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ASYMPTOTIC EXPANSIONS FOR CERTAIN MATHEMATICAL CONSTANTS AND SPECIAL FUNCTIONS

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For fixed real $b > 1$ and $\alpha > 0$, let $S_b^{[\alpha]}(n) = \sum_{k=1}^{n} b^k k^{-\alpha}$. Abel proved that $S_b^{[\alpha]}(n) \sim b^n \sum_{k=0}^{\infty} c_k n^{-(k+\alpha)}$ $(n \to \infty)$, and gave an explicit formula for determining the coefficients $c_k \equiv c_k(b, \alpha)$ in terms of Stirling numbers of the second kind. We here provide a recurrence relation for determining the coefficients c_k, without Stirling numbers. We also consider asymptotic expansions concerning Somos' quadratic recurrence constant, Glaisher-Kinkelin constant, Choi-Srivastava constants, and the Barnes G-function.

1. INTRODUCTION

Abel [**1**] derived a complete asymptotic expansion for a sequence of the following sum

$$S_b^{[\alpha]}(n) = \sum_{k=1}^{n} \frac{b^k}{k^{\alpha}} \tag{1}$$

as $n \to \infty$, for fixed real $b > 1$ and $\alpha > 0$. More precisely, Abel [**1**] established the following asymptotic expansion:

$$S_b^{[\alpha]}(n) \sim b^n \sum_{k=0}^{\infty} \binom{\alpha + k - 1}{k} \frac{a_k(b)}{n^{\alpha+k}} \qquad (n \to \infty), \tag{2}$$

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where $a_k(b)$ are given by

$$a_k(b) = b\sum_{j=0}^{k}\frac{j!\sigma(k,j)}{(b-1)^{j+1}}, \tag{3}$$

and $\sigma(k,j)$ denote Stirling numbers of the second kind. Stirling numbers of the second kind can be computed by the formula (see, e.g., [**54**, p. 99])

$$\sigma(k,j) = \frac{1}{j!}\sum_{i=1}^{j}(-1)^{j-i}\binom{j}{i}i^k.$$

Remark 1. *Hassani* [**34**] *proposed the following problem: Prove that*

$$\sum_{k=1}^{n}\frac{2^k-1}{k} = \frac{2^{n+1}}{n}(1+R_n), \quad \textit{where} \quad R_n \sim \frac{1}{n} \qquad (n\to\infty).$$

This problem has been proved by Giuliano [**31**] *Simic* [**51**]. *Abel* [**1**] *gave a complete asymptotic expansion for the sequence* $\{S_b(n)\}_{n\in\mathbb{N}}$ *of sums*

$$S_b(n) = \sum_{k=1}^{n}\frac{b^k-1}{k} \sim b^n\sum_{k=0}^{\infty}\frac{a_k(b)}{n^{k+1}} \qquad (n\to\infty),$$

as $n\to\infty$, *for fixed real* $b>1$. *The choice* $b=2$ *yields a solution of the problem* [**34**] *by Hassani. Also in* [**1**], *Abel derived a complete asymptotic expansion for* $S_b^{[\alpha]}(n)$, *and pointed out that* $S_b(n)$ *is asymptotically equivalent to* $S_b^{[1]}(n)$ *as* $n\to\infty$.

Alzer et al. [**3**] applied a classical series identity involving the psi function with a view to deriving series representations for a number of known mathematical constants. Chen and Srivastava [**18**] obtained several properties associated with inequalities and the logarithmically complete monotonicity of functions related to the gamma and psi functions and the Barnes G-function. Chen et al. [**21**] presented some properties associated with the monotonicity and the complete monotonicity of the psi function and establish the higher-order estimate for the familiar Euler-Mascheroni constant. Chen and Srivastava [**19**] established new analytical representations for the Euler-Mascheroni constant in terms of the psi function. Chen and Srivastava [**20**] established several further analytical representations for the Euler-Mascheroni constant in terms of the psi function.

In this paper, we aim to provide a recurrence relation for determining the coefficients of $n^{-(\alpha+k)}$ in Abel's expansion (2), without help of Stirling numbers of the second kind. We also consider asymptotic expansions concerning Somos' quadratic recurrence constant, Glaisher-Kinkelin constant, Choi-Srivastava constants, and the Barnes G-function.

