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SOME IDENTITIES RELATED TO EULERIAN POLYNOMIALS AND INVOLVING THE STIRLING NUMBERS

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In the paper, the authors establish two identities, which can be regarded as nonlinear differential equations, for the generating function of Eulerian polynomials, find two identities for the Stirling numbers of the second kind, present two identities for Eulerian polynomials and higher order Eulerian polynomials, and pose two open problems about summability of two finite sums involving the Stirling numbers of the second kind. Some of these conclusions meaningfully and significantly simplify several known results.

1. MOTIVATIONS

In [**6, 7**], Kims stated that Eulerian polynomials $A_n(t)$ for $n \ge 0$ can be generated by

$$\frac{1-t}{e^{x(t-1)}-t} = \sum_{n=0}^{\infty} A_n(t)\frac{x^n}{n!}, \quad t \ne 1$$

and that higher order Eulerian polynomials $A_n^{(\alpha)}(t)$ for integers $n \ge 0$ and real numbers $\alpha > 0$ can be generated by

$$\left[\frac{1-t}{e^{x(t-1)}-t}\right]^{\alpha} = \sum_{n=0}^{\infty} A_n^{(\alpha)}(t)\frac{x^n}{n!}, \quad t \ne 1.$$

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This generation of $A_n(t)$ is same as the one in [**1**, p. 2], but different from the one

$$\frac{1-u}{e^{t(u-1)}-u}=1+\sum_{n=1}^{\infty}\frac{A_n(u)}{u}\frac{t^n}{n!}$$

in [**2**, p. 244, Eq. [5j]].

In [**7**, Theorem 1], Kims established inductively and recurrently that the generating function

$$F(t,x)=\frac{1}{e^{x(t-1)}-t},\quad t\ne 1 \tag{1}$$

satisfies the nonlinear ordinary differential equation

$$\frac{\partial^n F(t,x)}{\partial x^n}=(1-t)^n\sum_{i=1}^{n+1}a_{i-1}(n,t)F^i(t,x),\quad n\in\{0\}\cup\mathbb{N}, \tag{2}$$

where

$$a_0(n,t)=a_0(n-1,t)=\dots=a_0(1,t)=a_0(0,t)=1 \tag{3}$$

and

$$a_i(n,t)=it\sum_{j=0}^{n-i}(i+1)^j a_{i-1}(n-j-1,t),\quad 1\le j\le n. \tag{4}$$

In [**7**, Theorems 2 and 3], Kims presented that

$$A_{k+n}(t)=(1-t)^{n+1}\sum_{i=1}^{n+1}a_{i-1}(n,t)\frac{A_k^{(i)}(t)}{(1-t)^i}$$

and

$$\sum_{j=0}^{\infty}t^j(j+1)^{k+n}=\frac{1}{(1-t)^k}\sum_{i=1}^{n+1}a_{i-1}(n,t)\frac{A_k^{(i)}(t)}{(1-t)^i}$$

for $k,n\in\{0\}\cup\mathbb{N}$. From (3) and (4), Kims derived inductively that

$$\begin{aligned}a_i(n,t)=i!t^i\sum_{j_{i-1}=0}^{n-i}\sum_{j_{i-2}=0}^{n-j_{i-1}-i}\cdots\sum_{j_1=0}^{n-j_{i-1}-\dots-j_2-i}&(i+1)^{j_{i-1}}\\ \times i^{j_{i-2}}\cdots 3^{j_1}\bigl(2^{n-j_{i-1}-j_{i-2}-\dots-j_1-i+1}-1\bigr)&\end{aligned} \tag{5}$$

for $1\le i\le n$.

It is clear that the above formulas (4) and (5) for $a_i(n,t)$ cannot be computed easily either by hand or by computer software. Can one find a simple expression for the quantities $a_i(n,t)$? For supplying a solution to this problem, the first and third authors of this paper obtained in [**29**] the following three theorems.

