seen that (5.68) does not need to be restricted to $l < v$ as in our earlier derivation leading to (5.67). Furthermore, although the $D_{l,v,w}$ depend upon the even integer values of the zeta function, the powers of π are always cancelled by the $(\pi/2)^{2i}$ factor in (3.41) and the factor of $(2/\pi)^{2j+2v}$ outside of the summation over i in (5.64).

For the $\ell = 1$ case of (5.59), which represents the class of sums to which the untwisted Dowker sum belongs, we can use the results of Table 4.4 to determine the solutions whenever $v + w \leq 15$. If we denote the quantities listed in the table as $T_v(m)$, i.e., $T_v(m) = S_{m,v,1}/(m^2 - 1)$, then we find that the sums are given by

$$(5.69) \quad S_{m,v,w,1}^{CC} = (m^2 - 1) \sum_{j=0}^v (-1)^{v-j} \binom{v}{j} T_{w+j}(m).$$

In particular, for $v = 6$, $w = 3$ and with the aid of the corresponding values in Table 4.4 the above result yields

$$(5.70) \quad \sum_{k=1}^{m-1} \cot^{12}\left(\frac{k\pi}{m}\right) \csc^6\left(\frac{k\pi}{m}\right) = \frac{1}{194896477400625} (m^2 - 1) (m^2 - 4) (-29787342748 + 1960688815m^2 + 3595494399m^4 - 275848135m^6 - 335395979m^8 + 98107275m^{10} - 10795297m^{12} + 438670m^{14}).$$

It should be mentioned here that both (5.63) and (5.70) have been checked for specific values of m by calculating the numerical or decimal values of the sums on

the lhs and comparing them with the decimal values from the rational quantities on the rhs's of both equations in Mathematica [46].

As in the case of $S_{v,w,2}^{CC}(m)$, the solutions of $S_{v,w,1}^{CC}(m)$ can also be tabulated for various values of v and w using (5.69), although as we shall see, these are presented here more for completeness than out of sheer necessity. Table 5.6 presents the solutions of $S_{v,w,1}^{CC}(m)$ for v ranging from 0 to 3 and w from 1 to 4. For good reason, it can be seen that they have a similar structure to the results in Table 5.5. In this instance the $v = 0$ results have a common factor of $(m-1)(m-2)$, while the $v > 0$ results have a common factor of $(m^2-1)(m^2-4)$. As in the previous table, the $v = 0$ solutions are polynomials in m of degree $2w$, while the $v > 0$ solutions are polynomials in even powers of m with degree $2v+2w$. The values in the numerator are also identical to those in Table 5.5, but there is no factor of 2^v in the numerator. However, the really interesting point is that the coefficients are related to the coefficients of the corresponding solutions in Table 5.5. That is, the results in both tables are related to each other by either

$$S_{v,w,2}^{CC}(m) = \frac{1}{2} S_{v,w,1}^{CC}(2m) .$$

or

$$(5.71) \quad S_{v,w,1}^{CC}(m) = 2 S_{v,w,2}^{CC}(m/2) .$$

Since the solutions of $S_{v,w,1}^{CC}(m)$ are polynomials in even powers of m with degree $2v+2w$, they can be represented by $S_{v,w,1}^{CC}(m) = \sum_{l=0}^{v+w} E_{l,v,w} m^{2l}$. However, from (5.71) we observe that $E_{l,v,w} = 2^{1-2l} D_{l,v,w}$. On the other hand, $S_{v,w,1}^{CC}(m)$ can be expressed in terms of the $q_{v,i}$ as

$$S_{v,w,1}^{CC}(m) = \sum_{j=0}^v (-1)^{v-j} \binom{v}{j} \sum_{i=0}^{w+j} q_{j+w,i} m^{2i} .$$

These solutions can also be written in terms of the symmetric polynomials over the quadratic powers of the integers from 1 to v by introducing (4.54) and (4.55) into the above result.

Another trigonometric power sum that can be studied with the results of the previous sections is

$$(5.72) \quad S_{v,w,\ell}^{TS}(m) := \sum_{k=1}^{m-1} \tan^{2v} \left(\frac{k\pi}{\ell m} \right) \sec^{2w} \left(\frac{k\pi}{\ell m} \right) ,$$

where the same conditions apply to v and w as in (5.59). For $\ell = 2$, if we replace k by $m-k$, then we find that $S_{v,w,2}^{TS}(m) = S_{v,w,2}^{CC}(m)$. Hence there is no need to analyze these sums.

On the other hand, for $\ell = 1$, there is a possibility that one of the values of k can produce powers of $\cos(\pi/2)$ in the denominator when m is an even integer.

v	w	$S_{v,w,2}^{CC}(m)$
0	1	$\frac{1}{3}(m-1)(m-2)$
0	2	$\frac{1}{45}(m-1)(m-2)(m^2+3m-13)$
0	3	$\frac{1}{945}(m-1)(m-2)(2m^4+6m^3-28m^2-96m+251)$
0	4	$\frac{1}{14175}(m-1)(m-2)(3m^6+9m^5-59m^4-195m^3+457m^2+1761m-3551)$
1	1	$\frac{1}{45}(m^2-1)(m^2-4)$
1	2	$\frac{1}{945}(m^2-1)(m^2-4)(2m^2-11)$
1	3	$\frac{1}{14175}(m^2-1)(m^2-4)(3m^4-35m^2+107)$
1	4	$\frac{1}{467775}(m^2-1)(m^2-4)(10m^6-181m^4+1145m^2-2549)$
2	1	$\frac{2}{945}(m^2-1)(m^2-4)(m^2+5)$
2	2	$\frac{1}{14175}(m^2-1)(m^2-4)(3m^4-5m^2-58)$
2	3	$\frac{2}{467775}(m^2-1)(m^2-4)(5m^6-41m^4-5m^2+491)$
2	4	$\frac{1}{638512875}(m^2-1)(m^2-4)(1382m^8-20390m^6+81711m^4+75415m^2-799618)$
3	1	$\frac{1}{14175}(m^2-1)(m^2-4)(3m^4+25m^2+92)$
3	2	$\frac{1}{467775}(m^2-1)(m^2-4)(10m^6+17m^4-175m^2-932)$
3	3	$\frac{1}{638512875}(m^2-1)(m^2-4)(1382m^8-107840m^6-6740m^4+61765m^2+540812)$
3	4	$\frac{1}{383107725}(m^2-1)(m^2-4)(84m^{10}-962m^8+1406m^6+13881m^4-8005m^2-165164)$

