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# REVISITING SIMULATION FUNCTIONS VIA INTERPOLATIVE CONTRACTIONS

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This paper is dedicated to Professor Gradimir V. Milovanovic on the occasion of his 70th anniversary.

In this paper, introduce the notion of an interpolative Hardy-Rogers type $\mathcal{Z}$-contraction and we revisit the renowned Hardy-Rogers contraction in the framework of interpolation. We investigate the existence of fixed points for such mappings in the context of metric spaces and list immediate consequences that covers some existing results in the literature.

## 1. Introduction and Preliminaries

In 2015 Khojasteh *et al.* [16], introduced an auxiliary function, *simulation function*, that covers and involves several existing contraction types in the literature. On the other hand, very recently, in [14], an interpolative contraction mappings are introduced to enrich fixed point theory. Interpolation theory is very deep theory and has been used widely in several research fields, see e.g. [17]. In this paper, we want to combine these two approaches and investigate the existence of fixed points that forms interpolative contractions in the framework of simulation functions in the context of complete metric spaces.

First of all, for the sake of completeness we recollect some basic definitions and results.

**Definition 0.1.** *(See [16]) A* simulation function *is a mapping* $\zeta : [0,\infty)\times[0,\infty) \to \mathbb{R}$ *satisfying the following conditions:*

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($\zeta_1$) *$\zeta(t,s) < s - t$ for all $t, s > 0$;*

($\zeta_2$) *if $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then*

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0. \tag{1}$$

($\zeta_3$) *$\zeta(0,0) = 0$;*

Due to the axiom ($\zeta_1$), we have

$$\zeta(t,t) < 0 \text{ for all } t > 0. \tag{2}$$

The definition above was refined by omitting the condition $\zeta(0,0) = 0$ Argoubi *et al.* [4]. Throughout the paper, the letter $\mathcal{Z}$ denotes the family of all functions $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ that satisfies only ($\zeta_1$) and ($\zeta_2$). From now on, a function $\zeta$ is called simulation function if $\zeta \in \mathcal{Z}$.

The following example is derived from [1].

**Example 0.1.** *Let $\phi_i : [0,\infty) \to [0,\infty)$ be continuous functions such that $\phi_i(t) = 0$ if and only if, $t = 0$. For $i = 1,2,3,4,5,6$, we define the mappings $\zeta_i : [0,\infty) \times [0,\infty) \to \mathbb{R}$, as follows*

(*i*) *$\zeta_1(t,s) = \phi_1(s) - \phi_2(t)$ for all $t, s \in [0,\infty)$, where $\phi_1, \phi_2 : [0,\infty) \to [0,\infty)$ are two continuous functions such that $\phi_1(t) = \phi_2(t) = 0$ if and only if $t = 0$ and $\phi_1(t) < t \leq \phi_2(t)$ for all $t > 0$.*

(*ii*) *$\zeta_2(t,s) = s - \dfrac{f(t,s)}{g(t,s)}t$ for all $t, s \in [0,\infty)$, where $f, g : [0,\infty)^2 \to (0,\infty)$ are two continuous functions with respect to each variable such that $f(t,s) > g(t,s)$ for all $t, s > 0$.*

(*iii*) *$\zeta_3(t,s) = s - \phi_3(s) - t$ for all $t, s \in [0,\infty)$.*

(*iv*) *$\zeta_4(t,s) = s\,\varphi(s) - t$ for all $s, t \in [0,\infty)$, where $\varphi : [0,\infty) \to [0,1)$ is a function such that $\limsup_{t\to r^+} \varphi(t) < 1$ for all $r > 0$.*

(*v*) *$\zeta_5(t,s) = \eta(s) - t$ for all $s, t \in [0,\infty)$, where $\eta : [0,\infty) \to [0,\infty)$ is an upper semi-continuous mapping such that $\eta(t) < t$ for all $t > 0$ and $\eta(0) = 0$.*

(*vi*) *$\zeta_6(t,s) = s - \int_0^t \phi(u)du$ for all $s, t \in [0,\infty)$, where $\phi : [0,\infty) \to [0,\infty)$ is a function such that $\int_0^\varepsilon \phi(u)du$ exists and $\int_0^\varepsilon \phi(u)du > \varepsilon$, for each $\varepsilon > 0$.*

*It is clear that each function $\zeta_i$ ($i = 1,2,3,4,5,6$) forms a simulation function.*

One can find more interesting examples of simulation functions in [1–3,5,11–13,16,18].

