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FORMULAS DERIVED FROM MOMENT GENERATING FUNCTIONS AND BERNSTEIN POLYNOMIALS

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The purpose of this paper is to provide some identities derived by moment generating functions and characteristics functions. By using functional equations of the generating functions for the combinatorial numbers $y_1(m, n; \lambda)$, defined in [**12**, p. 8, Theorem 1], we obtain some new formulas for moments of discrete random variable that follows binomial (Newton) distribution with an application of the Bernstein polynomials. Finally, we present partial derivative formulas for moment generating functions which involve derivative formula of the Bernstein polynomials.

1. INTRODUCTION

Characteristic functions and generating functions such as moment generating functions, ordinary generating functions, and exponential generating functions have been widely used in variety of fields (namely, probability theory, engineering, and variety branches of mathematics such as discrete mathematics, mathematical statistics, and mathematical physics). The motivation of this paper is to apply characteristic functions and generating functions to the special probability distributions. After these applications, we give some formulas and identities. Therefore, these distributions are not only the well-known Bernstein polynomials under the

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some restrictions, but aslo the Newton distribution, which is very important probability model when there are two possible outcomes. In addition, these applications, formulas and identities are also associated with well-known special numbers, special polynomials and moments of a random variable of the probability distribution. In recent years, there are many interesting and useful applications on these functions, combinatorial identities, special polynomials and numbers (*cf.* [**1**]-[**16**]; and the references cited therein).

We briefly introduce some well-known generating functions for the special numbers and polynomials that are used when deriving our identities and formulas.

Let X be a random variable of the probability distribution $f(x)$. The well-known characteristic function and moment generating function of the random variable X are given as follows, respectively:

$$K_x(t) = \mathbb{E}(e^{itx}),$$

and

$$M_x(t) = \mathbb{E}(e^{tx}),$$

where $\mathbb{E}(X)$ denotes the expected value or mean of the random variable X and $i^2 = -1$ (*cf.* [**10**, p. 10, Eq-(1.3.2)], [**15**, p. 112]; and the references cited therein). In [**10**, p. 15, Eq-(1.3.6) and Theorem 2.1.1], for distribution function $f(x)$, Lukacs gave various proprieties of the characteristic function $K_x(t)$:

$$K_x(t) = \int_{-\infty}^{\infty} f(x)\exp(itx)dx.$$

By using definition of the function $K_x(t)$, we have

$$K(0) = 1,$$

,

$$|K(t)| \leq 1$$

and

$$K_x(-t) = \overline{K_x(t)},$$

where $\overline{K_x(t)}$ is a the complex conjugate of $K_x(t)$.

The function $K_x(t)$ is uniformly continuous on $\mathbb{R}$, denotes the set of real numbers (*cf.* [**10**], [**15**]).

The λ-Stirling numbers are defined by

$$Fs(t, m; \lambda) = \frac{(\lambda e^t - 1)^m}{m!} = \sum_{n=0}^{\infty} S_2(n, m; \lambda)\frac{t^n}{n!}, \tag{1}$$

(*cf.* [**3**], [**11**], [**14**]; and the references therein). If $\lambda = 1$, then the λ-Stirling numbers reduce to the Stirling numbers of the second kind (*cf.* [**3**], [**2**], [**6**], [**8**], [**12**], [**?**]; and the references therein).

The poynomials $S_{m}^{n}\left(x;\lambda\right)$, so-called array type polynomials, are defined by

$$F_{A}\left(t,x,m;\lambda\right)=Fs(t,m;\lambda)e^{xt}=\sum_{n=0}^{\infty}S_{m}^{n}\left(x;\lambda\right)\frac{t^{n}}{n!}. \tag{2}$$

where λ is a complex or real numbers (*cf.* [**3**], [**6**], [**14**]; and the references therein). When $\lambda=1$ in equation (2), we have the the array polynomials:

$$S_{m}^{n}\left(x\right)=S_{m}^{n}\left(x;1\right).$$

Using equation (2), an explicit formula for the polynomials $S_{m}^{n}\left(x\right)$ is given by

$$S_{m}^{n}\left(x\right)=\frac{1}{m!}\sum_{j=0}^{m}(-1)^{m-j}\left(\begin{array}{c}m\\ j\end{array}\right)\left(x+j\right)^{n},$$

(*cf.* [**3**], [**4**], [**6**], [**14**]; and the references therein). Some basic properties of the polynomials $S_{m}^{n}\left(x\right)$ are given by

$$S_{n}^{n}\left(x\right)=1,$$

where $n\geq 0$, and

$$S_{0}^{n}\left(x\right)=x^{n}.$$

If $m>n$, then

$$S_{m}^{n}\left(x\right)=0$$

(*cf.* [**3**], [**4**], [**6**], [**14**]; and the references therein).