Theorem 1 ([**29**, Theorem 1])**.** *Eulerian polynomials* $A_n(t)$ *and higher order Eulerian polynomials* $A_n^{(\alpha)}(t)$ *can be computed by*

$$A_n(t) = \sum_{k=0}^{n} k!S(n,k)(t-1)^{n-k} \tag{6}$$

and

$$A_n^{(\alpha)}(t) = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} \Gamma(k+\alpha)S(n,k)(t-1)^{n-k}, \tag{7}$$

where $n \ge 0$ *is an integer,* $\alpha > 0$ *is a real number,* $S(n,k)$*, which can be generated by the exponential function*

$$\frac{(e^x-1)^k}{k!} = \sum_{n=k}^{\infty} S(n,k)\frac{x^n}{n!}$$

and can be computed by the explicit formula

$$S(n,k) = \frac{1}{k!}\sum_{\ell=1}^{k}(-1)^{k-\ell}\binom{k}{\ell}\ell^n,$$

stand for the Stirling numbers of the second kind, and $\Gamma(z)$ *denotes the classical Euler gamma function.*

Theorem 2 ([**29**, Theorem 2])**.** *The generating function* $F(t,x)$ *satisfies the nonlinear ordinary differential equations*

$$\frac{\partial^n F(t,x)}{\partial x^n} = (t-1)^n \sum_{i=0}^{n}\left[\sum_{k=i}^{n}(-1)^k k!S(n,k)\binom{k}{i}\right] t^i F^{i+1}(t,x) \tag{8}$$

and, generally,

$$\frac{\partial^n F^{\alpha}(t,x)}{\partial x^n} = \frac{(t-1)^n}{\Gamma(\alpha)} \sum_{i=0}^{n}\left[\sum_{k=i}^{n}(-1)^k \Gamma(k+\alpha)S(n,k)\binom{k}{i}\right] t^i F^{\alpha+i}(t,x), \tag{9}$$

where $n \ge 0$ *is an integer and* $\alpha > 0$ *is a real number.*

Theorem 3 ([**29**, Theorem 3])**.** *For* $n \in \mathbb{N}$ *and* $\alpha > 0$*, higher order Eulerian polynomials* $A_n^{(\alpha)}(t)$ *satisfy the recurrence relation*

$$\sum_{k=0}^{n}\binom{n}{k}\left[\sum_{\ell=0}^{n-k} S(n-k,\ell)\frac{\langle\alpha\rangle_\ell}{(1-t)^\ell}\right]\frac{A_k^{(\alpha)}(t)}{(t-1)^k} = 0, \tag{10}$$

where

$$\langle\alpha\rangle_n = \prod_{k=0}^{n-1}(\alpha-k) = \begin{cases}\alpha(\alpha-1)\cdots(\alpha-n+1), & n \ge 1\\ 1, & n = 0\end{cases}$$

is called the falling factorial. In particular, when $\alpha = 1$,

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left[\frac{S(n-k,0)}{(1-t)^{k-1}}+\frac{S(n-k,1)}{(1-t)^k}\right]A_k(t)=0.$$

It is easy to see that the equation (8) simplifies the one (2). As stated in [**29**, Remarks 1 and 2], comparing the equation (2) with (8) reveals that

$$a_i(n,t)=\left[\sum_{k=i}^{n}(-1)^k k!S(n,k)\binom{k}{i}\right]t^i, \quad 0\le i\le n. \tag{11}$$

This expression is simpler, more significant, and more meaningful than the one in (5). It is easier to compute the quantities in brackets of the nonlinear ordinary differential equations (8) and (9) than to compute the quantity $a_i(n,t)$ in (5).

In [**6**, Theorem 1], it was obtained inductively and recursively that the nonlinear differential equations

$$n!t^n(1-t)^nF^{n+1}(t,x)=\sum_{i=0}^{n}a_i(n)(1-t)^{n-i}\frac{\partial^i F(t,x)}{\partial x^i}, \quad n\in\mathbb{N} \tag{12}$$

have a solution $F(t,x)$ defined by (1) for $t\ne 1$, where $a_0(n)=(-1)^n n!$,

$$a_i(n)=(-1)^{n-i}n!H_{n,i}, \quad 1\le i\le n, \tag{13}$$

with $H_{n,0}=1$ for $n\in\mathbb{N}$, $H_{n,1}=H_n=\sum_{k=1}^{n}\frac{1}{k}$ for $n\in\mathbb{N}$, and $H_{n,i}=\sum_{k=i}^{n}\frac{H_{k-1,i-1}}{k}$ for $2\le i\le n$. For more information on $H_{n,i}$, please refer to [**43**, Remark 1] and closely related references therein. Therefore, the following results were derived in [**6**, Theorems 2 and 3]:

1. For $n,k\ge 0$,

$$n!t^nA_k^{(n+1)}(t)=\sum_{i=0}^{n}a_i(n)(1-t)^{n-i}A_{k+i}(t).$$

2. For $k\ge 1$ and $m\ge k+n$,

$$n!t^nA_k^{(n+1)}(t)=\sum_{m=0}^{k+n-1}\sum_{i=0}^{n}\sum_{\ell=0}^{m}(-1)^\ell\binom{n+k+1}{\ell}(m-\ell+1)^{k+i}a_i(n)t^m$$

and

$$\sum_{i=0}^{n}\sum_{\ell=0}^{m}(-1)^\ell\binom{n+k+1}{\ell}(m-\ell+1)^{k+i}a_i(n)=0$$

3. For $k=0$ and $m\ge n+1$,

$$n!t^nA_0^{(n+1)}(t)=\sum_{m=0}^{n}\sum_{i=0}^{n}\sum_{\ell=0}^{m}(-1)^\ell\binom{n+1}{\ell}(m-\ell+1)^{i}a_i(n)t^m$$

and

$$\sum_{i=0}^{n}\sum_{\ell=0}^{m}(-1)^{\ell}\binom{n+1}{\ell}(m-\ell+1)^{i}a_i(n)=0.$$

We observe that the equations (2) and (12) can be rewritten respectively as

$$\frac{1}{(1-t)^n}\frac{\partial^n F(t,x)}{\partial x^n}=\sum_{i=0}^{n}a_i(n,t)F^{i+1}(t,x),\quad n\in\{0\}\cup\mathbb{N}$$

and

$$n!t^nF^{n+1}(t,x)=\sum_{i=0}^{n}a_i(n)\frac{1}{(1-t)^i}\frac{\partial^i F(t,x)}{\partial x^i},\quad n\in\{0\}\cup\mathbb{N}. \tag{14}$$

Consequently, the equations (2) and (12) are essentially inversive to each other. This motivates us to consider two questions:

1. can one simplify the expression of the quantities $a_i(n)$ significantly and meaningfully?
2. what are the inversive ones of the equations (8) and (9)?

2. MAIN RESULTS AND THEIR PROOFS

Now we are in a position to state and prove our main results.

Theorem 4. *For $n\ge 0$, the function $F(t,x)$ defined by* (1) *satisfies nonlinear differential equations*

$$F^{n+1}(t,x)=\frac{1}{n!t^n}\sum_{i=0}^{n}\frac{s(n+1,i+1)}{(1-t)^i}\frac{\partial^i F(t,x)}{\partial x^i}, \tag{15}$$

where $s(n,k)$, which can be generated by

$$\frac{[\ln(1+x)]^k}{k!}=\sum_{n=k}^{\infty}s(n,k)\frac{x^n}{n!},\quad |x|<1,$$

stand for the Stirling numbers of the first kind.

Proof. In [**32**] and [**43**], it was obtained that

$$(-1)^{n+k}s(n,k)=(n-1)!H_{n-1,k-1},\quad n\ge k\ge 1. \tag{16}$$

Substituting the relations (13) and (16) into (12) or (14) yields

$$n!t^nF^{n+1}(t,x)=\sum_{i=0}^{n}(-1)^{n-i}n!H_{n,i}\frac{1}{(1-t)^i}\frac{\partial^i F(t,x)}{\partial x^i}$$

$$= \sum_{i=0}^{n} s(n+1, i+1) \frac{1}{(1-t)^i} \frac{\partial^i F(t,x)}{\partial x^i}$$

for $n \geq 0$. The proof of Theorem 4 is complete. □

Theorem 5. *For $n \geq 0$, the function $F(t,x)$ defined by* (1) *satisfies nonlinear differential equations*

$$\frac{\partial^n F(t,x)}{\partial x^n} = (1-t)^n \sum_{k=0}^{n} S(n+1, k+1) k! t^k F^{k+1}(t,x). \tag{17}$$