Suppose $(X,d)$ is a metric space, $T$ is a self-mapping on $X$ and $\zeta \in \mathcal{Z}$. We say that $T$ is a *$\mathcal{Z}$-contraction* with respect to $\zeta$ [16], if

$$\zeta(d(Tx,Ty),d(x,y)) \geq 0 \qquad \text{for all } x,y \in X. \tag{3}$$

Again $(\zeta_2)$, we have the following inequality

$$d(Tx,Ty) \neq d(x,y) \text{ for all distinct } x,y \in X. \tag{4}$$

Thus, we conclude that $T$ cannot be an isometry whenever $T$ is a $\mathcal{Z}$-contraction. In other words, if a $\mathcal{Z}$-contraction $T$ in a metric space has a fixed point, then it is necessarily unique.

**Theorem 0.1.** *Every $\mathcal{Z}$-contraction on a complete metric space has a unique fixed point.*

Recently an interesting fixed point result via interpolation was reported in [14]. More precisely, in [14], the notion of interpolative Kannan contraction was introduced as follows: For a metric space $(X,d)$, a mapping $T: X \to X$ is called an interpolative Kannan contraction if

$$d(Tx,Ty) \leq \lambda \left[d(x,Tx)\right]^{\alpha} \cdot \left[d(y,Ty)\right]^{1-\alpha}, \tag{5}$$

for all $x,y \in X$ with $x,y \in X \backslash Fix(T)$, where $Fix(T)$ is the set of all fixed point of $T$, $\lambda \in [0,1)$ and $\alpha \in (0,1)$. The main result in [14] is the following.

**Theorem 0.2** ( [14])**.** *Let $(X,d)$ be a complete metric space and $T$ be an interpolative Kannan type contraction. Then $T$ has a fixed point in $X$.*

For sake of completeness, we shall recollect one of the renowned generalizations of the Banach Contraction Principle [6] which is know as Hardy-Rogers contraction:

**Theorem 0.3.** *[8]. Let $(X,d)$ be a complete metric space. Let $T: X \to X$ be a given mapping such that*

$$d(Tx,Ty) \leq \alpha d(x,y) + \beta d(x,Tx) + \gamma d(y,Ty) + \delta[\frac{1}{2}(d(x,Ty) + d(y,Tx))],$$

*for all $x,y \in X$, where $\alpha,\beta,\gamma,\delta$ are non-negative reals such that $\alpha+\beta+\gamma+\delta<1$. Then $T$ has a unique fixed point in $X$.*

In this paper, we investigate interpolative type contractions by using the simulation function in the context of complete metric spaces. More precisely, we revisit the renowned Hardy-Rogers contraction in the framework of interpolation via simulation function.

## 2. Main results

We start with the following definition that is belong to Browder and Petrsyn [7].

**Definition 0.2.** *We say that a self-mapping $T : X \to X$ on a metric space $(X, d)$ is asymptotically regular* at a point $x \in X$ *if* $\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0$

On what follows we introduce the notion of the interpolative Hardy-Rogers type *$\mathcal{Z}$-contraction.*

**Definition 0.3.** *Let $T$ be a self-mapping defined on a metric space $(X, d)$. If there exist $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta + \gamma < 1$, and $\zeta \in \mathcal{Z}$ such that*

$$\zeta(d(Tx, Ty), C(x, y)) \geq 0, \tag{6}$$

*for all $x, y \in X \backslash Fix(T)$, where $Fix(T)$ is the set of all fixed point of $T$, and*

$$C(x,y) := [d(x,y)]^{\beta} \cdot [d(x,Tx)]^{\alpha} \cdot [d(y,Ty)]^{\gamma} \cdot \left[\frac{1}{2}(d(x,Ty) + d(y,Tx))\right]^{1-\alpha-\beta-\gamma}$$

*then we say that $T$ is an interpolative Hardy-Rogers type $\mathcal{Z}$-*contraction *with respect to $\zeta$.*