2. COEFFICIENTS OF ABEL'S EXPANSION

Theorem 1. *For reals $\alpha > 0$ and $b > 1$, the following asymptotic formula holds*

$$S_b^{[\alpha]}(n) \sim b^n \sum_{k=0}^{\infty} \frac{c_k}{n^{k+\alpha}} \qquad (n \to \infty), \tag{4}$$

where the coefficients $c_k \equiv c_k(b, \alpha)$ are given by the recurrence relation

$$c_0 = \frac{b}{b-1}, \quad c_k = \frac{1}{b-1} \sum_{j=0}^{k-1} c_j \binom{k+\alpha-1}{k-j} \qquad (k \geq 1). \tag{5}$$

Namely,

$$\begin{aligned} S_b^{[\alpha]}(n) \sim \frac{b^{n+1}}{n^\alpha} \biggl\{ & \frac{1}{b-1} + \frac{\alpha}{(b-1)^2 n} + \frac{\alpha(1+\alpha)(b+1)}{2(b-1)^3 n^2} \\ & + \frac{\alpha(1+\alpha)(2+\alpha)(b^2+4b+1)}{6(b-1)^4 n^3} + \dots \biggr\} \qquad (n \to \infty). \end{aligned} \tag{6}$$

Proof. Denote

$$S_n \equiv S_b^{[\alpha]}(n) \quad \text{and} \quad T_n = b^n \sum_{k=0}^{\infty} \frac{c_k}{n^{k+\alpha}}.$$

In view of (2), we can let $S_n \sim T_n$ and

$$\Delta S_n := S_n - S_{n-1} \sim T_n - T_{n-1} =: \Delta T_n \tag{7}$$

as $n \to \infty$, where c_k are real numbers to be determined.

It is easy to see that

$$\Delta S_n = b^n n^{-\alpha}. \tag{8}$$

Direct computation yields

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{c_k}{(n-1)^{k+\alpha}} &= \sum_{k=0}^{\infty} \frac{c_k}{n^{k+\alpha}} \left(1 - \frac{1}{n}\right)^{-k-\alpha} = \sum_{k=0}^{\infty} \frac{c_k}{n^{k+\alpha}} \sum_{j=0}^{\infty} (-1)^j \binom{-k-\alpha}{j} \frac{1}{n^j} \\ &= \sum_{k=0}^{\infty} \frac{c_k}{n^{k+\alpha}} \sum_{j=0}^{\infty} \binom{k+\alpha+j-1}{j} \frac{1}{n^j} = \sum_{k=0}^{\infty} \sum_{j=0}^{k} c_j \binom{k+\alpha-1}{k-j} \frac{1}{n^{k+\alpha}}. \end{aligned}$$

We then obtain

$$\begin{aligned} \Delta T_n &= \sum_{k=0}^{\infty} \left\{ b^n c_k - \sum_{j=0}^{k} b^{n-1} c_j \binom{k+\alpha-1}{k-j} \right\} \frac{1}{n^{k+\alpha}} \\ &= \frac{b^n (1 - \frac{1}{b}) c_0}{n^\alpha} + b^n \sum_{k=1}^{\infty} \left\{ c_k - \sum_{j=0}^{k} \frac{c_j}{b} \binom{k+\alpha-1}{k-j} \right\} n^{-(k+\alpha)}. \end{aligned} \tag{9}$$

Equating coefficients of the term $n^{-(k+\alpha)}$ $(k = 0, 1, 2, \ldots)$ on the right-hand sides of (8) and (9) yields $c_0 = \frac{b}{b-1}$ and

$$c_k = \sum_{j=0}^{k} \frac{c_j}{b} \binom{k+\alpha-1}{k-j} = \sum_{j=0}^{k-1} \frac{c_j}{b} \binom{k+\alpha-1}{k-j} + \frac{c_k}{b} \qquad (k \geq 1),$$

which gives the desired formula (5). The proof is complete.