The Stirling numbers of the second kind $S(n,k)$ satisfy the identities

$$\sum_{\ell=k}^{n} (-1)^\ell \binom{\ell}{k} \ell! S(n,\ell) = (-1)^n k! S(n+1, k+1), \quad n \geq k \geq 0 \tag{18}$$

and

$$\sum_{\ell=k+1}^{n} (-1)^\ell \binom{\ell-1}{k} \ell! S(n,\ell) = (-1)^n (k+1)! S(n, k+1), \quad n > k \geq 0. \tag{19}$$

Proof. Rewriting the equations in (15) as

$$\begin{pmatrix} F(t,x) \\ tF^2(t,x) \\ \vdots \\ n! t^n F^{n+1}(t,x) \end{pmatrix} = M_{(n+1)\times(n+1)} \begin{pmatrix} F(t,x) \\ \frac{1}{1-t} \frac{\partial F(t,x)}{\partial x} \\ \vdots \\ \frac{1}{(1-t)^n} \frac{\partial^n F(t,x)}{\partial x^n} \end{pmatrix},$$

where the $(n+1)\times(n+1)$ matrix

$$M_{(n+1)\times(n+1)} = \begin{pmatrix} s(1,1) & 0 & 0 & \cdots & 0 \\ s(2,1) & s(2,2) & 0 & \cdots & 0 \\ s(3,1) & s(3,2) & s(3,3) & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ s(n+1,1) & s(n+1,2) & s(n+1,3) & \cdots & s(n+1,n+1) \end{pmatrix}.$$

Since the inverse matrix

$$M_{(n+1)\times(n+1)}^{-1} = \begin{pmatrix} S(1,1) & 0 & 0 & \cdots & 0 \\ S(2,1) & S(2,2) & 0 & \cdots & 0 \\ S(3,1) & S(3,2) & S(3,3) & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ S(n+1,1) & S(n+1,2) & S(n+1,3) & \cdots & S(n+1,n+1) \end{pmatrix},$$

see [**2**, p. 213, eq. [5c]], it follows immediately that

$$\begin{pmatrix} F(t,x) \\ \frac{1}{1-t}\frac{\partial F(t,x)}{\partial x} \\ \frac{1}{(1-t)^2}\frac{\partial^2 F(t,x)}{\partial x^2} \\ \vdots \\ \frac{1}{(1-t)^n}\frac{\partial^n F(t,x)}{\partial x^n} \end{pmatrix} = M_{(n+1)\times(n+1)}^{-1} \begin{pmatrix} F(t,x) \\ tF^2(t,x) \\ 2!t^2F^3(t,x) \\ \vdots \\ n!t^nF^{n+1}(t,x) \end{pmatrix},$$

that is,

$$\frac{1}{(1-t)^n}\frac{\partial^n F(t,x)}{\partial x^n} = \sum_{k=1}^{n+1} S(n+1,k)(k-1)!t^{k-1}F^k(t,x), \quad n \ge 0.$$

This is equivalent to (17). Comparing (17) with (8) leads to (18).

In [**47**, p. 118, Eq. (9.18)], it was proved that

$$\sum_{j=\alpha}^{n}(-1)^{n-j}\binom{j-1}{\alpha-1}j!S(n,j) = \alpha!S(n,\alpha), \quad n \ge \alpha \ge 0. \tag{20}$$

By virtue of (18) and (20), we obtain

$$\sum_{\ell=k}^{n}(-1)^{\ell}\left[\binom{\ell}{k} - \binom{\ell-1}{k-1}\right]\ell!S(n,\ell) = (-1)^n k![S(n+1,k+1) - S(n,k)].$$

Since the recurrence relations

$$\binom{x}{j} = \binom{x-1}{j} + \binom{x-1}{j-1}, \quad x \in \mathbb{C}, \quad j \in \{0\} \cup \mathbb{N}$$

and

$$S(n+1,k) = kS(n,k) + S(n,k-1), \quad 0 \le k-1 \le n,$$

see [**47**, p. 8, Eq. (1.27)] and [**47**, p. 114, Eq. (9.1)], we arrive at the identity (19) straightforwardly. The proof of Theorem 5 is complete. □