**Lemma 0.1.** *On a metric space $(X, d)$, every Hardy-Rogers type $\mathcal{Z}$-*contraction *with respect to $\zeta$ is asymptotically regular*

*Proof.* Let $x$ be an arbitrary point of a metric space $(X, d)$ and let $T : X \to X$ be a Hardy-Rogers type *$\mathcal{Z}$-contraction* with respect to $\zeta \in \mathcal{Z}$. If there exists some $p \in \mathbb{N}$ such that $T^p x = T^{p-1}x$, then $y = T^{p-1}x$ is a fixed point of $T$, that is, $Ty = y$. Consequently, we have that $T^n y = y$ for all $n \in \mathbb{N}$, so

$$\begin{aligned} d(T^n x, T^{n+1}x) &= d(T^{n-p+1}T^{p-1}x, T^{n-p+2}T^{p-1}x) = d(T^{n-p+1}y, T^{n-p+2}y) \\ &= d(y, y) = 0, \end{aligned}$$

for sufficient large $n \in \mathbb{N}$. Thus, we conclude that

$$\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0.$$

So $T$ is asymptotically regular at $x$. On the contrary, suppose that $T^n x \neq T^{n-1}x$ for all $n \in \mathbb{N}$, that is,

$$d(T^n x, T^{n-1}x) > 0 \quad \text{for all } n \in \mathbb{N}.$$

On what follows, from (6) and $(\zeta_1)$, we have that, for all $n \in \mathbb{N}$,

$$0 \leq \zeta(d(T^{n+1}x, T^n x), C(T^n x, T^{n-1}x)) < C(T^n x, T^{n-1}x) - d(T^{n+1}x, T^n x).$$

In particular,

$$d(T^{n+1}x, T^n x) < C(T^n x, T^{n-1}x) \quad \text{for all } n \in \mathbb{N}, \text{ where} \tag{7}$$

$$\begin{aligned}
C(T^n x, T^{n-1}x) &= \left[d\left(T^n x, T^{n-1}x\right)\right]^{\beta} \cdot \left[d\left(T^n x, T^{n+1}x\right)\right]^{\alpha} \cdot \left[d\left(T^{n-1}x, T^n x\right)\right]^{\gamma} \\
&\cdot \left[\tfrac{1}{2}(d\left(T^n x, T^n x\right) + d\left(T^{n-1}x, T^{n+1}x\right))\right]^{1-\alpha-\beta-\gamma} \\
&\le \left[d\left(T^n x, T^{n-1}x\right)\right]^{\beta} \cdot \left[d\left(T^n x, T^{n+1}x\right)\right]^{\alpha} \cdot \left[d\left(T^{n-1}x, T^n x\right)\right]^{\gamma} \\
&\cdot \left[\tfrac{1}{2}(d\left(T^{n-1}x, T^n x\right) + d\left(T^n x, T^{n+1}x\right))\right]^{1-\alpha-\beta-\gamma}.
\end{aligned} \tag{8}$$

Note that for the assumption $d\left(T^{n-1}x, T^n x\right) < d\left(T^n x, T^{n+1}x\right)$, the expression (7) turns into

$$\begin{aligned}
C(T^n x, T^{n-1}x) &\le \left[d\left(T^n x, T^{n-1}x\right)\right]^{\beta} \cdot \left[d\left(T^n x, T^{n+1}x\right)\right]^{\alpha} \\
&\cdot \left[d\left(T^{n-1}x, T^n x\right)\right]^{\gamma} \cdot \left[d\left(T^n x, T^{n+1}x\right)\right]^{1-\alpha-\beta-\gamma}.
\end{aligned} \tag{9}$$

Thus, the inequality (7) together with (9) yields that

$$\left[d\left(T^n x, T^{n+1}x\right)\right]^{\beta+\gamma} < \left[d\left(T^{n-1}x, T^n x\right)\right]^{\beta+\gamma}. \tag{10}$$

It is a contradiction with assumption. Hence, we have

$$d\left(T^{n-1}x, T^n x\right) < d\left(T^n x, T^{n+1}x\right) \text{ for all } n \in \mathbb{N}$$

On account of the inequality above, we deduce that the sequence $\{d(T^n x, T^{n-1}x)\}$ is a monotonically decreasing of non-negative real numbers. Thus, there exists $\ell \in [0, \infty)$ such that $\lim_{n\to\infty} d(T^n x, T^{n+1}x) = \ell \ge 0$. We shall prove that $\ell = 0$. Suppose, on the contrary, that $\ell > 0$. It is easy to see that $\lim_{n\to\infty} C(T^n x, T^{n+1}x) = \ell$.

