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A GENERATION METHOD FOR COMPLETELY MONOTONE FUNCTIONS

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In this article we present technique how to produce completely monotone functions using linear functionals and already known families of completely monotone functions. After that, using mean value theorems, we construct means of Cauchy type that have monotonicity properties.

1. Introduction

A function f is said to be *completely monotone* on an open interval $I \subset (0,\infty)$ if it has derivatives of all orders there and satisfies

$$(1) \qquad (-1)^n f^{(n)}(x) \geq 0 \quad \text{for all} \quad x \in I \quad \text{and} \quad n = 0,1,2,\ldots.$$

The class of all completely monotone functions on I is denoted by $\mathcal{CM}(I)$. It is obvious that $\mathcal{CM}(I) \subset \mathcal{C}(I)$, where $\mathcal{C}(I)$ denotes class off all convex functions on I. This inclusion can be refined, as we will show at the end of this introductory part.

One of the most important features of completely monotone functions is their integral representation: It is well known (see [**10**, p. 161]) that a necessary and sufficient condition for f to be completely monotone on I is that

$$(2) \qquad f(x) = \int_0^\infty e^{-tx} d\sigma(t), \quad x \in I$$

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for some non-decreasing function $\sigma : (0,\infty) \to \mathbb{R}$ and the integral converges for $x \in I$. In particular, we conclude that a non-identically zero completely monotone function f cannot vanish for any $x \in I$.

Completely monotone function are included in a slightly larger class of convex functions.
A function $f : I \to \mathbb{R}$ is *exponentially convex* on I if it is continuous and

$$\sum_{i,j=1}^{n} \xi_i \xi_j f\left(\frac{x_i + x_j}{2}\right) \geq 0, \tag{3}$$

for all $n \in \mathbb{N}$ and all choices $\xi_i \in \mathbb{R}$ and every $x_i \in I,\ 1 \leq i \leq n$.

If we take $n = 1$, we find $f(x) \geq 0$. Then, taking $n = 2$ we have the inequality

$$\xi_1^2 f(x) + 2\xi_1\xi_2 f\left(\frac{x+y}{2}\right) + \xi_2^2 f(y) \geq 0, \tag{4}$$

for all $\xi_1,\ \xi_2 \in \mathbb{R}$, and all $x, y \in I$.
In particular, if we take $\xi_1 = 1,\ \xi_2 = -1$ in (4) we have

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2},$$

i.e. exponentially convex functions are convex, since f is also continous.
Also, quadratic form (4) implies

$$f^2\left(\frac{x+y}{2}\right) \leq f(x)f(y), \text{ for all } x, y \in I. \tag{5}$$

Hence, exponentially convex functions are also *log-convex* functions. Let us denote $\mathcal{EC}(I)$ and $\mathcal{LC}(I)$, classes of exponentially convex and log-convex functions on I, respectively.
Now, we observe that (5) gives us

$$f\left(\frac{1}{2^n}x_0 + \left(1 - \frac{1}{2^n}\right)y\right) \leq f(x_0)^{1/2^n} f(y)^{1-1/2^n}.$$

So, if $f(x_0) = 0$, for some $x_0 \in I$, then

$$f\left(\frac{1}{2^n}x_0 + \left(1 - \frac{1}{2^n}\right)y\right) = 0,$$

and then $f(y) = \lim_n f\left(\frac{1}{2^n}x_0 + \left(1 - \frac{1}{2^n}\right)y\right) = 0$, concluding $f \equiv 0$, on I. Hence, similarly as in the completely monotone case, we conclude that a non-identically zero exponentially convex function f cannot vanish for any $x \in I$.