Theorem 6. *For $n \ge 0$, Eulerian polynomials $A_n(t)$ and higher order Eulerian polynomials $A_k^{(\alpha)}(t)$ satisfy*

$$\sum_{k=0}^{n}\frac{s(n,k)}{(t-1)^k}A_k(t) = \frac{n!}{(t-1)^n} \tag{21}$$

and

$$\sum_{k=0}^{n}\frac{s(n,k)}{(t-1)^k}A_k^{(\alpha)}(t) = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)}\frac{1}{(t-1)^n}. \tag{22}$$

Proof. Theorem 12.1 in [**47**, p. 171] reads that, if b_α and a_k are a collection of constants independent of n, then

$$a_n = \sum_{\alpha=0}^{n} S(n,\alpha) b_\alpha \quad \text{if and only if} \quad b_n = \sum_{k=0}^{n} s(n,k) a_k.$$

The identity (6) can be rearranged as

$$\frac{A_n(t)}{(t-1)^n} = \sum_{k=0}^{n} S(n,k) \frac{k!}{(t-1)^k}.$$

Consequently, it follows that

$$\frac{n!}{(t-1)^n} = \sum_{k=0}^{n} s(n,k) \frac{A_k(t)}{(t-1)^k}$$

which can be rewritten as (21).

Similarly, the identity (7) can also be reformulated as

$$\frac{\Gamma(\alpha) A_n^{(\alpha)}(t)}{(t-1)^n} = \sum_{k=0}^{n} S(n,k) \frac{\Gamma(k+\alpha)}{(t-1)^k}$$

and, consequently,

$$\frac{\Gamma(n+\alpha)}{(t-1)^n} = \sum_{k=0}^{n} s(n,k) \frac{\Gamma(\alpha) A_k^{(\alpha)}(t)}{(t-1)^k}$$

which can be rearranged as (22). The required proof is complete. □

3. REMARKS

In this section, among other things, we give several remarks about our main results and pose two open problems.

Remark 1. The expressions (11) and (13) for $a_i(n,t)$ and $a_i(n)$ can be meaningfully and significantly simplified as

$$a_i(n,t) = (-1)^n i! S(n+1, i+1) t^i, \quad 0 \le i \le n$$

and

$$a_i(n) = s(n+1, i+1).$$

As a result, those main results in [**6, 7**], mentioned in the first section, can be meaningfully and significantly simplified. For the sake of saving the space and shortening the length of this paper, we do not rewrite them in details here. This answers the first question posed in the first section of this paper.

Remark 2. The identity (15) in Theorem 4 partially answers the second question posed in the first section of this paper.

Remark 3. Theorem 3 is a simplified and reformulated version of Theorem 3 in [**29**].

Remark 4. The identity (17) in Theorem 5 simplifies the one (8) in Theorem 2.

Remark 5. To the best of our knowledge, identities (18) and (19) are new.

Remark 6. Motivated by identities (18), (19), and (20), we naturally pose another open problem: for $n \ge k \ge 0$ and $\alpha > 0$, is the finite sum

$$\sum_{k=i}^{n}(-1)^k\binom{k}{i}\Gamma(k+\alpha)S(n,k)$$

in the bracket of the identity (9) summable? The solution of this problem can be used to partially answer the second question posed in the first section of this paper.

Remark 7. Theorem 12.2 in [**47**, p. 171] states that, if b_j and a_k are a collection of constants which are independent of n and if α is a nonnegative integer such that $\alpha \ge n$, then

$$a_n = \sum_{j=0}^{\alpha} S(j,n)b_j \quad \text{if and only if} \quad b_n = \sum_{k=0}^{\alpha} s(k,n)a_k.$$

Motivated by the above mentioned Theorems 12.1 and 12.2 in [**47**, p. 171] and by identities (18), (19), and (20), we naturally pose an open problem more: is the finite sum

$$\sum_{\ell=0}^{n-k} S(n-k,\ell)\langle\alpha\rangle_\ell t^\ell$$

in the bracket of the identity (10), or equivalently,

$$\sum_{\ell=0}^{n} S(n,\ell)\langle\alpha\rangle_\ell t^\ell,$$

summable?

Remark 8. For some new results on the gamma function $\Gamma(z)$, please refer to [**8, 18, 28, 30, 36**] and closely related references therein.