Since $T$ is Hardy-Rogers type $\mathcal{Z}$-*contraction* with respect to $\zeta \in \mathcal{Z}$, by $(\zeta_2)$, we have

$$0 \le \limsup_{n\to\infty} \zeta(d(T^{n+1}x, T^n x), C(T^n x, T^{n-1}x)) < 0,$$

which is a contradiction. Thus, $\ell = 0$ and this proves that

$$\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0.$$

Hence, $T$ is an asymptotically regular mapping at $x$. □

**Remark 0.1.** *In the proof of the previous result we have proved that if $T : X \to X$ is a Hardy-Rogers type $\mathcal{Z}$-*contraction *on a metric space $(X, d)$ and $\{x_{n+1} = T^n x_0\}$ is a Picard sequence of $T$, then*

$$\begin{aligned}
&\textit{either} \quad \textit{there exists } k_0 \in \mathbb{N} \textit{ such that } x_{k_0} \textit{ is a fixed point of } T \\
&\textit{or} \quad 0 < d(T^{n+1}x, T^n x) < d(T^n x, T^{n-1}x) \qquad \textit{for all } n \in \mathbb{N}.
\end{aligned} \tag{11}$$

Now, we show that every Picard sequence $\{x_n\}$ generated by a Hardy-Rogers type $\mathcal{Z}$-*contraction* is always bounded.

**Lemma 0.2.** *Let a self-mapping $T$ on a metric space $(X, d)$ form a Hardy-Rogers type $\mathcal{Z}$-*contraction *with respect to $\zeta$. If $\{x_n\}$ is a Picard sequence generated by $T$, then $\{d(x_n, x_m) : n, m \in \mathbb{N}\}$ is bounded.*

*Proof.* Start with an arbitrary initial point $x_0 \in X$ we built a iterative sequence $\{x_n\}$ which is defined recursively by $x_{n+1} = Tx_n$ for all non-negative integer $n$. If there exists some $n \geq 0$ and $p \geq 1$ such that $x_{n+p} = x_n$, then the set $\{x_n : n \in \mathbb{N}\}$ is finite, so it is bounded. Hence, assume that $x_{n+p} \neq x_n$ for all $n \geq 0$ and $p \geq 1$. In this case, by Remark 0.1, we have that:

$$0 < d(x_{n+1}, x_n) < d(x_n, x_{n-1}) \quad \text{for all } n \in \mathbb{N}. \tag{12}$$

Notice that by Lemma 0.1,

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{13}$$

In particular, there exists $n_0 \in \mathbb{N}$ such that

$$d(x_{n+1}, x_n) < 1 \qquad \text{for all } n \geq n_0.$$

We shall prove that $\{x_n : n \in \mathbb{N}\}$ is bounded by the method of *Reductio ad Absurdum.* We assume that the set

$$D = \{d(x_m, x_n) : m > n\}.$$

is not bounded. Thus, one can find a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} d(x_{n_{k+1}}, x_{n_k}) \neq 0$. Indeed, since $D$ is unbounded, there exist $n_1, n_0$ with $n_1 > n_0$ such that $d(x_{n_1}, x_{n_0}) > 1$. If $n_1$ is the smallest natural number, greater than $n_0$, verifying this property, then we can suppose that

$$d(x_p, x_{n_0}) \leq 1 \quad \text{for all } p \in \{n_0, n_0+1, \ldots, n_1-1\}.$$

Again, as $D$ is not bounded, there exists $n_2 > n_1$ such that

$$d(x_{n_2}, x_{n_1}) > 1 \quad \text{and} \quad d(x_p, x_{n_1}) \leq 1 \quad \text{for all } p \in \{n_1, n_1+1, \ldots, n_2-1\}.$$

