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IMPROVEMENTS OF ASYMPTOTIC APPROXIMATION FORMULAS FOR THE FACTORIAL FUNCTION

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Asymptotic expansions of the gamma function are studied and new accurate approximations for the factorial function are given.

1. INTRODUCTION AND MOTIVATION

One of the most beautiful formulas in mathematics is the classical Stirling approximation formula for the factorial function

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

This is in fact a shortening of the Stirling asymptotic expansion which has been known for centuries [**1, 7**]:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \exp\left(\frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \frac{1}{1680n^7} + \dots\right). \tag{1.1}$$

Another form of this expansion is often also credited to Stirling, but it is due to Laplace:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \frac{571}{2488320n^4} + \dots\right). \tag{1.2}$$

Some other variations of these formulas include the Wehmeier square root formula which has been rediscovered by other methods by Batir and Mortici in [**2, 9**]

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \sqrt{1 + \frac{1}{6n} + \frac{1}{72n^2} - \frac{31}{6480n^3} - \frac{139}{155520n^4} + \dots}, \tag{1.3}$$

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and the famous very accurate Ramanujan formula which is related to one of his open problems solved a decade ago by Karatsuba in [**6**]

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \sqrt[6]{1+\frac{1}{2n}+\frac{1}{8n^2}+\frac{1}{240n^3}-\frac{11}{1920n^4}+\ldots}. \tag{1.4}$$

There are also similar formulas with 4th and 8th root in Batir, Mortici [**3, 10**].

There is an interesting website [**8**], where Luschny gave a unique overview of all known approximation formulas for the factorial function. He compared their numerical precision through the number of exact decimal digits (EDD) which are defined by

$$\mathrm{EDD}(n) = -\log_{10}\left|1-\frac{\text{formula(n)}}{n!}\right|$$

for some value of n. We will present only a part of his tables and show the precision of some of the best asymptotic formulas. The (i)-th column is the EDD of the given approximation formula using the series up to the i-th order term. The bold ones are the most accurate.

Formula	n	(0)	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Laplace	100	3.1	6.5	8.6	11.7	13.1	16.2	17.2	20.4	21.1
Wehmeier	100	3.1	6.2	8.6	11.4	13.1	15.9	17.2	20.0	21.1
Ramanujan	100	3.1	5.7	9.2	11.0	13.3	15.4	17.3	19.5	21.1
De Moivre	100	**3.4**	**7.1**	8.6	**12.0**	13.1	16.4	17.2	20.5	22.8
Gosper	100	3.1	6.2	8.5	11.9	13.1	**17.5**	17.2	**21.2**	21.1
Nemes	100	3.1	6.2	**10.1**	10.9	**14.9**	15.2	**19.4**	19.2	**23.0**
Laplace	1000	4.1	8.5	11.6	15.6	18.1	22.2	24.2	28.3	30.1
Wehmeier	1000	4.1	8.2	11.6	15.4	18.1	21.9	24.2	28.0	30.1
Ramanujan	1000	4.1	7.7	12.2	15.0	18.3	21.4	24.3	27.5	30.1
De Moivre	1000	**4.4**	**9.1**	11.6	**16.0**	18.1	22.4	24.3	28.6	29.1
Gosper	1000	4.1	8.2	11.4	15.9	18.1	**23.1**	24.2	**29.7**	30.1
Nemes	1000	4.1	8.2	**13.1**	14.9	**19.7**	21.2	**26.9**	27.2	**33.5**
Laplace	10000	5.1	10.5	14.6	19.6	23.1	28.2	31.2	36.3	39.1
Wehmeier	10000	5.1	10.2	14.6	19.3	23.1	27.9	31.2	36.0	39.1
Ramanujan	10000	5.1	9.7	15.2	19.0	23.3	27.4	31.3	35.5	39.1
De Moivre	10000	**5.4**	**11.1**	14.6	**20.0**	23.1	28.5	31.2	36.6	37.1
Gosper	10000	5.1	10.2	14.4	19.9	23.1	**29.1**	31.2	**37.6**	39.1
Nemes	10000	5.1	10.2	**16.1**	18.9	**24.7**	27.2	**33.7**	35.2	**42.1**