All other similarities follow from, as we will show, the inclusion $\mathcal{CM}(I) \subset \mathcal{EC}(I)$; suppose $h \in \mathcal{CM}(I)$, $n \in \mathbb{N}$, $\xi_i \in \mathbb{R}$ and $x_i \in I$, $1 \le i \le n$. Then

$$\sum_{i,j=1}^{n} \xi_i \xi_j h\left(\frac{x_i + x_j}{2}\right) = \sum_{i,j=1}^{n} \xi_i \xi_j \int_0^\infty e^{-t\frac{x_i+x_j}{2}} d\sigma_h(t) = \int_0^\infty \left(\sum_{i=1}^{n} e^{-\frac{tx_i}{2}}\right)^2 d\sigma_h(t) \ge 0.$$

Hence, $g \in \mathcal{EC}(I)$ and we showed

$$\mathcal{CM}(I) \subset \mathcal{EC}(I) \subset \mathcal{LC}(I) \subset \mathcal{C}(I).$$

We finish this overview section with a characterization of exponential convexity (see [**1**, p. 211]): The function $f \in \mathcal{EC}(I)$ if and only if

$$f(x) = \int_{-\infty}^{\infty} e^{tx} d\sigma(t), \quad x \in I \tag{6}$$

for some non-decreasing function $\sigma : \mathbb{R} \to \mathbb{R}$.

2. Examples and counterexamples

In this section we list some elementary and special functions belonging to above mentioned classes of functions.

Example 1 (Exponential function)**.** *Very first, simple example.*

(i) For $c > 0$, and $\alpha \le 0$, the function $x \mapsto ce^{\alpha x}$ belongs to $\mathcal{CM}((0,\infty))$. In a view of unilateral Laplace transform (2) *we have there*

$$\sigma(t) = \begin{cases} c, & t \in [-\alpha, \infty); \\ 0, & t \in (0, -\alpha). \end{cases}$$

(ii) For $c > 0$ and $\alpha > 0$, $x \mapsto ce^{\alpha x}$ belongs to $\mathcal{EM}(\mathbb{R})$, but not to $\mathcal{CM}(I)$, for any open interval I. In a view of bilateral Laplace transform (6) *we have there*

$$\sigma(t) = \begin{cases} c, & t \in [\alpha, \infty); \\ 0, & t \in (-\infty, \alpha). \end{cases}$$

Example 2 (Power function)**.** *For $I = (0,\infty)$, $c > 0$ we consider the function $x \mapsto cx^\beta$. Of course, it has only sense to cosider the case $\beta \le 0$, since for $\beta > 0$ the function is not even in $\mathcal{LC}(I)$.*

(i) For $c > 0$, and $\alpha < 0$, the function $x \mapsto cx^{-\alpha}$ belongs to $\mathcal{CM}(I)$, since

$$x^{-\alpha} = \int_0^\infty e^{-xt} \frac{t^{\alpha-1}}{\Gamma(\alpha)} dt. \tag{7}$$

Again, we can rewrite (7) *in the form of unilateral Laplace transform* (2) *for*

$$\sigma(t) = \frac{t^\alpha}{\alpha\Gamma(\alpha)}, \; t > 0.$$

(ii) The function $x \mapsto cx^{-\alpha}$ *is clearly also in* $\mathcal{EC}(I)$*, and we can rewrite* (7) *in the form of bilateral Laplace transform* (6) *and now*

$$\sigma(u) = \begin{cases} \frac{-c(-u)^{\alpha}}{\alpha\Gamma(\alpha)}, & u \in (-\infty, 0); \\ 0, & u \in [0, \infty). \end{cases}$$

Now $\lim_{u \to -\infty} \sigma(u) = -\infty$*, and* σ *is increasing since* $\sigma'(u) = c\frac{(-u)^{\alpha-1}}{\Gamma(\alpha)} > 0$*, for* $u \in (-\infty, 0)$*, and* $\sigma(u) = 0$*, for* $u \in [0, \infty)$*.*

Before we proceed to further examples, let us refer to a couple of theorems that can be used in generation of completely monotone (exponentially convex) functions from existing ones.

Theorem 1. *If* $f \in \mathcal{CM}(I)$ *(resp.* $\mathcal{EM}(I)$*) then* $f^{(2k)} \in \mathcal{CM}(I)$ *(resp.* $\mathcal{EM}(I)$*), for any* $k \in \mathbb{N}$*.*

The proof of the theorem follows from integral representations (2) and (6). In particular, it follows that no polynomial is completely monotone (exponentially convex) function.

Theorem 2. *The set* $\mathcal{EC}(I)$ *is a convex cone, i.e.*

(i) $$\alpha f_1 + \beta f_2 \in \mathcal{EC}(I) \quad \text{for all } \alpha,\ \beta \geq 0 \text{ and } f_1,\ f_2 \in \mathcal{EC}(I),$$

which is closed under multiplication i.e.