Recursively, we can get a partial subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that, for all $k \geq 1$,

$$d(x_{n_{k+1}}, x_{n_k}) > 1 \quad \text{and} \quad d(x_p, x_{n_k}) \leq 1 \quad \text{for all } p \in \{n_k, n_k+1, \ldots, n_{k+1}-1\}.$$

Hence, by the triangular inequality, we have that, for all $k$,

$$1 < d(x_{n_{k+1}}, x_{n_k}) \leq d(x_{n_{k+1}}, x_{n_{k+1}-1}) + d(x_{n_{k+1}-1}, x_{n_k}) \leq d(x_{n_{k+1}}, x_{n_{k+1}-1}) + 1. \tag{14}$$

Letting $k \to \infty$ in (14) and taking (13) into account, we obtain

$$\lim_{k\to\infty} d(x_{n_{k+1}}, x_{n_k}) = 1.$$

By (12), we have $d(x_{n_{k+1}}, x_{n_k}) \leq d(x_{n_{k+1}-1}, x_{n_k-1})$. Therefore using the triangular inequality we obtain

$$\begin{aligned} 1 < d(x_{n_{k+1}}, x_{n_k}) &\leq d(x_{n_{k+1}-1}, x_{n_k-1}) \leq d(x_{n_{k+1}-1}, x_{n_k}) + d(x_{n_k}, x_{n_k-1}) \\ &\leq 1 + d(x_{n_k}, x_{n_k-1}). \end{aligned}$$

Letting $k \to \infty$ and using (13) we obtain

$$\lim_{k\to\infty} d(x_{n_{k+1}-1}, x_{n_k-1}) = 1.$$

Since $T$ is a Hardy-Rogers type $\mathcal{Z}$-*contraction* with respect to $\zeta \in \mathcal{Z}$, for all $k$, we have

$$0 \leq \zeta(d(Tx_{n_{k+1}-1}, Tx_{n_k-1}), C(x_{n_{k+1}-1}, x_{n_k-1})) < C(x_{n_{k+1}-1}, x_{n_k-1}) - d(x_{n_{k+1}}, x_{n_k})$$

which is equivalent to

$$d(x_{n_{k+1}}, x_{n_k}) < C(x_{n_{k+1}-1}, x_{n_k-1}), \tag{15}$$

where

$$\begin{aligned} C(x_{n_{k+1}-1}, x_{n_k-1}) &= \left[d\left(x_{n_{k+1}-1}, x_{n_k-1}\right)\right]^{\beta} \cdot \left[d\left(x_{n_{k+1}-1}, Tx_{n_{k+1}-1}\right)\right]^{\alpha} \\ &\cdot \left[d\left(x_{n_k-1}, Tx_{n_k-1}\right)\right]^{\gamma} \\ &\cdot \left[\tfrac{1}{2}(d\left(x_{n_{k+1}-1}, Tx_{n_k-1}\right) + d\left(x_{n_k-1}, Tx_{n_{k+1}-1}\right))\right]^{1-\alpha-\beta-\gamma} \end{aligned}.$$

Letting $k \to \infty$ in the inequality (15), we find that

$$1 = \lim_{k\to\infty} d(x_{n_{k+1}}, x_{n_k}) \leq 0 \tag{16}$$

is a contradiction. This proves that $D = \{d(x_m, x_n) : m > n\}$ is bounded. □

We can now state the main result of this paper.

**Theorem 0.4.** *Let $(X, d)$ be a complete metric metric space and $T$ be an interpolative Hardy-Rogers type $\mathcal{Z}$-*contraction *with respect to $\zeta$. Then there exists $u \in X$ such that $Tu = u$.*

*Proof.* Start with an arbitrary initial point $x_0 \in X$, we construct the Picard sequence $\{x_n = T^n x_0\}_{n\geq 0}$. Incase of a sequence $\{x_n\}$ contains a fixed point of $T$, the proof is completed. So, we assume that $\{x_n\}$ has no fixed point of $T$. Accordingly, due to Lemma 0.1 together with Remark 0.1 we derive that

$$0 < d(x_{n+1}, x_n) < d(x_n, x_{n-1}) \quad \text{for all } n \in \mathbb{N}. \tag{17}$$