Table 5.6: $S_{v,w,1}^{CC}(m)$ for various values of v and w

Therefore, this value of k needs to be excluded when $m = 2n$ and n is a non-zero integer. Thus, the above sum needs to be modified to

$$(5.73) \quad S_{v,w,1}^{TS}(2n) = \sum_{\substack{k=1 \\ k \neq n}}^{2n-1} \tan^{2v} \left(\frac{k\pi}{2n} \right) \sec^{2w} \left(\frac{k\pi}{2n} \right).$$

We now split the above sum into two separate sums, the first ranging from $k = 1$ to $n - 1$ and the second, from $k = n + 1$ to $2n$. In the second sum we replace k by $2n - k$, which yields the first sum again. Consequently, (5.73) reduces to

$$S_{v,w,1}^{TS}(2n) = 2 \sum_{k=1}^{n-1} \tan^{2v} \left(\frac{k\pi}{2n} \right) \sec^{2w} \left(\frac{k\pi}{2n} \right),$$

By replacing k by $n - k$, we find that $S_{v,w,1}^{TS}(2n) = 2S_{v,w,2}^{CC}(n)$. The case of $m = 2n + 1$ can also be reduced, but it yields a second trigonometric power sum with an alternating summand, i.e., a summand with an extra phase factor of $(-1)^k$. These types of sums will be addressed in the future work mentioned previously. Finally, we add that the $\ell = 1$, $w = 0$ and $m = 2n + 1$ case of (5.72) has been studied by Shevelev and Moses in [41], where they give the polynomial values of the sum for the first five values of v .

6. CONCLUSION

In this paper we have presented a novel integral approach for evaluating the Gardner-Fisher sum or $S_{v,2}(m)$ as defined by either (1.1) or (1.2), which is not only computationally expedient compared with other methods, but has also enabled us to quantify the coefficients of the polynomials as evidenced by (3.39) to (3.40). In so doing, we have been able to relate the generalized cosecant numbers of [28] to the symmetric polynomials $s(v, n)$ over the set of quadratic powers, $\{1, 2^2, 3^2, \dots, (v-1)^2\}$, via (3.44), which has been achieved formally in Section 3 or by matching the general result given here by (3.12) with the empirically determined results of (43) and (44) obtained in [27].

To demonstrate the versatility of our integral approach, it was extended to situations where the argument of $\pi/2$ in the cosecant power of the Gardner-Fisher sum was replaced by π/ℓ . Consequently, we were able to evaluate the sums for the $\ell = 1$ case or $S_{v,1}(m)$, which is known as the untwisted Dowker sum [9] and has been studied extensively by Cvijović and Srivastava [7]. The latter authors obtain a general result for the sum, which is given by another unwieldy sum whose summand is a product of Bernoulli numbers and the esoteric Nörlund polynomials as in (4.56). As a result, one is unable to ascertain the number-theoretical forms for the coefficients of the polynomials in $S_{v,1}(m)$. Hence one has to rely on a software package such as Mathematica [46] to generate the final forms for the untwisted Dowker sum. Nevertheless, the five values presented in [7] agree with the first five results in Table 4.4. Furthermore, by comparing their form for the untwisted Dowker sum with our (4.53), we are able to express the specific Nörlund polynomials in their result either in terms of the symmetric polynomials $s(v, n)$ presented here or in terms of the generalized cosecant numbers. An interesting by-product of the integral approach is that we were able to cast even integer values of the Hurwitz zeta function in terms of ratios of the generalized cosecant numbers. Furthermore, by employing the results in Sections 2-4, we were able to study more intricate sums involving products of powers of cotangent and tangent with powers of cosecant and secant respectively. The coefficients of the polynomial solutions in m to these sums were also found to be dependent upon the generalized cosecant numbers.

This paper has resulted in a cross-fertilization of the fields of classical analysis, number theory and computational/experimental mathematics. Moreover, it represents the introductory work of a more ambitious programme where the concepts and methods presented here and in [27, 28] are to be extended beyond the trigonometric power sums appearing in [1] and [6]. Included in this investigation will be the cases where the summands alternate in sign due to a phase factor of $(-1)^k$. In addition, this investigation will involve the extra cosine factor that appears in the twisted Dowker sums as in (4.47) as well as its sine analogue. Once again, the existing results for some of these sums are in a similar unwieldy form as (4.56) and therefore, are not able to provide the interesting number-theoretical behaviour associated with the polynomial solutions.

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Carlos M. da Fonseca

University of Primorska

FAMNIT

Glagoljaška 8, 6000 Koper

Slovenia

E-mail: *carlos.dafonseca@famnit.upr.si*

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M. Lawrence Glasser

Department of Physics

Clarkson University

Potsdam, NY 13699-5820

USA

E-mail: *laryg@clarkson.edu*

Victor Kowalenko

Department of Mathematics and Statistics

The University of Melbourne

Victoria 3010

Australia

E-mail: *vkowa@unimelb.edu.au*