(ii) $$f_1 \cdot f_2 \in \mathcal{EC}(I) \quad \text{for all } f_1, f_2 \in \mathcal{EC}(I),$$

and under pointwise convergence:

(iii) $$\text{if } \lim_{n \to \infty} f_n(x) = f(x) \text{ exist for all } x \in I, \text{ then } f \in \mathcal{EC}(I).$$

Theorem 2 is proved in [**5**] and is also valid in the case of class $\mathcal{CM}(I)$, $I = (0, \infty)$, as can be found in the book [**9**, p. 5]. It is interesting to point out here that part (iii) in Schilling's book [**9**] is proved via weakly convergence of measures, which is consistent with general probabilistic approach to complete monotonicity in that book.

Theorem 3. *Let* $f \in \mathcal{CM}(I)$ *and let the power series*

$$\varphi(y) = \sum_{i=0}^{\infty} a_i y^i$$

converge for all y *in the range of the function* f*. If* $a_i \geq 0$*, for all* $i = 0, 1, 2, \ldots$*, then* $\varphi \circ f \in \mathcal{CM}(I)$*.*

The proof of the theorem obviously follows from application of Theorem 2.

Example 3 (Special functions)**.** *Let* $I = (0, \infty)$.

(i) Gamma function $\Gamma(x) = \int_0^\infty e^{-s} s^{x-1} ds$ *is in* $\mathcal{EC}(I)$ *since*

$$\sum_{i,j=1}^{n} \xi_i \xi_j \Gamma\left(\frac{x_i + x_j}{2}\right) = \int_0^\infty e^{-s} \left(\sum_{i=1}^{n} s^{\frac{x_i - 1}{2}}\right)^2 ds \geq 0.$$

Γ *function is not in* $\mathcal{CM}(I)$ *since, for example,* $\Gamma'(1) = -\gamma < 0$ *and* $\Gamma'(2) = 1 - \gamma > 0$, *where* $\gamma \approx 0.57721$ *is Euler constant.*

(ii) Hurwitz $\zeta-$*function*

$$\zeta(s, q) = \sum_{i=1}^{\infty} \frac{1}{(i+q)^s}.$$

With the application of Theorem 3 we conclude that the function $s \mapsto \zeta(s, q)$ *belongs to* $\mathcal{CM}((1, \infty))$, *and the function* $q \mapsto \zeta(s, q)$ *belongs to* $\mathcal{CM}((0, \infty))$.

(iii) Lerch transcendent

$$\Phi(s, a, z) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}.$$

Since

$$\Phi(s, a, z) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-at}}{1 - ze^{-t}} dt,$$

(a) the function $s \mapsto \Phi(s, a, z)$ *is in* $\mathcal{CM}(I)$, *for* $0 < z < 1,\ a > 0$;

(b) the function $a \mapsto \Phi(s, a, z)$ *is in* $\mathcal{CM}(I)$, *for either* $0 < z < 1,\ s > 0$, *or* $z = 1,\ s > 1$;

(c) the function $z \mapsto \Phi(s, a, z)$ *is in* $\mathcal{EM}((0, 1))$, *for* $s > 1,\ a > 0$, *but it is not in* $\mathcal{CM}((0, 1))$.

*Let us observe that the Lerch transcendent can be constructed using maximization of Shannon entropy, under some constraints, as it is showed in [**4**]. It is showed there functions in* (*a*) *and* (*b*) *part belong to* $\mathcal{EC}(I)$.

Composition of two completely monotone functions is not, in general, completely monotone function.

Example 4. *Let* $I = (0, \infty)$, $f(x) = x^{-1}$, $g(x) = x^{-3}$. *Then* $f,\ g \in \mathcal{CM}(I)$ *and* $f(g(x)) = x^3$ *and this is not completely monotone function, according to the remark after Theorem 1.*

However, if the nested function is so called *Bernstein function* then the next theorem is valid (see [**2**, p. 83.] and [**9**, p. 19.]). The proof of the theorem relies on well-known formula for the $n-$th derivative of the composition due to Faa di Bruno.