3. SOMOS' QUADRATIC RECURRENCE CONSTANT

Somos' quadratic recurrence constant is defined by

$$\sigma = \sqrt{1\sqrt{2\sqrt{3\cdots}}} = \prod_{n=1}^{\infty} n^{1/2^n} = 1.66168794\ldots. \tag{10}$$

The constant σ arises in the study of the asymptotic behaviour of the sequence

$$g_0 = 1, \quad g_n = n g_{n-1}^2, \quad n \in \mathbb{N} := \{1, 2, 3, \ldots\}, \tag{11}$$

with first few terms

$$g_0 = 1, \quad g_1 = 1, \quad g_2 = 2, \quad g_3 = 12, \quad g_4 = 576, \quad g_5 = 1658880, \ \ldots.$$

This sequence behaves as follows (see [**30**, p. 446]):

$$g_n \sim \frac{\sigma^{2^n}}{n} \left(1 + \frac{2}{n} - \frac{1}{n^2} + \frac{4}{n^3} - \frac{21}{n^4} + \frac{138}{n^5} - \frac{1091}{n^6} + \cdots\right)^{-1} \qquad (n \to \infty). \tag{12}$$

The constant σ appears in important problems from pure and applied analysis, which is a motivation of a large number of papers (see, e.g., [**12, 14, 16, 33, 35, 39, 40, 41, 43, 47, 48, 52**]).

Nemes [**43**] studied the coefficients in the asymptotic expansion (12) and developed some recurrence relations. Chen [**12**] presented the following asymptotic expansion:

$$g_n \sim \frac{\sigma^{2^n}}{n} \left(1 - \frac{2}{n} + \frac{5}{n^2} - \frac{16}{n^3} + \frac{66}{n^4} - \frac{348}{n^5} + \cdots\right), \tag{13}$$

which yields

$$g_n \sim \frac{\sigma^{2^n}}{n} \exp\left\{-\frac{2}{n} + \frac{3}{n^2} - \frac{26}{3n^3} + \frac{75}{2n^4} - \frac{1082}{5n^5} + \cdots\right\}. \tag{14}$$

In this section, we provide a recurrence relation for determining the coefficients of n^{-k} in the expansion (14), asserted in Theorem 2.

Theorem 2. *As $n \to \infty$, the following asymptotic expansion holds*

$$g_n \sim \frac{\sigma^{2^n}}{n} \exp\left(\sum_{k=1}^{\infty} \frac{p_k}{n^k}\right), \tag{15}$$

where the coefficients p_k are given by the recurrence relation

$$p_1 = -2, \quad p_k = (-1)^k \left\{ \frac{2}{k} + \sum_{j=1}^{k-1} (-1)^j p_j \binom{k-1}{k-j} \right\} \qquad (k \geq 2). \tag{16}$$

Proof. In view of (14), we can let

$$g_n \sim \frac{\sigma^{2^n}}{n} \exp\left(\sum_{k=1}^{\infty} \frac{p_k}{n^k}\right) \quad \text{as} \quad n \to \infty,$$

where p_k ($k \in \mathbb{N}$) are real numbers to be determined. Denote

$$U_n = \sum_{k=1}^{n} \frac{\ln k}{2^k} + \frac{\ln n}{2^n} - \ln \sigma \quad \text{and} \quad V_n = \frac{1}{2^n} \sum_{k=1}^{\infty} \frac{p_k}{n^k}.$$

Noting that $g_n = \prod_{k=1}^{n} k^{2^{n-k}}$, we can let $U_n \sim V_n$ and

$$\Delta U_n := U_{n+1} - U_n \sim V_{n+1} - V_n =: \Delta V_n \quad \text{as} \quad n \to \infty. \tag{17}$$

It is easy to see that

$$\Delta U_n = \frac{1}{2^n} \ln\left(1 + \frac{1}{n}\right) = \frac{1}{2^n} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} n^{-k}. \tag{18}$$

Direct computation yields

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{p_k}{(n+1)^k} &= \sum_{k=1}^{\infty} \frac{p_k}{n^k} \left(1 + \frac{1}{n}\right)^{-k} = \sum_{k=1}^{\infty} \frac{p_k}{n^k} \sum_{j=0}^{\infty} \binom{-k}{j} \frac{1}{n^j} \\ &= \sum_{k=1}^{\infty} \frac{p_k}{n^k} \sum_{j=0}^{\infty} (-1)^j \binom{k+j-1}{j} \frac{1}{n^j} \\ &= \sum_{k=1}^{\infty} \sum_{j=1}^{k} p_j (-1)^{k-j} \binom{k-1}{k-j} \frac{1}{n^k}. \end{aligned} \tag{19}$$