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{18}$$

We assert that the sequence $\{x_n\}$ is Cauchy. On account that Lemma 0.2, we guarantee that $\{d(x_m, x_n) : m, n \in \mathbb{N}\}$ is bounded. Consider the sequence $\{S_n\} \subset [0, \infty)$ given by:

$$S_n = \sup\left(\{d(x_i, x_j) : i \geq j \geq n\}\right) \quad \text{for all } n \in \mathbb{N}.$$

It is easy to notice that the sequence $\{S_n\}$ is a monotonically non-increasing of non-negative real numbers. Thus, we conclude that this sequence is convergent, that is, there exists $S \geq 0$ such that $\lim_{n\to\infty} S_n = S$. We claim that $S = 0$. We shall use the method of *Reductio ad Absurdum* to prove our claim. Suppose, on the contrary, that $S > 0$. Then, by definition of $S_n$, for every $k \in \mathbb{N}$ there exists $n_k, m_k \in \mathbb{N}$ such that $m_k > n_k \geq k$ and

$$S_k - \frac{1}{k} < d(x_{m_k}, x_{n_k}) \leq S_k.$$

Hence, we find

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = S. \tag{19}$$

By using (17) and the triangular inequality, we have, for all $k$,

$$d(x_{m_k}, x_{n_k}) \leq d(x_{m_k-1}, x_{n_k-1}) \leq d(x_{m_k-1}, x_{m_k}) + d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_k-1}).$$

Letting $k \to \infty$ in the above inequality and using (18) and (19), we derive that

$$\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k-1}) = S. \tag{20}$$

Hence, we have Due to fact that $T$ is a Hardy-Rogers type $\mathcal{Z}$*-contraction* with respect to $\zeta \in \mathcal{Z}$ we have

$$\begin{aligned} 0 &\leq \zeta(d(Tx_{m_k}, Tx_{n_k}), C(x_{m_k}, x_{n_k})) \\ &= \zeta(d(x_{m_k-1}, x_{n_k-1}), C(x_{m_k}, x_{n_k})) < 0, \end{aligned}$$

which implies

$$d(x_{m_k-1}, x_{n_k-1}) \leq C(x_{m_k}, x_{n_k}) \tag{21}$$

where

$$\begin{aligned} C(x_{m_k}, x_{n_k}) &= [d(x_{m_k}, x_{n_k})]^{\beta} \cdot [d(x_{m_k}, Tx_{m_k})]^{\alpha} \\ &\cdot [d(x_{n_k}, Tx_{n_k})]^{\gamma} \\ &\cdot \left[\tfrac{1}{2}(d(x_{m_k}, Tx_{n_k}) + d(x_{n_k}, Tx_{m_k}))\right]^{1-\alpha-\beta-\gamma}. \end{aligned}$$

Letting $k \to \infty$ in the inequality (21), we find that

$$S = \lim_{k\to\infty} d(x_{m_k-1}, x_{n_k-1}) \leq \lim_{k\to\infty} C(x_{m_k}, x_{n_k}) = 0, \tag{22}$$

is a contradiction. Thus, we deduce that $S = 0$ and, hence, $\{x_n\}$ is a Cauchy sequence. Since $(X, q)$ is a complete metric space, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$.

We shall show that the point $u$ is a fixed point of $T$ reasoning by contradiction. Suppose that $Tu \neq u$, that is, $d(u, Tu) > 0$. Hence we have

$$\lim_{n\to\infty} d(Tx_n, Tu) = \lim_{n\to\infty} d(x_{n+1}, Tu) = d(u, Tu) > 0.$$

Therefore, there is $n_0 \in \mathbb{N}$ such that

$$d(Tx_n, Tu) > 0 \quad \text{for all } n \geq n_0.$$

In particular, $Tx_n \neq Tu$. This also means that $x_n \neq u$ for all $n \geq n_0$. As $d(Tx_n, Tu) > 0$ and $d(x_n, u) > 0$, axiom $(\zeta_2)$ and property (6) imply that, for all $n \geq n_0$,

$$0 \leq \zeta(d(Tx_n, Tu), d(x_n, u)) < d(x_n, u) - d(Tx_n, Tu).$$

In particular, $0 \leq d(Tx_n, Tu) \leq d(x_n, u)$ for all $n \geq n_0$, which means that

$$\lim_{n\to\infty} d(x_{n+1}, Tu) = \lim_{n\to\infty} d(Tx_n, Tu) = 0.$$

Therefore, $\{x_n\}$ converges, at the same time, to $u$ and to $Tu$. By the uniqueness of the limit, $u = Tu$, which contradicts $Tu \neq u$. As a consequence, $u$ is a fixed point of $T$. □

## 3. Consequences

In this section, we give some immediate consequence of our main result. The following corollary is the main result of [10].