Table 1: Precision of various asymptotic formulas listed on Luschny's website

As one can see, the most accurate formula for the small number of terms, i.e. taking zero, one or three terms of the expansion, is the De Moivre formula, sometimes called the "n-half" formula:

$$n! \sim \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}}\left(1+\frac{1}{2(n+\frac{1}{2})}+\frac{1}{8(n+\frac{1}{2})^2}+\frac{1}{240(n+\frac{1}{2})^3}+\ldots\right). \tag{1.5}$$

Using an odd number of terms (greater than 3), the best formula is the Gosper formula which has a separate square root term in front of the remaining part of the expansion, as follows

$$n! \sim \sqrt{2\pi}\left(\frac{n}{e}\right)^n \sqrt{n+\frac{1}{6}}\left(1+\frac{1}{144n^2}-\frac{23}{6480n^3}+\dots\right), \tag{1.6}$$

and taking the series up to an even term, the most accurate formula is the modification of the Gosper formula due to Nemes [**11**]

$$n! \sim \sqrt{2\pi}\left(\frac{n}{e}\right)^n \sqrt{n+\frac{1}{6}}\left(1+\frac{1}{144(n+\frac{1}{4})^2}-\frac{1}{12960(n+\frac{1}{4})^3}+\dots\right). \tag{1.7}$$

All these mentioned formulas used to be considered separately and the connection between them was not clear. Until recently there were not any attempts to find a general procedure to calculate coefficients in this expansions.

In paper [**4**], Burić and Elezović were the first to study this problem and gave a general expansion for the factorial function by introducing the parameter m. By using properties of asymptotic power series, they proved the following expansion

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left[\sum_{k=0}^{\infty} P_k\, n^{-k}\right]^{1/m}, \tag{1.8}$$

where the sequence (P_n) satisfies the simple recursive relation

$$P_n = \frac{m}{n}\sum_{k=1}^{\lfloor (n+1)/2\rfloor} \frac{B_{2k}}{2k} P_{n-2k+1}, \qquad n \geq 1, \tag{1.9}$$

with $P_0 = 1$. B_n are Bernoulli numbers.

This allows for an easy calculation of coefficients in all of the previous expansions. Moreover, the authors explained the role of number 6 in the Ramanujan expansion. Their result easily led to "n-half" formulas and they also derived a generalization of the Gosper formula, for details see [**4**].

After them, in a recent paper [**13**], Wang gave a unified approach to most of the formulas mentioned here and on Luschny's website. He presented a general formula for calculating coefficients in these expansions by using Bell polynomials.

The main aim of this paper is to analyse the known asymptotic expansions and derive new approximation formulas, numerically more precise than the one presented on Luschny's website. In the next sections we will present formulas derived from the general expansion obtained by Burić and Elezović in [**4**]. Namely, we will introduce "n-quarter" formulas which are not known in the literature.

The coefficients in the asymptotic expansions are calculated very easily through efficient recursive algorithms from paper [**4**]. All calculations in our paper are done with *Mathematica*.

2. STIRLING TYPE FORMULAS

The starting point in studying approximation formulas for the factorial function is an asymptotic expansion of the logarithm of gamma function [**7**, p. 32]:

$$\log\Gamma(x+t) \sim (x+t-\tfrac{1}{2})\log x - x + \tfrac{1}{2}\log(2\pi) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}B_{n+1}(t)}{n(n+1)} x^{-n},$$

where $B_n(t)$ are Bernoulli polynomials. Taking $t=1$, the Stirling expansion (1.1) immediately follows.

Using this expansion and manipulation with asymptotic series, in [**4**] the authors proved the following theorem.