Theorem 4. *Let f and g be functions such that $f(g(x))$ is defined for all $x > 0$. If f and g' are completely monotone, then $f \circ g$ is also completely monotone.*

The previous theorem is not valid if f is exponentially convex: if we take $f(x) = e^x$ and for Bernstein function $g(x) = \log(x+1)$, then $x \mapsto f(g(x)) = x + 1$ is not exponentially convex according the comment after Theorem 1.

Theorem 4 can be also used in generation of completely monotone functions, as next examples show.

Example 5. *Let f and g be completely monotone functions. Then*

$$f\left(a + b\int_0^x g(t)dt\right),$$

where a and b are nonnegative constants, also completely monotone. In particular, the following functions are completely monotone on $I = (0, \infty)$

(i) $x \mapsto f(ax^\alpha + b),\ a \geq 0,\ b \geq 0,\ 0 \leq \alpha \leq 0;$

(ii) $x \mapsto f(a + b \ln x),\ a \geq 0,\ b \geq 0;$

(iii) $x \mapsto f(a + b\sqrt{x}),\ a \geq 0,\ b \geq 0;$

(iv) $x \mapsto f(1 - e^{-x});$

(v) $x \mapsto f(\mathrm{arctg}\sqrt{x}).$

*Some of these examples are taken from [**7**].*

We end this section with the example of function that is in $\mathcal{LC}(I)$ but not in $\mathcal{EC}(I)$.

Example 6. *Let $I = (0, 1)$. The function $x \mapsto f(x) = e^{x^3 - x}$ is in $\mathcal{LC}(I)$ but not in $\mathcal{EC}(I)$.*
Log-convexity is obvious. Since $f^{(4)}(1/5) < 0$, it is not in $\mathcal{EC}(I)$, according Theorem 1.

3. A generation method for completely monotone functions

For $n \in \mathbb{N}$, let us denote by $K_n[a, b]$ all functions from $C[a, b]$ that are $n-$convex. Hence, $f \in K_n[a, b]$ if $[x_0, x_1, \ldots, x_n; f] \geq 0$ for any choice of mutually different numbers $x_i \in [a, b],\ i = 0, \ldots, n$. Here $[x_0, x_1, \ldots, x_n; f]$ denotes divided difference of the function f in the knots $x_0, x_1, \ldots, x_n \in [a, b]$.
In the sequel, we consider linear functionals $A : C[a, b] \to \mathbb{R}$ that have the property

$$f \in K_n[a, b] \Rightarrow Af \geq 0. \tag{8}$$

One obvious example of a linear functional that has property (8) is $f \mapsto [x_0, x_1, \ldots, x_n; f]$.

In the sequel we assume $0 < a < b < \infty$.

Theorem 5. *Let $f \in C^n[a,b]$ and let $A : C[a,b] \to \mathbb{R}$ be a linear functional which satisfies property* (8). *Then there exists $\xi \in [a,b]$ such that*

$$Af = f^{(n)}(\xi) A g_0, \tag{9}$$

where $g_0(x) = x^n/n!$.

Proof. Let $m = \min_{x\in[a,b]} f^{(n)}(x),\ M = \max_{x\in[a,b]} f^{(n)}(x)$. Let us observe that the function $\varphi(x) = M\frac{x^n}{n!} - f(x) = Mg_0(x) - f(x)$ is $n-$convex function since $\varphi^{(n)}(x) = M - f^{(n)}(x) \geq 0$. Hence, $A\varphi \geq 0$ and we conclude

$$Af \leq MAg_0.$$

Similarly,

$$mAg_0 \leq Af \leq MAg_0.$$

Now we have (9). □

If we denote $p_i(x) = x^i,\ i \in \mathbb{N}_0$, then from Theorem 5 it follows

$$Ap_i = 0,\ i = 0, 1 \ldots, n-1.$$

Corollary 1. *Let $A : C[a,b] \to \mathbb{R}$ be a linear functional that satisfies* (8) *and $Ap_n > 0$.*