**Corollary 0.1.** *[10] Let $(X, d)$ be a complete metric space and $T : X \to X$ be a mapping such that*

$$d(Tx, Ty) \leq \lambda C(x, y) \quad \textit{for all } x, y \in X,$$

*where $(C(x, y))$ is defined as in Definition 0.3 and $\lambda \in [0, 1)$. Then $T$ has a unique fixed point in $X$.*

*Proof.* The result follows from Theorem 0.1 taking into account that $T$ is a $\mathcal{Z}$-contraction with respect to $\zeta_B \in \mathcal{Z}$, where $\zeta_B$ is defined by $\zeta_B(t, s) = \lambda s - t$ for all $s, t \in [0, \infty)$ . (see Example 0.1). □

**Corollary 0.2.** *Let $(X, d)$ be a complete metric space and let $T : X \to X$ be a mapping satisfying the following condition:*

$$d(Tx, Ty) \leq C(x, y) - \varphi(C(x, y)) \qquad \textit{for all } x, y \in X,$$

*where $(C(x, y))$ is defined as in Definition 0.3 and $\varphi : [0, \infty) \to [0, \infty)$ is a lower semi continuous function and $\varphi^{-1}(0) = \{0\}$. Then $T$ has a unique fixed point in $X$.*

*Proof.* The result follows from Theorem 0.1 taking into account that $T$ is a $\mathcal{Z}$-contraction with respect to $\zeta_R \in \mathcal{Z}$, where $\zeta_R$ is defined by $\zeta_R(t, s) = s - \varphi(s) - t$ for all $s, t \in [0, \infty)$ (see Example 0.1). □

**Corollary 0.3.** *Let $(X, d)$ be a complete metric space and $T : X \to X$ be a mapping. Suppose that for every $x, y \in X$,*

$$d(Tx, Ty) \leq \eta(C(x, y))$$

*for all $x, y \in X$, where $(C(x, y))$ is defined as in Definition 0.3 and $\eta : [0, +\infty) \to [0, +\infty)$ be an upper semi continuous mapping such that $\eta(t) < t$ for all $t > 0$ and $\eta(0) = 0$. Then $T$ has a unique fixed point.*

*Proof.* The result follows from Theorem 0.1 taking into account that $T$ is a $\mathcal{Z}$-contraction with respect to $\zeta_{BW} \in \mathcal{Z}$, where $\zeta_{BW}$ is defined by $\zeta_{BW}(t, s) = \eta(s) - t$ for all $s, t \in [0, \infty)$ (see Example 0.1). □

**Corollary 0.4.** *Let $(X, d)$ be a complete metric space and let $T : X \to X$ be a mapping satisfying the following condition:*

$$\int_0^{d(Tx,Ty)} \phi(t)dt \leq C(x, y) \qquad \text{for all } x, y \in X,$$

*where $(C(x, y))$ is defined as in Definition 0.3 and $\phi : [0, \infty) \to [0, \infty)$ is a function such that $\int_0^\epsilon \phi(t)qt$ exists and $\int_0^\epsilon \phi(t)dt > \epsilon$, for each $\epsilon > 0$. Then $T$ has a unique fixed point in $X$.*

*Proof.* The result follows from Theorem 0.1 taking into account that $T$ is a $\mathcal{Z}$-contraction with respect to $\zeta_K \in \mathcal{Z}$, where $\zeta_K$ is defined by

$$\zeta_K(t, s) = s - \int_0^t \phi(u)qu \qquad \text{for all } s, t \in [0, \infty)$$

(see Example 0.1). □

## Conclusion

It is clear that the list of consequences in the above section is not complete. In the section above, we give only the fundamental consequences. On the other hand, regarding Example 0.1, one can deduce more results. Furthermore, by changing the terms in $(C(x, y))$ in Definition 0.3, we get more consequences.

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