We then obtain

$$\Delta V_n = \frac{1}{2^n} \sum_{k=1}^{\infty} \left\{ \sum_{j=1}^{k} \frac{p_j}{2} (-1)^{k-j} \binom{k-1}{k-j} - p_k \right\} \frac{1}{n^k}. \tag{20}$$

Equating coefficients of the term n^{-k} on the right-hand sides of (18) and (20) yields

$$\frac{(-1)^{k-1}}{k} = \sum_{j=1}^{k} \frac{p_j}{2}(-1)^{k-j}\binom{k-1}{k-j} - p_k \qquad (k \geq 1).$$

For $k = 1$ we obtain $p_1 = -2$, and for $k \geq 2$ we have

$$\frac{(-1)^{k-1}}{k} = \sum_{j=1}^{k-1} \frac{p_j}{2}(-1)^{k-j}\binom{k-1}{k-j} - \frac{1}{2}p_k \qquad (k \geq 2),$$

which gives the desired formula (16). The proof is complete.

4. BARNES G-FUNCTION

The double gamma function Γ_2 and the multiple gamma functions Γ_n were introduced and investigated by Barnes in a series of papers [**4, 5, 6, 7**]. Barnes applied these functions in the theory of elliptic functions and theta functions. Nonetheless, except possibly for the citations of Γ_2 in the exercises by Whittaker and Watson [**57**, p.264] and also by Gradshteyn and Ryzhik [**32**, p.661, Entry 6.441 (4); p.937, Entry 8.333], these functions were revived only in about the middle of the 1980s in the study of determinants of Laplacians on the n-dimensional unit sphere S^n (see, e.g., [**24, 38, 44, 49, 55, 56**]). The theory of the double gamma function has indeed found interesting applications in many other recent investigations (see, for details, [**53, 54**]).

Barnes [**4**] defined the double gamma function (or Barnes G-function) $\Gamma_2 = 1/G$ satisfying each of the following properties:

(i) $G(z+1) = \Gamma(z)G(z)$, for all complex z,

(ii) $G(1) = 1$,

(iii) As $n \to \infty$,

$$\begin{aligned}\ln G(z+n+2) = {} & \frac{n+1+z}{2}\ln(2\pi) + \left(\frac{n^2}{2} + n + \frac{5}{12} + \frac{z^2}{2} + (n+1)z\right)\ln n \\ & - \frac{3n^2}{4} - n - nz - \ln A + \frac{1}{12} + O(n^{-1}),\end{aligned} \tag{21}$$

where Γ is the gamma function and A is called the Glaisher-Kinkelin constant defined by

$$\ln A = \lim_{n\to\infty}\left\{\ln\left(\prod_{k=1}^{n} k^k\right) - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}\right)\ln n + \frac{n^2}{4}\right\}, \tag{22}$$

the numerical value of A being $1.28242713\ldots$.

The Glaisher-Kinkelin constant A can be expressed as follows (see [**29**])

$$A = \lim_{n\to\infty} n^{-n^2/2-n/2-1/12} e^{n^2/4} \prod_{k=1}^{n} k^k, \tag{23}$$

$$\frac{e^{1/12}}{A} = \lim_{n\to\infty} \frac{G(n+1)}{n^{n^2/2-1/12}(2\pi)^{n/2}e^{-3n^2/4}}, \tag{24}$$

and (see [**23**, p. 129, Eq. (3.22)])

$$A = e^{\frac{1}{12}-\zeta'(-1)} = (2\pi)^{1/12}\left[e^{\gamma\pi^2/6-\zeta'(2)}\right]^{1/(2\pi^2)}, \tag{25}$$

where $\zeta'(z)$ is the derivative of the Riemann zeta function $\zeta(z)$ (see [**27**]). The constant A has drawn attention in many works (for example) [**13, 17, 23, 26, 27**] as well as in [**4**]. Finch shared a section in his book [**30**, pp. 135–138] to introduce this Glaisher-Kinkelin constant A. The Glaisher-Kinkelin constant A plays an important role in the study of Barnes G-function (for details, see, e.g., [**54**, Section 1.4]).