Let k be a nonnegative integer. Let λ be a complex number. The combinatorial numbers $y_{1}(n,k;\lambda)$ are defined by

$$F_{y_{1}}\left(t,k;\lambda\right)=\frac{1}{k!}\left(\lambda e^{t}+1\right)^{k}=\sum_{n=0}^{\infty}y_{1}(n,k;\lambda)\frac{t^{n}}{n!}. \tag{3}$$

By using equation (3), we have the following well-known formula for combinatorial numbers $y_{1}(n,k;\lambda)$:

$$y_{1}(n,k;\lambda)=\frac{1}{k!}\sum_{j=0}^{k}\left(\begin{array}{c}k\\ j\end{array}\right)j^{n}\lambda^{j}$$

(*cf.* [**12**]).

2. Moment generating function related to binomial (Newton) type distribution including the Bernstein polynomials

In this section, we firstly illustrate that the moment generating function of binomial (Newton) type distribution is related to well-known the Bernstein polynomials. Thus, using this moment generating function, we derive the partial derivative formulas.

The Bernstein polynomials are defined by

$$B_k^n(x;a,b) = \binom{n}{k} \left(\frac{x-a}{b-a}\right)^k \left(\frac{b-x}{b-a}\right)^{n-k}, \tag{4}$$

where a and b are real numbers (cf. [**5**, Chapter 5, pp. 299-306], [**7**], [**9**], [**13**]). Now,assume that $0 \leq \frac{x-a}{b-a} \leq 1$ and $0 \leq \frac{b-x}{b-a} \leq 1$, then equation (4) reduces to binomial type distribution (cf. [**9**], [**13**]). We note that when $a = 0$ and $b = 1$, equation (4) reduces to the binomial distribution for $0 \leq x \leq 1$.

Moment generating function can be desrcibed as

$$M_X(t, x : n; a, b) = \sum_{k=0}^{n} e^{kt} B_k^n(x; a, b). \tag{5}$$

Substituting (4) into the above equation, we have

$$M_X(t, x : n; a, b) = \left(e^t \frac{x-a}{b-a} + \frac{b-x}{b-a}\right)^n. \tag{6}$$

Substituting $a = 0$ and $b = 1$ into the above equation, we also have the well-known moment generating function for the Binomial distribution:

$$M_X(t, x : n; 0, 1) = \left(xe^t + 1 - x\right)^n$$

(c*f.* [**10**], [**15**, p. 100]).

2.1. Partial Derivative formulas

In this section, we briefly summarize the derivation of the partial derivative formulas for moment generating function that involves the Bernstein polynomials.

$$\frac{\partial^l}{\partial x^l} \left\{M_X(t, x : n; a, b)\right\} = \sum_{k=0}^{n} e^{kt} \frac{d^l}{dx^l} \left\{B_k^n(x; a, b)\right\}. \tag{7}$$

Next, the following modifying Theorem. 2.6. in [**13**, p. 7, 2.6. Theorem]

$$\frac{\partial^l}{\partial x^l} \left\{B_k^n(x; a, b)\right\} = \frac{n!}{(n-l)!} \sum_{j=0}^{l} (-1)^{l-j} \binom{l}{j} B_{k-j}^{n-l}(x; a, b)$$

is substituted into (7) to obtain the following differential equation:

$$\frac{\partial^l}{\partial x^l}\left\{M_X(t,x:n;a,b)\right\} = \frac{n!}{(n-l)!}\sum_{j=0}^{l}(-1)^{l-j}\left(\begin{array}{c} l \\ j \end{array}\right)\sum_{k=0}^{n} e^{kt}B_{k-j}^{n-j}(x;a,b),$$

where $B_{k-j}^{n-j}(x;a,b)=0$ if $k-j<0$ or $n-j<k-j$. Substituting $a=0$, $b=1$ and $l=1$ into the above equation, we have

$$\frac{\partial}{\partial x}\left\{M_X(t,x:n;0,1)\right\} = \sum_{k=0}^{n} e^{kt}\frac{d}{dx}\left\{B_k^n(x)\right\}.$$