(i) *If $f, g \in C^n[a,b]$, then there exists $\xi \in [a,b]$ such that*

$$\frac{f^{(n)}(\xi)}{g^{(n)}(\xi)} = \frac{Af}{Ag}. \tag{10}$$

(ii) *Let I be any open interval in $(0,\infty)$. Assume that $\mathbf{F} = \{f_t : t \in I\}$ is the family of $n-$time differentiable functions on $[a,b]$, such that $t \mapsto f_t^{(n)}(x)$ is in $\mathcal{CM}(I)$, for any $x \in \mathbb{R}$. Then $t \mapsto Af_t$ is also belongs to $\mathcal{CM}(I)$;*

(iii) *for any $p \leq u,\ q \leq v,\ p,q,u,v \in I$, we have*

$$M_{p,q}(A,\mathbf{F}) \leq M_{u,v}(A,\mathbf{F}) \tag{11}$$

where

$$M_{p,q}(A,\mathbf{F}) = \begin{cases} \left(\frac{Af_p}{Af_q}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\frac{\frac{d}{dp}(Af_p)}{Af_p}\right), & p = q. \end{cases} \tag{12}$$

Proof. (i) We define an auxiliary function $\phi(t) = f(t)Af - g(t)Ag$. By Theorem 5 there exists $\xi \in [a,b]$ such that

$$A\phi = \phi^{(n)}(\xi)Ag_0.$$

Since $A\phi = 0$, the result follows.
(ii) This is direct consequence of theorem Theorem 5.
(iii) Since $t \mapsto Af_t$ is also in $\mathcal{LC}(I)$ we have (see [**8**, p. 2])

$$\frac{\log Af_q - \log Af_p}{q-p} \le \frac{\log Af_v - \log Af_u}{v-u}, \tag{13}$$

for $p \le u,\ q \le v; p \ne q, u \ne v$, that is in fact (11). The case $p = q$ in (11) we get after we pass with the limit $\lim\limits_{\substack{p\to q\\ p\le u}}$ in (13).

□

Corollary 2. *Let I be an open interval in $(0,\infty)$, $a, b \in \mathbb{R}$ and $A : C[a,b] \to \mathbb{R}$ a linear functional which satisfies (8). Let $\mathbf{F} = \{f_t : t \in I\}$ be a family of functions in $C^n[a,b]$. If*

$$a \le \left(\frac{\frac{d^n f_p}{dx^n}}{\frac{d^n f_q}{dx^n}}\right)^{\frac{1}{p-q}}(x) \le b, \tag{14}$$

for $x \in [a,b],\ p, q \in I$, then $M_{p,q}(A,\mathbf{F})$ is a mean.

If condition (14) is not fulfilled, we will call $M_{p,q}(A,\mathbf{F})$ a *quasi mean.*

Remark 1. *In some examples that will follow, we will have very simple recognition means among quasi means:*

$$\left(\frac{\frac{d^n f_p}{dx^n}}{\frac{d^n f_q}{dx^n}}\right)^{\frac{1}{p-q}}(x) = x,\ \ x \in [a,b],\ p \ne q.$$

Example 7. *Let $I = (0,\infty)$ and let family $\mathbf{F} = \{f_t : t \in I\}$ of functions defined on $[a,b],\ 0 < a < b < \infty$, by*

$$f_t(x) = \begin{cases} \frac{(-1)^n x^{-t+n}}{(t-1)(t-2)\cdots(t-n)}, & t \notin \{1,\dots,n\}; \\ \frac{(-1)^{j-1} x^{n-j} \ln x}{(j-1)!(n-j)!}, & t = j \in \{1,2\dots,n\}. \end{cases} \tag{15}$$

Since $\frac{d^n}{dx^n} f_t(x) = x^{-t}$, $t \mapsto \frac{d^n}{dx^n} f_t(x)$ is from $\mathcal{CM}(I)$ by Example 2. From Corollary 1 we then conclude complete monotonicity of the function $t \mapsto Af_t = [x_0, x_1, \dots, x_n; f_t]$.

Next we evaluate explicitly expression $M_{p,q}(A,\mathbf{F})$ introduced in (12) with monotonicity property (11), for the linear functional $Af = [x_0, x_1, \dots, x_n; f]$. Since

linear functional A is only dependent on $x_0, x_1, \dots, x_n$, so we use abbreviation $M_{p,q}(\mathbf{x}, \mathbf{F})$.
First

$$\left(\frac{\frac{d^n f_p}{dx^n}}{\frac{d^n f_q}{dx^n}}\right)^{\frac{1}{p-q}}(x) = \frac{1}{x},\ x \in [a, b],$$

so $M_{p,q}(\mathbf{x}, \mathbf{F})$ are just quasi means.