Theorem 2.1. *The logarithm of gamma function has the following asymptotic expansion*

$$\log\Gamma(x+t) \sim (x+t-\tfrac{1}{2})\log x - x + \tfrac{1}{2}\log(2\pi) + \frac{1}{m}\log\left(\sum_{n=0}^{\infty} P_n(t)x^{-n}\right) \tag{2.10}$$

where polynomials $P_n(t)$ are defined by $P_0(t)=1$ and

$$P_n(t) = \frac{m}{n}\sum_{k=1}^{n} \frac{(-1)^{k+1}B_{k+1}(t)}{k+1} P_{n-k}(t), \qquad n \geq 1. \tag{2.11}$$

Taking $t=1$ we obtain the result (1.8) already mentioned in the Introduction. In [**4**] the authors studied this expansion for various choices of parameter m but they only considered natural values and presented formulas for $m=12$ and $m=24$ which are more precise than the Ramanjuan formula ($m=6$) when including more terms into the expansion, but these formulas are still not as good as Nemes or Gosper formula.

However, since this expansion is valid for all real $m \neq 0$, we can find even better formulas. In the Figure 1 (on the left), one can see the value for 100! and the value of asymptotic expansion for the factorial function (up to two terms) depending on the parameter m (x-axis). On the right, we can see the precision EDD for 100! also depending on the parameter m.

We can see that a better formula can be obtained for $m=5.4$ or an even better with $m=5.37$ (found numerically). Then we have the following expansion

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{179}{400n} + \frac{32041}{320000n^2} + \frac{7339}{384000000n^3} + \ldots\right)^{\frac{1}{5.37}}. \tag{2.12}$$

In the next table we will show that this is the most precise formula for approximating the factorial function when taking just two terms into an expansion and with just two decimal digits of parameter m.

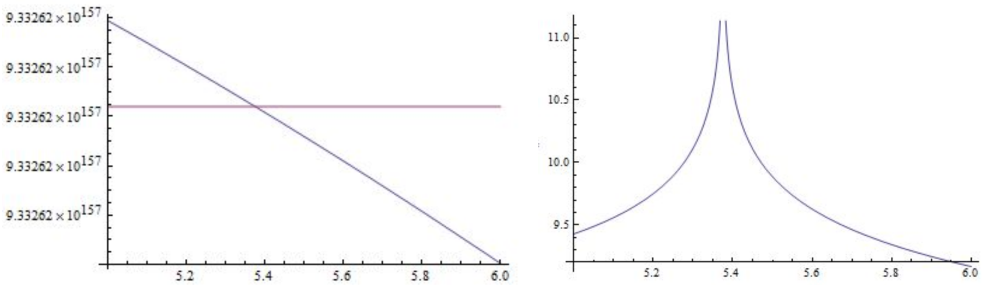

Figure 1: Precision of the general asymptotic formula for 100!

In the same way, we can find optimal values of m when taking more terms in the asymptotic expansion. Their numerical precision is presented in Table 2. Notations in the table are identical to those explained in the Introduction.

We see that by taking more even terms, the formulas are more precise with larger values of m. If we compare them with those in Table 1, we see that these formulas are more numerically precise than the Nemes formula. But of course, one can argue that formulas such as (2.12) and for other real values of m (we can achieve better precision by taking value of m by three or more decimal digits) are artificial and not applicable in a real computing sense. We do agree with that statement, but we find them very interesting primarily from the theoretical point of view since this is the first time one can investigate and study asymptotic formulas in detail and find various optimal formulas for approximating any value of factorial function.