The following integral representation for Barnes G-function was established by Ferreira and López [**29**, Theorem 1]: For $|\mathrm{Arg}(z)| < \pi$,

$$\begin{aligned} \ln G(z+1) = \frac{1}{4}z^2 + z\ln\Gamma(z+1) - \left(\frac{1}{2}z^2 + \frac{1}{2}z + \frac{1}{12}\right)\ln z - \ln A \\ + \sum_{k=1}^{N-1} \frac{B_{2k+2}}{2k(2k+1)(2k+2)z^{2k}} + R_N(z) \quad (N = 1, 2, \ldots), \end{aligned} \tag{26}$$

where B_{2k+2} are the Bernoulli numbers. The remainder $R_N(z)$ is for $\Re(z) > 0$ given by

$$R_N(z) = \int_0^\infty \left(\frac{t}{e^t - 1} - \sum_{k=0}^{2N} \frac{B_k}{k!} t^k\right) \frac{e^{-zt}}{t^3} dt. \tag{27}$$

Estimates for $|R_N(z)|$ are also found by Ferreira and López [**29**], showing that the expansion is indeed an asymptotic expansion of $\ln G(z+1)$ in sectors of the complex plane cut along the negative real axis. Pedersen [**45**, Theorem 1.1] proved that for any $N \geq 1$, the function $x \mapsto (-1)^N R_N(x)$ is completely monotonic on $(0, \infty)$. Other asymptotic relations (avoiding the $\ln\Gamma$ term) has been obtained by Ruijsenaars [**50**] and investigated by Pedersen [**46**], Koumandos [**36**] and Koumandos and Pedersen [**37**]. Some upper and lower bounds for the double gamma function were derived in terms of the gamma, psi and polygamma functions, see [**8, 9, 10, 18**]. Chen [**11**] and Mortici [**42**] established the inequalities and asymptotic expansions for $\ln A$ in (22). Chen and Lin [**17**] and Chen [**13**] presented a class of asymptotic expansions related to Glaisher-Kinkelin constant and the Barnes G-function.

As $x \to \infty$, the Stirling formula for Barnes G-function can be found (see [**53**, p. 26]):

$$\ln G(x+1) = \frac{x}{2}\ln(2\pi) - \ln A + \frac{1}{12} - \frac{3x^2}{4} + \left(\frac{x^2}{2} - \frac{1}{12}\right)\ln x + O(x^{-1}). \tag{28}$$

In this section, we develop the formula (28) to produce a complete asymptotic expansion given by Theorem 3.

Theorem 3. *As $x \to \infty$, the following asymptotic expansion holds*

$$\ln G(x+1) = \frac{x}{2}\ln(2\pi) - \ln A + \frac{1}{12} - \frac{3x^2}{4} + \left(\frac{x^2}{2} - \frac{1}{12}\right)\ln x + \sum_{k=1}^{\infty}\frac{q_k}{x^k}, \tag{29}$$

where the coefficients q_k are given by the recurrence relation

$$q_1 = 0, \quad q_k = \frac{(-1)^{k+1}}{k}\left\{\sum_{j=1}^{k-1} q_j(-1)^j\binom{k}{k-j+1} + \frac{(-1)^k B_{k+2}}{(k+1)(k+2)} + \frac{(k+6)(k-1)}{12(k+3)(k+2)(k+1)}\right\} \quad \text{for} \quad k \geq 2. \tag{30}$$

Here B_n denote Bernoulli numbers. Namely,

$$\ln G(x+1) \sim \frac{x}{2}\ln(2\pi) - \ln A + \frac{1}{12} - \frac{3x^2}{4} + \left(\frac{x^2}{2} - \frac{1}{12}\right)\ln x - \frac{1}{240x^2} + \frac{1}{1008x^4} - \frac{1}{1440x^6} + \frac{1}{1056x^8} - \frac{691}{327600x^{10}} + \cdots. \tag{31}$$