Remark 9. For some recent development of the Stirling numbers of the first and second kinds, please refer to [**3, 4, 5, 9, 12, 13, 14, 15, 17, 26, 27, 41**] and closely related references therein.

Remark 10. The motivation of this paper is same as the one in [**10, 11, 16, 19, 20, 21, 22, 23, 24, 25, 27, 29, 31, 32, 33, 34, 37, 38, 39, 40, 42, 43, 44, 45, 46, 48**] and closely related references therein.

Remark 11. This paper is a slightly revised version of the preprint [**35**].

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REFERENCES

1. P. Barry, *Eulerian polynomials as moments, via exponential Riordan arrays*, J. Integer Seq. **14** (2011), no. 9, Article 11.9.5, 14 pages.
2. L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, Revised and Enlarged Edition, D. Reidel Publishing Co., Dordrecht and Boston, 1974; Available online at `https://doi.org/10.1007/978-94-010-2196-8`.
3. B.-N. Guo, I. Mező, and F. Qi, *An explicit formula for the Bernoulli polynomials in terms of the r-Stirling numbers of the second kind*, Rocky Mountain J. Math. **46** (2016), no. 6, 1919–1923; Available online at `https://doi.org/10.1216/RMJ-2016-46-6-1919`.
4. B.-N. Guo and F. Qi, *Explicit formulae for computing Euler polynomials in terms of Stirling numbers of the second kind*, J. Comput. Appl. Math. **272** (2014), 251–257; Available online at `https://doi.org/10.1016/j.cam.2014.05.018`.
5. B.-N. Guo and F. Qi, *Some identities and an explicit formula for Bernoulli and Stirling numbers*, J. Comput. Appl. Math. **255** (2014), 568–579; Available online at `https://doi.org/10.1016/j.cam.2013.06.020`.
6. D. S. Kim and T. Kim, *Revisit nonlinear differential equations associate with Eulerian polynomials*, Bull. Korean Math. Soc. **54** (2017), no. 4, 1185–1194; Available online at `https://doi.org/10.4134/BKMS.b160276`.
7. T. Kim and D. S. Kim, *Some identities of Eulerian polynomials arising from nonlinear differential equations*, Iran. J. Sci. Technol. Trans. Sci. (2017), in press; Available online at `https://doi.org/10.1007/s40995-016-0073-0`.
8. F. Qi, *A completely monotonic function involving the gamma and trigamma functions*, Kuwait J. Sci. **43** (2016), no. 3, 32–40.
9. F. Qi, *A new formula for the Bernoulli numbers of the second kind in terms of the Stirling numbers of the first kind*, Publ. Inst. Math. (Beograd) (N.S.) **100(114)** (2016), 243–249; Available online at `https://doi.org/10.2298/PIM150501028Q`.
10. F. Qi, *A simple form for coefficients in a family of nonlinear ordinary differential equations*, Adv. Appl. Math. Sci. **17** (2018), no. 8, 555–561.
11. F. Qi, *A simple form for coefficients in a family of ordinary differential equations related to the generating function of the Legendre polynomials*, Adv. Appl. Math. Sci. **17** (2018), in press; ResearchGate Preprint (2017), available online at `https://doi.org/10.13140/RG.2.2.27365.09446`.
12. F. Qi, *An explicit formula for the Bell numbers in terms of the Lah and Stirling numbers*, Mediterr. J. Math. **13** (2016), no. 5, 2795–2800; Available online at `https://doi.org/10.1007/s00009-015-0655-7`.

13. F. Qi, *Diagonal recurrence relations, inequalities, and monotonicity related to the Stirling numbers of the second kind*, Math. Inequal. Appl. **19** (2016), no. 1, 313–323; Available online at https://doi.org/10.7153/mia-19-23.

14. F. Qi, *Diagonal recurrence relations for the Stirling numbers of the first kind*, Contrib. Discrete Math. **11** (2016), no. 1, 22–30; Available online at http://hdl.handle.net/10515/sy5wh2dx6 and https://doi.org/10515/sy5wh2dx6.

15. F. Qi, *Explicit formulas for computing Bernoulli numbers of the second kind and Stirling numbers of the first kind*, Filomat **28** (2014), no. 2, 319–327; Available online at https://doi.org/10.2298/FIL1402319O.