Formula	n	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
m=1	100	**6.5**	8.6	**11.7**	13.1	**16.2**	17.2	**20.4**	21.1
m=5.37	100	5.7	**11.2**	11.0	13.3	15.5	17.3	19.6	21.1
m=12.00	100	5.4	8.0	11.2	**14.0**	15.3	17.6	19.3	21.3
m=18.03	100	5.2	7.6	10.1	12.9	15.2	**21.5**	19.3	21.5
m=24.44	100	5.1	7.3	9.7	12.2	14.8	17.4	19.3	**25.0**
m=1	1000	**8.5**	11.6	**15.6**	18.1	**22.2**	24.2	**28.3**	30.1
m=5.37	1000	7.7	**14.6**	15.0	18.3	21.4	24.3	27.5	30.1
m=12.00	1000	7.4	11.0	15.2	**19.0**	21.3	24.6	27.3	30.3
m=17.92	1000	7.2	10.5	14.1	17.9	21.2	**27.8**	27.3	30.5
m=24.21	1000	7.1	10.3	13.7	17.2	20.8	24.4	27.3	**33.7**
m=1	10000	**10.5**	14.6	**19.6**	23.1	**28.2**	31.2	**36.3**	39.1
m=5.37	10000	9.7	**17.5**	19.0	23.3	27.4	31.3	35.5	39.1
m=12.00	10000	9.4	14.0	19.2	**24.0**	27.3	31.6	35.3	39.3
m=17.90	10000	9.2	13.5	18.1	22.9	27.2	**34.7**	35.3	39.6
m=24.19	10000	9.1	13.3	17.7	22.2	26.8	31.4	35.3	**43.1**

Table 2: Precision of Stirling type formulas depending on the real values of m

3. N-HALF AND N-QUARTER FORMULAS

We saw in the previous section that we cannot obtain better precision when taking odd number of terms in classical Stirling type formulas. Table 1 from Luschny's website gives us a hint for the next step.

From (2.10) we easily derive "n-half" formulas taking $t = \frac{1}{2}$ and $x = n + \frac{1}{2}$.

Corollary 3.2. *It holds*

$$n! \sim \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}} \left[\sum_{k=0}^{\infty} P_k\,(n+\tfrac{1}{2})^{-k}\right]^{1/m}, \tag{3.13}$$

where (P_n) *is a sequence defined by* $P_0 = 1$ *and*

$$P_n = \frac{m}{n}\sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \frac{(2^{-2k+1}-1)B_{2k}}{2k} P_{n-2k+1}, \quad n \geq 1. \tag{3.14}$$

Choice $m = 1$ leads to the De Moivre expansion (1.5). By analysing formulas for various choices of parameter m, we found out that the best choices are the following: $m = 14$, $m = 22$, $m = 35$ and $m = 47$. Their precision is presented in Table 3 and here are the first few terms in their expansions:

$$n! \sim \sqrt{2\pi}\left(\frac{N}{e}\right)^{N} \sqrt[14]{1 - \frac{7}{12N} + \frac{49}{288N^2} + \frac{49}{51840N^3} + \ldots}, \tag{3.15}$$

$$n! \sim \sqrt{2\pi}\left(\frac{N}{e}\right)^{N} \sqrt[22]{1 - \frac{11}{12N} + \frac{121}{288N^2} - \frac{3883}{51840N^3} + \ldots}, \tag{3.16}$$

$$n! \sim \sqrt{2\pi}\left(\frac{N}{e}\right)^{N} \sqrt[35]{1 - \frac{35}{24N} + \frac{1225}{1152N^2} - \frac{35819}{82944N^3} + \ldots}, \tag{3.17}$$

$$n! \sim \sqrt{2\pi}\left(\frac{N}{e}\right)^{N} \sqrt[47]{1 - \frac{47}{24N} + \frac{2209}{1152N^2} - \frac{471739}{414720N^3} + \ldots}, \tag{3.18}$$

where $N = n + \frac{1}{2}$. This was not discovered by the authors in the paper [**4**].

By comparing these results with those from Table 1, we can see that "n-half" formulas lead to better numerical precision than the ordinary expansions. Moreover, taking an even number of terms, these new formulas are better than the Nemes formula (1.7), except when taking 8 terms in the expansion. For an odd number of terms, De Moivre formula remains better.

Of course, we can achieve better precision by considering the real values of parameter m but we leave this to the interested reader, it can easily be found in the same way as in the previous section.

Nemes used shift $n + \frac{1}{4}$ in his expansion, which motivated us to introduce new formulas that we shall call "n-quarter" formulas. They follow from (2.10) by taking $x = n + \frac{1}{4}$ and $t = \frac{3}{4}$.