In order to deduce all limit cases for these quasi means we have to introduce some notation.

By $V(\mathbf{x}; f)$, where $\mathbf{x} = (x_0, x_1, \dots, x_n)$, we denote

$$V(\mathbf{x}; f) := \begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} & f(x_0) \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} & f(x_1) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} & f(x_n) \end{vmatrix}$$

Particulary, for $f(t) = t^r \ln^k t$ we will denote

$$V(\mathbf{x}; r, k) := \begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} & x_0^r \ln^k x_0 \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} & x_1^r \ln^k x_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} & x_n^r \ln^k x_n \end{vmatrix}.$$

Similarly, we denote

$$W(\mathbf{x}; r, k) := \begin{vmatrix} 1 & \ln x_0 & \ln^2 x_0 & \cdots & \ln^{n-1} x_0 & x_0^r \ln^k x_0 \\ 1 & \ln x_1 & \ln^2 x_1 & \cdots & \ln^{n-1} x_1 & x_1^r \ln^k x_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \ln x_n & \ln^2 x_n & \cdots & \ln^{n-1} x_n & x_n^r \ln^k x_n \end{vmatrix},$$

First, we can rewrite our linear functional in the new notation

$$[x_0, x_1, \dots, x_n; f] = \frac{V(\mathbf{x}; f)}{V(\mathbf{x}; n, 0)}. \tag{16}$$

$M_{p,q}(\mathbf{x}, \mathbf{F})$, completed with all expressions in limit cases:
(17)

$$M_{p,q}(\mathbf{x}, \mathbf{F}) = \begin{cases} \left(\frac{\prod_{k=1}^n (q-k)}{\prod_{k=1}^n (p-k)} \frac{V(\mathbf{x}; -p+n, 0)}{V(\mathbf{x}; -q+n, 0)}\right)^{\frac{1}{p-q}}, & (p-q)\prod_{k=1}^n [(q-k)(p-q)] \neq 0; \\ \left(\frac{\prod_{k=1}^n (q-k)}{(-1)^{j-1}(j-1)!(n-j)!} \ \frac{V(\mathbf{x}; n-j, 1)}{V(\mathbf{x}; -q+n, 0)}\right)^{\frac{1}{j-q}}, & q \neq p = j \in \{1, 2, \dots, n\}; \\ \left((-1)^{k-j}\binom{n}{j}\binom{n}{k}^{-1} \ \frac{V(\mathbf{x}; n-j, 1)}{V(\mathbf{x}; n-k, 1)}\right)^{\frac{1}{j-k}}, & p = k \neq j = q,\ k, j \in \{1, 2, \dots, n\}; \\ \exp\left(\frac{V(\mathbf{x}; -q+n, 1)}{V(\mathbf{x}; -q+n, 0)} - \sum_{k=1}^n \frac{1}{q-k}\right), & p = q \notin \{1, 2, \dots, n\}; \\ \exp\left(\frac{V(\mathbf{x}; -q+n, 2)}{2V(\mathbf{x}; -q+n, 1)} - \sum_{\substack{k=1 \\ k \neq q}}^n \frac{1}{q-k}\right), & p = q \in \{1, 2, \dots, n\}. \end{cases}$$

It is interesting that quasi means (17) *can be converted into means of numbers* $x_0, x_1, \dots, x_n$ *if we replace* $x_i \to \frac{1}{x_i}$, $i = 0, 1, \dots, n$, *since*

$$\min\{x_0, x_1, \dots, x_n\} \le M_{p,q}(\mathbf{x}^{-1}, \mathbf{F}) \le \max\{x_0, x_1, \dots, x_n\}$$

where $\mathbf{x}^{-1}$ *stands for the vector* $(1/x_0, 1/x_1, \dots, 1/x_n)$.
Means $M_{p,q}(\mathbf{x}^{-1}, \mathbf{F})$ *are known in the literature as* generalized Stolarsky means *(see [**6**]). One last comment about form of the family* (15)*: among all particular solutions of the differential equation* $\frac{d^n}{dx^n} f_t(x) = x^{-t}$ *we omitted those with polynomial part since our functional vanish on it.*