16. F. Qi, *Explicit formulas for the convolved Fibonacci numbers*, ResearchGate Working Paper (2016), available online at https://doi.org/10.13140/RG.2.2.36768.17927.

17. F. Qi, *Integral representations and properties of Stirling numbers of the first kind*, J. Number Theory **133** (2013), no. 7, 2307–2319; Available online at https://doi.org/10.1016/j.jnt.2012.12.015.

18. F. Qi, *Limit formulas for ratios between derivatives of the gamma and digamma functions at their singularities*, Filomat **27** (2013), no. 4, 601–604; Available online at https://doi.org/10.2298/FIL1304601Q.

19. F. Qi, *Notes on several families of differential equations related to the generating function for the Bernoulli numbers of the second kind*, Turkish J. Anal. Number Theory **6** (2018), no. 2, 40–42; Available online at https://doi.org/10.12691/tjant-6-2-1.

20. F. Qi, *Simple forms for coefficients in two families of ordinary differential equations*, Glob. J. Math. Anal. **6** (2018), no. 1, 7–9; Available online at https://doi.org/10.14419/gjma.v6i1.9778.

21. F. Qi, *Simplification of coefficients in two families of nonlinear ordinary differential equations*, ResearchGate Preprint (2017), available online at https://doi.org/10.13140/RG.2.2.30196.24966.

22. F. Qi, *Simplifying coefficients in a family of nonlinear ordinary differential equations*, Acta Comment. Univ. Tartu. Math. (2018), in press; ResearchGate Preprint (2017), available online at https://doi.org/10.13140/RG.2.2.23328.07687.

23. F. Qi, *Simplifying coefficients in a family of ordinary differential equations related to the generating function of the Laguerre polynomials*, ResearchGate Preprint (2017), available online at https://doi.org/10.13140/RG.2.2.13602.53448.

24. F. Qi, *Simplifying coefficients in a family of ordinary differential equations related to the generating function of the Mittag–Leffler polynomials*, ResearchGate Preprint (2017), available online at https://doi.org/10.13140/RG.2.2.27758.31049.

25. F. Qi, *Simplifying coefficients in differential equations related to generating functions of reverse Bessel and partially degenerate Bell polynomials*, Bol. Soc. Paran. Mat. (2019), in press; ResearchGate Preprint (2017), available online at https://doi.org/10.13140/RG.2.2.19946.41921.

26. F. Qi and B.-N. Guo, *A closed form for the Stirling polynomials in terms of the Stirling numbers*, Tbilisi Math. J. **10** (2017), no. 4, 153–158; Available online at https://doi.org/10.1515/tmj-2017-0053.

27. F. Qi and B.-N. Guo, *A diagonal recurrence relation for the Stirling numbers of the first kind*, Appl. Anal. Discrete Math. **12** (2018), no. 1, 153–165; Available online at https://doi.org/10.2298/AADM170405004Q.

28. F. Qi and B.-N. Guo, *An inequality involving the gamma and digamma functions*, J. Appl. Anal. **22** (2016), no. 1, 49–54; Available online at `https://doi.org/10.1515/jaa-2016-0005`.

29. F. Qi and B.-N. Guo, *Explicit formulas and recurrence relations for higher order Eulerian polynomials*, Indag. Math. **28** (2017), no. 4, 884–891; Available online at `https://doi.org/10.1016/j.indag.2017.06.010`.

30. F. Qi and B.-N. Guo, *Integral representations and complete monotonicity of remainders of the Binet and Stirling formulas for the gamma function*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM **111** (2017), no. 2, 425–434; Available online at `https://doi.org/10.1007/s13398-016-0302-6`.

31. F. Qi and B.-N. Guo, *Some properties of the Hermite polynomials and their squares and generating functions*, Preprints **2016**, 2016110145, 14 pages; Available online at `https://doi.org/10.20944/preprints201611.0145.v1`.

32. F. Qi and B.-N. Guo, *Viewing some ordinary differential equations from the angle of derivative polynomials*, Iran. J. Math. Sci. Inform. **14** (2019), no. 2, in press; Preprints **2016**, 2016100043, 12 pages; Available online at `https://doi.org/10.20944/preprints201610.0043.v1`.