Formula	n	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
m=1	100	**7.1**	8.6	**12.0**	13.1	**16.4**	17.2	**20.5**	21.1
m=14	100	5.9	**10.2**	11.0	13.4	15.4	17.4	19.5	21.2
m=22	100	5.7	8.5	11.1	**14.9**	15.3	17.6	19.3	21.3
m=35	100	5.5	7.9	10.7	13.6	15.5	**19.3**	19.2	21.6
m=47	100	5.4	7.6	10.1	12.9	15.6	17.9	19.3	**24.2**
m=1	1000	**9.1**	11.6	**16.0**	18.1	**22.4**	24.3	**28.6**	30.1
m=14	1000	7.9	**13.2**	15.0	18.4	21.4	24.2	27.5	30.1
m=22	1000	7.7	11.5	15.1	**20.1**	21.3	24.6	27.3	30.2
m=35	1000	7.5	10.9	14.7	18.6	21.5	**27.6**	27.3	30.6
m=47	1000	7.4	10.6	14.1	17.9	21.6	24.9	27.3	**32.1**
m=1	10000	**11.1**	14.6	**20.0**	23.1	**28.5**	31.2	**36.6**	39.1
m=14	10000	9.9	**16.2**	19.0	23.4	27.4	31.3	35.5	39.1
m=22	10000	9.7	14.5	19.1	**25.1**	27.3	31.6	35.3	39.2
m=35	10000	9.5	13.9	18.7	23.6	27.5	**34.1**	35.3	39.6
m=47	10000	9.4	13.6	18.1	22.9	27.6	31.9	35.3	**41.0**

Table 3: Precision of n-half formulas depending on the integer value of m

Corollary 3.3. *It holds*

$$n! \sim \sqrt{2\pi}(n+\tfrac{1}{4})^{\frac{1}{4}}\left(\frac{n+\frac{1}{4}}{e}\right)^{n+\frac{1}{4}}\left[\sum_{k=0}^{\infty} P_k\,(n+\tfrac{1}{4})^{-k}\right]^{1/m}, \tag{3.19}$$

where (P_n) *is a sequence defined by* $P_0 = 1$ *and*

$$P_n = \frac{m}{n}\sum_{k=1}^{n}\frac{(2^{-k}-1)B_{k+1}-(k+1)2^{-(k+1)}E_k}{(k+1)2^{k+1}}P_{n-k}, \quad n \ge 1, \tag{3.20}$$

where B_k *are Bernoulli numbers and* E_k *Euler numbers.*

In a similar way as for "n-half" formulas, we analysed their numerical precision and found the optimal integer values of parameter m. The results are presented in Table 4 and here are the beginnings of several asymptotic expansions from the table:

$$n! \sim \sqrt{2\pi}N^{\frac{1}{4}}\left(\frac{N}{e}\right)^{N}\left(1-\frac{1}{96N}+\frac{145}{18432N^2}+\frac{1867}{26542080N^3}+\dots\right), \tag{3.21}$$

$$n! \sim \sqrt{2\pi}N^{\frac{1}{4}}\left(\frac{N}{e}\right)^{N}\sqrt{1-\frac{1}{48N}+\frac{73}{4608N^2}-\frac{77}{3317760N^3}+\dots}, \tag{3.22}$$

$$n! \sim \sqrt{2\pi}N^{\frac{1}{4}}\left(\frac{N}{e}\right)^{N}\sqrt[29]{1-\frac{29}{96N}+\frac{5017}{18432N^2}-\frac{1821577}{26542080N^3}+\dots}, \tag{3.23}$$

where $N = n+\frac{1}{4}$.