Since

$$\frac{d}{dx}\left\{B_k^n(x)\right\} = n\left(B_{k-1}^{n-1}(x) - B_k^{n-1}(x)\right)$$

(*cf*. [**5**], [**13**]), we get

$$(1-x)\frac{\partial}{\partial x}\left\{M_X(t,x:n;0,1)\right\} = (x-x^2)\sum_{k=0}^{n}\frac{\partial}{\partial t}\left\{e^{kt}B_k^n(x)\right\} - M_X(t,x:n;0,1).$$

Therefore,

$$(x-1)\frac{\partial}{\partial x}\left\{M_X(t,x:n;0,1)\right\} + (x-x^2)\frac{\partial}{\partial t}\left\{M_X(t,x:n;0,1)\right\} = M_X(t,x:n;0,1).$$

The moments are given by

$$G_X(t,x:m,n;a,b) = \frac{\partial^m}{\partial x^m}\left\{M_X(t,x:n;a,b)\right\}.$$

Substituting $t=0$ into the above equation, we have

$$\begin{aligned} \mathbb{E}_X(X^m &: n;a,b) = m_m(x:m,n) \\ &= G_X(0,x:n;a,b) \\ &= \sum_{k=0}^{n} k^m B_k^n(x;a,b). \end{aligned}$$

The characteristic function is given by

$$K_X(t,x:n) = \sum_{k=0}^{n} e^{kti}B_k^n(x;a,b). \tag{8}$$

Substituting (4) into the above equation, we get

$$K(t,x:n;a,b) = \left(e^{it}\frac{x-a}{b-a} + \frac{b-x}{b-a}\right)^n. \tag{9}$$

By rearranging the above equation, we obtain

$$K_X(t,x:n;a,b)=\sum_{k=0}^{n}\sum_{v=0}^{k}\left(\begin{array}{c}k\\v\end{array}\right)B_k^n(x;a,b)\left(\cos t\right)^v\left(i\sin t\right)^{k-v}. \tag{10}$$

Since

$$i\sin t=\sinh t,$$

equation (10) can be rewritten as by the following theorem:

Theorem 1. *Let a and b are real numbers and n be nonegative integer. We assume that $0\leq\frac{x-a}{b-a}\leq 1$ and $0\leq\frac{b-x}{b-a}\leq 1$. Then we have*

$$K_X(t,x:n;a,b)=\sum_{k=0}^{n}\sum_{v=0}^{k}\left(\begin{array}{c}k\\v\end{array}\right)B_k^n(x;a,b)\left(\cos t\right)^v\left(\sinh t\right)^{k-v}.$$

3. Combinatorial Identities and Formulas

In this section, using functional equations of the generating functions for the combinatorial numbers and moments, we derive new formulas and combinatorial identities for moments, the combinatorial numbers $y_1(n,k;\lambda)$ and the Bernstein polynomials.

Theorem 2. *Let a and b are real numbers and n be nonegative integer. Then we have*

$$y_1\left(m,n;\frac{x-a}{b-x}\right)=\frac{1}{n!}\left(\frac{b-a}{b-x}\right)^n\sum_{k=0}^{n}k^mB_k^n(x;a,b)$$

Proof. By equation (5), we have

$$M_X(t,x:n;a,b)=\sum_{m=0}^{\infty}\sum_{k=0}^{n}k^mB_k^n(x;a,b)\frac{t^m}{m!}. \tag{11}$$

Combining (3) with (6), we get the following equation:

$$M_X(t,x:n;a,b)=n!\left(\frac{b-x}{b-a}\right)^nF_{y_1}\left(t,n;\frac{x-a}{b-x}\right). \tag{12}$$

By using the above equation, we obtain

$$M_X(t,x:n;a,b)=n!\left(\frac{b-x}{b-a}\right)^n\sum_{m=0}^{\infty}y_1\left(m,n;\frac{x-a}{b-x}\right)\frac{t^m}{m!}. \tag{13}$$

Combining (11) with (13) yields the assertion of the theorem. □

Theorem 3. *Let a and b are real numbers and n be nonnegative integer. We assume that $0 \leq \frac{x-a}{b-a} \leq 1$ and $0 \leq \frac{b-x}{b-a} \leq 1$. Then we have*

$$\mathbb{E}(X^m : n; a, b) = n! \left(\frac{b-x}{b-a}\right)^n y_1 \left(m, n; \frac{x-a}{b-x}\right). \tag{14}$$