Proof. Denote

$$P(x) = \ln G(x+1) - \left\{\frac{x}{2}\ln(2\pi) - \ln A + \frac{1}{12} - \frac{3x^2}{4} + \left(\frac{x^2}{2} - \frac{1}{12}\right)\ln x\right\}$$

and

$$Q(x) = \sum_{k=1}^{\infty}\frac{q_k}{x^k}.$$

In view of (28), we can let $P(x) \sim Q(x)$ and

$$\Delta P(x) := P(x+1) - P(x) \sim Q(x+1) - Q(x) =: \Delta Q(x) \tag{32}$$

as $x \to \infty$, where q_k are real numbers to be determined.

Noting that $\ln G(x+1) - \ln G(x) = \ln\Gamma(x)$ and

$$\ln\Gamma(x+1) \sim x\ln x - x + \ln\sqrt{2\pi x} + \sum_{k=1}^{\infty}\frac{B_{k+1}}{k(k+1)x^k}. \tag{33}$$

(see [**2**, p. 257, Eq. (6.1.40)]), we obtain

$$\Delta P(x) = \ln\Gamma(x+1) - x\ln x + x - \ln\sqrt{2\pi x} + \frac{2x+3}{4} - \left(\frac{x^2}{2} + x + \frac{5}{12}\right)\ln\left(1+\frac{1}{x}\right) \sim \sum_{k=3}^{\infty}\left\{\frac{B_{k+1}}{k(k+1)} + \frac{(-1)^{k-1}(k+5)(k-2)}{12(k+2)(k+1)k}\right\}\frac{1}{x^k}. \tag{34}$$

By (19), we have

$$\Delta Q(x) = \sum_{k=1}^{\infty}\left\{\sum_{j=1}^{k} q_j(-1)^{k-j}\binom{k-1}{k-j} - q_k\right\}\frac{1}{x^k} = \sum_{k=1}^{\infty}\left\{\sum_{j=1}^{k-1} q_j(-1)^{k-j}\binom{k-1}{k-j}\right\}\frac{1}{x^k} = -\frac{q_1}{x^2} + \sum_{k=3}^{\infty}\left\{\sum_{j=1}^{k-1} q_j(-1)^{k-j}\binom{k-1}{k-j}\right\}\frac{1}{x^k}. \tag{35}$$

Equating coefficients of the term x^{-k} on the right-hand sides of (34) and (35) yields $q_1 = 0$ and for $k \geq 3$,

$$\frac{B_{k+1}}{k(k+1)} + \frac{(-1)^{k-1}(k+5)(k-2)}{12(k+2)(k+1)k} = \sum_{j=1}^{k-1} q_j(-1)^{k-j}\binom{k-1}{k-j},$$

$$\frac{B_{k+1}}{k(k+1)} + \frac{(-1)^{k-1}(k+5)(k-2)}{12(k+2)(k+1)k} = \sum_{j=1}^{k-2} q_j(-1)^{k-j}\binom{k-1}{k-j} - (k-1)q_{k-1},$$

$$q_{k-1} = \frac{(-1)^k}{k-1}\left\{\sum_{j=1}^{k-2} q_j(-1)^j\binom{k-1}{k-j} + \frac{(-1)^{k+1}B_{k+1}}{k(k+1)} + \frac{(k+5)(k-2)}{12(k+2)(k+1)k}\right\},$$

which can be written as (30). The proof is complete.

5. GLAISHER-KINKELIN AND CHOI-SRIVASTAVA CONSTANTS

Choi and Srivastava (see [**26**, p.102] and [**27**]) introduced two mathematical constants B and C (analogous to the Glaisher-Kinkelin constant A) defined by

$$\ln B = \lim_{n\to\infty}\left\{\sum_{k=1}^{n} k^2\ln k - \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)\ln n + \frac{n^3}{9} - \frac{n}{12}\right\} \tag{36}$$

and

$$\ln C = \lim_{n\to\infty}\left\{\sum_{k=1}^{n} k^3 \ln k - \left(\frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}-\frac{1}{120}\right)\ln n + \frac{n^4}{16}-\frac{n^2}{12}\right\} \tag{37}$$

for which the approximate numerical values are given by

$$B = 1.03091675\ldots \quad \text{and} \quad C = 0.97955746\ldots.$$

Like the expression of the Glaisher-Kinkelin constant A in (25), the constants B and C are also known to be expressible in terms of special values of the derivative of the Riemann zeta function $\zeta(s)$ as follows (see [**27**] and [**28**, Eq. (1.9)]):

$$\ln B = -\zeta'(-2) \quad \text{and} \quad \ln C = -\frac{11}{720} - \zeta'(-3). \tag{38}$$