Formula	n	(0)	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
m=1	100	**4.0**	**6.1**	10.2	10.9	**15.1**	15.2	**19.0**	19.2	**22.8**
m=2	100	4.0	6.1	**10.6**	10.9	14.7	15.2	18.8	19.2	22.6
m=29	100	4.0	6.0	8.6	**12.9**	14.0	15.4	17.9	19.3	21.8
m=41	100	4.0	6.0	8.5	11.2	14.2	**15.5**	17.8	19.4	21.7
m=100	100	4.0	5.9	8.0	10.2	12.5	14.8	17.1	**21.5**	21.6
m=1	1000	**5.0**	**8.1**	13.2	14.9	**20.7**	21.2	**26.3**	27.2	**32.1**
m=2	1000	5.0	8.1	**13.9**	14.9	19.8	21.2	25.9	27.2	31.9
m=29	1000	5.0	8.0	11.6	**16.9**	19.0	21.4	24.9	27.3	30.8
m=41	1000	5.0	8.0	11.5	15.1	19.2	**21.4**	24.8	27.3	30.7
m=100	1000	5.0	7.9	11.0	14.2	17.5	20.8	24.1	**29.2**	30.6
m=1	10000	**6.0**	**10.1**	16.2	18.9	**25.8**	21.2	**33.4**	35.2	**41.2**
m=2	10000	6.0	10.1	**16.9**	18.9	24.8	27.2	33.0	35.2	40.9
m=29	10000	6.0	10.0	14.6	**20.9**	24.0	27.4	31.9	35.3	39.8
m=41	10000	6.0	10.0	14.5	19.1	24.2	**27.4**	31.8	35.3	36.7
m=100	10000	6.0	9.9	14.0	18.2	22.5	26.8	31.1	**37.2**	39.6

Table 4: Precision of n-quarter formulas depending on the integer value of m

We can see that "n-quarter" formulas are most accurate for the small number of terms, even the first simple formula for $m = 1$ is better than the De Moivre formula! Taking expansions up to two or four terms, $m = 2$ and $m = 29$ both lead to better precision than the Nemes formula. If we take more terms, they are close to the Gosper and Nemes formula, but "n-half" formulas are still better.

4. FINAL REMARKS AND CONCLUSION

The Stirling asymptotic expansion for the factorial function has been known for centuries, but its improvements and modifications are still subject of many research papers and are studied by mathematicians all around the world. A new approach to these formulas was shown in the paper [**4**], where the authors introduced the parameter m and presented a general procedure for obtaining various asymptotic expansions with the main result in Theorem 2.1. More recently, a unified approach to this topic was presented in a paper [**13**], while a detailed comparison of all formulas can be found on Luschny's website [**8**].

The introducing of the general parameter m gives a new interesting view on this subject and it enables the derivation of many new accurate approximation formulas for the factorial function. Although taking real values of m in a way leads to artificial formulas, e.g. the formula (2.12) and others from Table 2, it is interesting to analyse asymptotic expansions in this way.

We can also easily derive "n-half" and "n-quarter" formulas which are more accurate (for some value of the parameter m) than the already known formulas.

We would especially like to single out this simple "n-quarter" formula

$$n! \sim \sqrt{2\pi}(n + \tfrac{1}{4})^{\frac{1}{4}} \left(\frac{n + \frac{1}{4}}{e} \right)^{n+\frac{1}{4}} \tag{4.24}$$

which is, to our knowledge, when comparing the results from Table 4 to Table 1 from [**8**], the most accurate asymptotic formula for factorial function without additional terms, and also the most accurate formula when taking two terms in the expansion:

$$n! \sim \sqrt{2\pi} N^{\frac{1}{4}} \left(\frac{N}{e} \right)^N \sqrt{1 - \frac{1}{48N} + \frac{73}{4608N^2}} \tag{4.25}$$

with $N = n + \frac{1}{4}$. These "n-quarter" formulas are also the most accurate ones when taking two more terms, but when taking 5 of more terms in the expansions, the Gosper and Nemes asymptotic formulas are still the best ones.

Finally, we would also like to note that continued fraction formulas for the factorial function are simpler, better and computationally less expensive than asymptotic formulas. Interested readers can refer to Luschny's website [**8**] for a detailed report and comparison of all approximations formulas for the factorial function. There, one can also find efficient and powerful continued fraction formulas derived by Luschny.

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