Proof. Since

$$M_X(t, x : n; a, b) = \mathbb{E}(e^{tx} : n; a, b),$$

we have

$$\begin{aligned} M_X(t, x \quad : \quad n; a, b) &= \sum_{m=0}^{\infty} \left(\sum_{k=0}^{n} k^m B_k^n(x; a, b) \right) \frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty} \mathbb{E}(X^m; n; a, b) \frac{t^m}{m!}. \end{aligned}$$

Combining the above equation with (13), and after some elementary computations, we get the assertion of the theorem. □

For $m = 1$ and 2, we compute few values of the numbers $\mathbb{E}(X^m : n; a, b)$ given by equation (14). Computing the expected value or mean, $\mu_X(n; a, b)$ and variance $\sigma_X^2(n; a, b)$ of the random variable X with the help of formula in (14) as follows:

Substituting $m = 1$ into (14), we have

$$\begin{aligned} \mu_X(n; a, b) &= \mathbb{E}(X : n; a, b) = n! \left(\frac{b-x}{b-a}\right)^n y_1 \left(1, n; \frac{x-a}{b-x}\right) \\ &= n \left(\frac{x-a}{b-a}\right) \end{aligned}$$

and $m = 2$ into (14), we have

$$\sigma_X^2(n; a, b) = \mathbb{E}(X^2 : n; a, b) - \mathbb{E}^2(X : n; a, b) = \frac{n\,(x-a)\,(b-x)}{(b-a)^2}$$

(*cf.* [**13**]).

Setting $a = 0$ and $b = 1$, we easily see that

$$\mu_X(n; 0, 1) = \mu_X = nx$$

and

$$\sigma_X^2(n; 0, 1) = \sigma_X^2 = nx(1-x)$$

(cf. [**5**, Chapter 5, pp. 299-306], [**9**], [**13**], [**15**, p. 77]).

Theorem 4. *Let a and b are real numbers and n be nonnegative integer. We assume that $0 \leq \frac{x-a}{b-a} \leq 1$ and $0 \leq \frac{b-x}{b-a} \leq 1$. Then we have*

$$\mathbb{E}(X^m : n; a, b) = n! \left(\frac{b-x}{b-a}\right)^n S_2 \left(m, n; \frac{x-a}{b-x}\right).$$

Proof. Combining (9) with (1), we get the following equation:

$$K_X(t,x:n;a,b)=n!\left(\frac{b-x}{b-a}\right)^n Fs\left(it,n;\frac{x-a}{b-x}\right). \tag{15}$$

From the above equation, we have

$$K_X(t,x:n;a,b)=n!\left(\frac{b-x}{b-a}\right)^n \sum_{m=0}^{\infty} i^m S_2\left(m,n;\frac{x-a}{b-x}\right)\frac{t^m}{m!}. \tag{16}$$

On the other hand, using (9), we also have

$$\begin{aligned} K_X(t,x \quad &: \quad n;a,b)=\sum_{m=0}^{\infty}\left(\sum_{k=0}^{n} k^m B_k^n(x;a,b)\right)\frac{(ti)^m}{m!} \\ &= \sum_{m=0}^{\infty} i^m \mathbb{E}(X^m:n;a,b)\frac{t^m}{m!} \end{aligned}$$

Combining the above equation with (16), we arrive the assertion of the theorem. □

By combining Theorem 1, Theorem 3 and Theorem 4, we have

$$K_X(t,x:n;a,b)=\sum_{m=0}^{\infty}\sum_{k=0}^{n}\sum_{v=0}^{k}\binom{k}{v} i^m v!(k-v)!2^{m-k}B_k^n(x;a,b)$$

$$\times\sum_{c=0}^{m}\binom{m}{c} S_{k-v}^{c}\left(\frac{-k}{2}\right) y_1(m-c,n:1)\frac{t^m}{m!}.$$

Combining the above equation with (9) yields the following theorem after some elementary computations:

Theorem 5. *Let a and b are real numbers and n be nonegative integer. Then we have*

$$S_2\left(m,n;\frac{x-a}{x-b}\right)=\left(\frac{b-x}{b-a}\right)^{-n}\sum_{k=0}^{n}\sum_{v=0}^{k}\binom{k}{v}\frac{v!(k-v)!2^{m-k}}{n!}B_k^n(x;a,b)$$

$$\times\sum_{c=0}^{m}\binom{m}{c} S_{k-v}^{c}\left(\frac{-k}{2}\right) y_1(m-c,n:1).$$

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