33. F. Qi, D. Lim, and A.-Q. Liu, *Explicit expressions related to degenerate Cauchy numbers and their generating function*, HAL archives (2018), available online at `https://hal.archives-ouvertes.fr/hal-01725045`.

34. F. Qi, D. Lim, and B.-N. Guo, *Explicit formulas and identities for the Bell polynomials and a sequence of polynomials applied to differential equations*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM (2018), in press; Available online at `https://doi.org/10.1007/s13398-017-0427-2`.

35. F. Qi, D. Lim, and B.-N. Guo, *Some identities relating to Eulerian polynomials and involving Stirling numbers*, Preprints **2017**, 2017080004, 10 pages; Available online at `https://doi.org/10.20944/preprints201708.0004.v1`.

36. F. Qi and M. Mahmoud, *Bounding the gamma function in terms of the trigonometric and exponential functions*, Acta Sci. Math. (Szeged) **83** (2017), no. 1-2, 125–141; Available online at `https://doi.org/10.14232/actasm-016-813-x`.

37. F. Qi, D.-W. Niu, and B.-N. Guo, *Simplification of coefficients in differential equations associated with higher order Frobenius–Euler numbers*, Preprints **2017**, 2017080017, 7 pages; Available online at `https://doi.org/10.20944/preprints201708.0017.v1`.

38. F. Qi, D.-W. Niu, and B.-N. Guo, *Simplifying coefficients in differential equations associated with higher order Bernoulli numbers of the second kind*, Preprints **2017**, 2017080026, 6 pages; Available online at `https://doi.org/10.20944/preprints201708.0026.v1`.

39. F. Qi, D.-W. Niu, and B.-N. Guo, *Some identities for a sequence of unnamed polynomials connected with the Bell polynomials*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM **112** (2018), in press; Available online at `https://doi.org/10.1007/s13398-018-0494-z`.

40. F. Qi, X.-L. Qin, and Y.-H. Yao, *The generating function of the Catalan numbers and lower triangular integer matrices*, Preprints **2017**, 2017110120, 12 pages; Available online at `https://doi.org/10.20944/preprints201711.0120.v1`.

41. F. Qi, X.-T. Shi, and F.-F. Liu, *Several identities involving the falling and rising factorials and the Cauchy, Lah, and Stirling numbers*, Acta Univ. Sapientiae Math. **8** (2016), no. 2, 282–297; Available online at https://doi.org/10.1515/ausm-2016-0019.

42. F. Qi, J.-L. Wang, and B.-N. Guo, *Notes on a family of inhomogeneous linear ordinary differential equations*, Adv. Appl. Math. Sci. **17** (2018), no. 4, 361–368.

43. F. Qi, J.-L. Wang, and B.-N. Guo, *Simplifying and finding nonlinear ordinary differential equations*, ResearchGate Working Paper (2017), available online at https://doi.org/10.13140/RG.2.2.28855.32166.

44. F. Qi, J.-L. Wang, and B.-N. Guo, *Simplifying differential equations concerning degenerate Bernoulli and Euler numbers*, Trans. A. Razmadze Math. Inst. **172** (2018), no. 1, 90–94; Available online at http://dx.doi.org/10.1016/j.trmi.2017.08.001.

45. F. Qi and J.-L. Zhao, *Some properties of the Bernoulli numbers of the second kind and their generating function*, Bull. Korean Math. Soc. **55** (2018), in press; Available online at https://doi.org/10.4134/BKMS.b180039.

46. F. Qi, Q. Zou, and B.-N. Guo, *Some identities and a matrix inverse related to the Chebyshev polynomials of the second kind and the Catalan numbers*, Preprints **2017**, 2017030209, 25 pages; Available online at https://doi.org/10.20944/preprints201703.0209.v2.

47. J. Quaintance and H. W. Gould, *Combinatorial Identities for Stirling Numbers*. The unpublished notes of H. W. Gould. With a foreword by George E. Andrews. World Scientific Publishing Co. Pte. Ltd., Singapore, 2016.

48. J.-L. Zhao, J.-L. Wang, and F. Qi, *Derivative polynomials of a function related to the Apostol–Euler and Frobenius–Euler numbers*, J. Nonlinear Sci. Appl. **10** (2017), no. 4, 1345–1349; Available online at https://doi.org/10.22436/jnsa.010.04.06.

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