As the Euler-Mascheroni constant γ is involved in the classical gamma function Γ, the constants A, B and C have appeared naturally in the theory of the multiple gamma functions Γ_n (see, e.g., [**54**, Section 1.4]) and play their respective roles, for example ([**53**, p. 39, p. 247], [**25**, p. 523, Eq. (2.50)], [**23**]).

Chen [**11**] established the asymptotic expansions related to the Glaisher-Kinkelin constant A and Choi-Srivastava constants B and C. Mortici [**42**] dealt with the same problem. Recently, Cheng and Chen [**22**] and Chen and Choi [**15**] established new asymptotic expansions of the Glaisher-Kinkelin and Choi-Srivastava constants. For example, by using Bernoulli numbers, Chen [**11**] established the asymptotic expansions related to the constants A, B and C.

In this section, we provide a recurrence relation for determining the coefficients of each asymptotic expansion related to the constants A, B and C, without Bernoulli numbers.

Theorem 4. *As $n \to \infty$, the following asymptotic expansion holds*

$$\sum_{k=1}^{n} k \ln k - \left(\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}\right)\ln n + \frac{n^2}{4} \sim \ln A - \sum_{k=2}^{\infty}\frac{\lambda_k}{n^k}, \tag{39}$$

where the coefficients λ_k are given by the recurrence relation

(40)

$$\lambda_2 = -\frac{1}{720},\ \lambda_k = \frac{(-1)^{k+1}}{k}\left\{\sum_{j=2}^{k-1}\lambda_j(-1)^j\binom{k}{k-j+1} + \frac{k(k-1)}{12(k+1)(k+2)(k+3)}\right\}$$

for $k \geq 3$. Namely,

$$\sum_{k=1}^{n} k \ln k - \left(\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}\right)\ln n + \frac{n^2}{4} \sim \ln A + \frac{1}{720n^2} - \frac{1}{5040n^4} + \frac{1}{10080n^6} - \frac{1}{9504n^8} + \ldots. \tag{41}$$

Proof. Denote

$$I_n = \sum_{k=1}^{n} k \ln k - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}\right) \ln n + \frac{n^2}{4} - \ln A \quad \text{and} \quad J_n = -\sum_{k=2}^{\infty} \frac{\lambda_k}{n^k}.$$

We can let $I_n \sim J_n$ and

$$\Delta I_n := I_{n+1} - I_n \sim J_{n+1} - J_n =: \Delta J_n$$

as $n \to \infty$, where λ_k are real numbers to be determined.

We obtain, after some elementary transformations, that

(42)
$$\Delta I_n = \frac{n}{2} + \frac{1}{4} - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}\right) \ln\left(1 + \frac{1}{n}\right) = -\sum_{k=3}^{\infty} \frac{(-1)^{k-1}(k-1)(k-2)}{12k(k+1)(k+2)} n^{-k}.$$

Direct computation yields

$$\begin{aligned}\sum_{k=2}^{\infty} \frac{\lambda_k}{(n+1)^k} &= \sum_{k=2}^{\infty} \frac{\lambda_k}{n^k} \left(1 + \frac{1}{n}\right)^{-k} = \sum_{k=2}^{\infty} \frac{\lambda_k}{n^k} \sum_{j=0}^{\infty} \binom{-k}{j} \frac{1}{n^j} \\ &= \sum_{k=2}^{\infty} \frac{\lambda_k}{n^k} \sum_{j=0}^{\infty} (-1)^j \binom{k+j-1}{j} \frac{1}{n^j} = \sum_{k=2}^{\infty} \sum_{j=2}^{k} \lambda_j (-1)^{k-j} \binom{k-1}{k-j} \frac{1}{n^k}.\end{aligned}$$