Example 8. *Let* $0 < a < b < \infty, I = (0, \infty)$ *and a family* $\mathbf{F} = \{f_t : t \in I\}$ *of functions defined by*

$$f_t(x) = (-1)^n \frac{e^{-tx}}{t^n} \tag{18}$$

Since $\frac{d^n}{dx^n} f_t(x) = e^{-tx}$, $t \mapsto \frac{d^n}{dx^n} f_t(x)$ *is from* $\mathcal{CM}(I)$ *by Example 2. From Corollary 1 we then conclude complete monotonicity of the function* $t \mapsto Af_t = [x_0, x_1, \dots, x_n; f_t]$.
$M_{p,q}(\mathbf{x}, \mathbf{F})$ *introduced in* (12)*, in this particular case, take the form:*

$$M_{p,q}(\mathbf{x}, \mathbf{F}) = \begin{cases} \left(\frac{[x_0, x_1, \dots, x_n; f_p]}{[x_0, x_1, \dots, x_n; f_q]} \right)^{\frac{1}{p-q}}, & p \ne q; \\ \exp\left(\frac{[x_0, x_1, \dots, x_n; g_p]}{[x_0, x_1, \dots, x_n; f_p]} \right), & p = q; \\ \exp\left(-\frac{x_0 + x_1 + \cdots + x_n}{n+1} \right), & p = q = 0, \end{cases}$$

where $g_p(x) = \frac{(-1)^{n+1}}{p^{n+1}} e^{-px}(xp^2 - n)$.

We observe here that $\left(\frac{\frac{d^n f_p}{dx^n}}{\frac{d^n f_q}{dx^n}} \right)^{\frac{1}{p-q}} (-\log x) = x$, *so after the substitution* $x_i \to -\log x_i$, $i = 0, 1, \dots, n$ *in the above expressions we will get means for the numbers* $x_0, x_1, \dots, x_n$ *:*

$$M_{p,q}(\log \mathbf{x}^{-1}, \mathbf{F}) = \begin{cases} \left(\frac{[-\log x_0, -\log x_1, \dots, -\log x_n; f_p]}{[-\log x_0, -\log x_1, \dots, -\log x_n; f_q]} \right)^{\frac{1}{p-q}}, & p \ne q; \\ \exp\left(\frac{[-\log x_0, -\log x_1, \dots, -\log x_n; g_p]}{[-\log x_0, -\log x_1, \dots, -\log x_n; f_p]} \right), & p = q; \\ \sqrt[n+1]{x_0 x_1 \cdots x_n}, & p = q = 0, \end{cases}$$

where we denoted $\log \mathbf{x}^{-1} = (\log x_0^{-1}, \log x_1^{-1}, \dots, \log x_n^{-1})$.

The further production of complete monotonicity and means is now quite easy because we can take the family of functions to be $\mathbf{F} = \{f_t : t \in I\}$

$$f_t(x) = \frac{e^{-x\sqrt{t}}}{(-\sqrt{t})^n}, \quad I = (0, \infty), \quad \text{or} \quad f_t(x) = \frac{t^{-x}}{(-\ln t)^n} \quad I = (1, \infty); \dots$$

In this technique of the production of complete monotonicity there is also no limitation on the linear functional $A : C[a,b] \to \mathbb{R}$ with property (8). For example we can use *Steffensen inequality* and define a linear functional

$$Af = \int_{b-\lambda}^{b} f(t)dt - \int_{a}^{b} f(t)g(t)dt,$$

(g is integrable on $[a,b]$, $0 \leq g \leq 1$ and $\lambda = \int_a^b g(t)dt$). All we need, in this case, is a family of functions $\mathbf{F} = \{f_t : t \in I\}$ such that the function $t \mapsto f_t'(x)$ is completely monotone.

Similarly, for the *Jensen* functional

$$Af = \frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)$$

we need a family of functions $\mathbf{F} = \{f_t : t \in I\}$ such that the function $t \mapsto f_t''(x)$ is completely monotone on I implying that $t \mapsto Af_t$ will be also completely monotone.

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