We then obtain

(43)
$$\begin{aligned}\Delta J_n &= -\sum_{k=2}^{\infty} \left\{ \sum_{j=2}^{k} \lambda_j (-1)^{k-j} \binom{k-1}{k-j} - \lambda_k \right\} \frac{1}{n^k} \\ &= -\sum_{k=3}^{\infty} \left\{ \sum_{j=2}^{k-1} \lambda_j (-1)^{k-j} \binom{k-1}{k-j} \right\} n^{-k}.\end{aligned}$$

Equating coefficients of the term n^{-k} on the right-hand sides of (42) and (43) yields

$$\frac{(-1)^{k-1}(k-1)(k-2)}{12k(k+1)(k+2)} = \sum_{j=2}^{k-1} \lambda_j (-1)^{k-j} \binom{k-1}{k-j} \qquad (k \geq 3).$$

For $k = 3$ we obtain $\lambda_2 = -\frac{1}{720}$, and for $k \geq 4$ we have

$$\frac{(-1)^{k-1}(k-1)(k-2)}{12k(k+1)(k+2)} = \sum_{j=2}^{k-2} \lambda_j (-1)^{k-j} \binom{k-1}{k-j} - (k-1)\lambda_{k-1} \qquad (k \geq 4),$$

$$\lambda_{k-1} = \frac{(-1)^k}{k-1} \left\{ \sum_{j=2}^{k-2} \lambda_j (-1)^j \binom{k-1}{k-j} + \frac{(k-1)(k-2)}{12k(k+1)(k+2)} \right\} \qquad (k \geq 4),$$

which can be written as (40). The proof is complete.

Following the same method as was used in the proof of Theorem 4, we can prove the following Theorems 5 and 6. We here omit the proofs.

Theorem 5. *As $n \to \infty$, the following asymptotic expansion holds*

$$\sum_{k=1}^{n} k^2 \ln k - \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right) \ln n + \frac{n^3}{9} - \frac{n}{12} \sim \ln B + \sum_{k=1}^{\infty} \frac{\mu_k}{n^k}, \tag{44}$$

where the coefficients μ_k are given by the recurrence relation

(45)

$$\mu_1 = -\frac{1}{360},\ \mu_k = \frac{(-1)^{k+1}}{k} \left\{ \sum_{j=1}^{k-1} \mu_j (-1)^j \binom{k}{k-j+1} - \frac{k}{6(k+2)(k+3)(k+4)} \right\}$$

for $k \geq 2$. Namely,

$$\sum_{k=1}^{n} k^2 \ln k - \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right) \ln n + \frac{n^3}{9} - \frac{n}{12} \sim \ln B - \frac{1}{360n} + \frac{1}{7560n^3} - \frac{1}{25200n^5} + \frac{1}{33264n^7} - \frac{691}{16216200n^9} + \cdots. \tag{46}$$

Theorem 6. *As $n \to \infty$, the following asymptotic expansion holds*

$$\sum_{k=1}^{n} k^3 \ln k - \left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} - \frac{1}{120}\right) \ln n + \frac{n^4}{16} - \frac{n^2}{12} \sim \ln C - \sum_{k=2}^{\infty} \frac{\nu_k}{n^k}, \tag{47}$$

where the coefficients ν_k are given by the recurrence relation

(48)

$$\nu_2 = \frac{1}{5040} \quad \textit{and}$$

$$\nu_k = \frac{(-1)^{k+1}}{k} \left\{ \sum_{j=2}^{k-1} \nu_j (-1)^j \binom{k}{k-j+1} - \frac{k(k-1)(k+13)}{120(k+1)(k+3)(k+4)(k+5)} \right\}$$

for $k \geq 3$. Namely,

(49)

$$\sum_{k=1}^{n} k^3 \ln k - \left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} - \frac{1}{120}\right) \ln n + \frac{n^4}{16} - \frac{n^2}{12} \sim \ln C - \frac{1}{5040n^2} + \frac{1}{33600n^4} - \frac{1}{66528n^6} + \frac{691}{43243200n^8} - \frac{1}{34320n^{10}} + \cdots.$$

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