

ORTHOGONAL DIRICHLET POLYNOMIALS WITH CONSTANT WEIGHT

*Dedicated to Academician Professor Gradimir Milovanović
on the occasion of his 70th birthday.*

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Let $\{\lambda_j\}_{j=1}^{\infty}$ be a sequence of distinct positive numbers. We analyze the orthogonal Dirichlet polynomials $\{\psi_{n,T}\}$ formed from linear combinations of $\{\lambda_j^{-it}\}_{j=1}^n$, associated with constant (or Legendre) weight on $[-T, T]$. Thus

$$\frac{1}{2T} \int_{-T}^T \psi_{n,T}(t) \overline{\psi_{m,T}(t)} dt = \delta_{mn}.$$

Moreover, we analyze how these polynomials behave as T varies.

1. Introduction

Throughout, let

(1) $\{\lambda_j\}_{j=1}^{\infty}$ be a sequence of distinct positive numbers.

Given $m \geq 1$, a *Dirichlet polynomial of degree $\leq n$* [16], [17] associated with this sequence of exponents has the form

$$\sum_{n=1}^m a_n \lambda_n^{-it} = \sum_{n=1}^m a_n e^{-i(\log \lambda_n)t},$$

where $\{a_n\} \subset \mathbb{C}$. We denote the set of all such polynomials by \mathcal{L}_n .

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The traditional orthogonal Dirichlet polynomials are just the “monomials” $\{\lambda_n^{-it}\}$ themselves. Indeed, in the theory of almost-periodic functions [1], [2], heavy use is made of orthogonality in the mean:

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \lambda_j^{-it} \overline{\lambda_k^{-it}} dt = \delta_{jk}.$$

In the hope that a more standard orthogonality relation might have some advantages, the author [6], investigated Dirichlet orthogonal polynomials associated with the arctangent density. Thus $\phi_n \in \mathcal{L}_n$ has positive leading coefficient, and

$$\int_{-\infty}^{\infty} \phi_n(t) \overline{\phi_m(t)} \frac{dt}{\pi(1+t^2)} = \delta_{mn}, \quad m, n \geq 1.$$

These Dirichlet orthogonal polynomials admit a very simple explicit expression, at least when $0 < \lambda_1 < \lambda_2 < \dots$: for $n \geq 2$,

$$\phi_n(t) = \frac{\lambda_n^{1-it} - \lambda_{n-1}^{1-it}}{\sqrt{\lambda_n^2 - \lambda_{n-1}^2}}.$$

These orthonormal polynomials have been applied in several questions by Weber and Dimitrov as well as the author [4], [7], [15], [17], [18], [19]. In a subsequent paper [8], the author considered orthogonal Dirichlet polynomials for the Laguerre weight, though it turned out that much of the material there was already subsumed by Müntz orthogonal polynomials [3]. Müntz orthogonal polynomials have also been a topic investigated by Gradimir Milovanovic [10], [11], [12], to whom this paper is dedicated.

Very recently [9], we investigated Dirichlet orthogonal polynomials for rational weights

$$w(t) = \sum_{m=1}^L \frac{c_m}{\pi(1+(b_m t)^2)}$$

and appropriately chosen $\{c_j\}$. Here $L \geq 1$, and $1 = b_1 < b_2 < \dots < b_L$. We obtained a simple explicit determinantal expression for the orthonormal polynomials, but could only resolve positivity of the weight for the case $L = 2$.

In this paper, we let $T > 0$, and consider $\psi_{n,T} \in \mathcal{L}_n$, with positive leading coefficient $\gamma_{n,T}$, such that

$$(2) \quad (\psi_{n,T}, \psi_{m,T})_T = \frac{1}{2T} \int_{-T}^T \psi_{n,T}(t) \overline{\psi_{m,T}(t)} dt = \delta_{mn}.$$

We are especially interested in how $\psi_{n,T}$ behaves as T varies, and in particular how it behaves as $T \rightarrow \infty$. Next, define the n th reproducing kernel

$$(3) \quad K_{n,T}(u, v) = \sum_{j=1}^n \psi_{j,T}(u) \overline{\psi_{j,T}(v)}.$$

In the sequel, we also use

$$\mathbb{S}(u) = \frac{\sin u}{u}.$$

From the simple relation

$$(4) \quad (\lambda_j^{-it}, \lambda_k^{-it})_T = \frac{1}{2T} \int_{-T}^T (\lambda_j/\lambda_k)^{-it} dt = \mathbb{S}(T \log(\lambda_j/\lambda_k)),$$

and standard determinantal representations for orthonormal functions with respect to a given inner product, we see that

$$(5) \quad \psi_{n,T}(x) = \frac{(-1)^{n+1}}{\sqrt{A_{n-1,T} A_{n,T}}} \times \det \begin{bmatrix} \lambda_1^{-ix} & \lambda_2^{-ix} & \lambda_3^{-ix} & \cdots & \lambda_n^{-ix} \\ 1 & \mathbb{S}(T \log \lambda_1/\lambda_2) & \mathbb{S}(T \log \lambda_1/\lambda_3) & \cdots & \mathbb{S}(T \log \lambda_1/\lambda_n) \\ \mathbb{S}(T \log \lambda_2/\lambda_1) & 1 & \mathbb{S}(T \log \lambda_2/\lambda_3) & \cdots & \mathbb{S}(T \log \lambda_2/\lambda_n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{S}(T \log \lambda_{n-1}/\lambda_1) & \mathbb{S}(T \log \lambda_{n-1}/\lambda_2) & \mathbb{S}(T \log \lambda_{n-1}/\lambda_3) & \cdots & \mathbb{S}(T \log \lambda_{n-1}/\lambda_n) \end{bmatrix},$$

so the leading coefficient of $\psi_{n,T}(x)$ is

$$(6) \quad \gamma_{n,T} = \sqrt{\frac{A_{n-1,T}}{A_{n,T}}},$$

where

$$(7) \quad A_{n,T} = \det [\mathbb{S}(T \log \lambda_j/\lambda_k)]_{1 \leq j,k \leq n}.$$

It follows easily from the determinantal expression and the fact that $\lim_{x \rightarrow \infty} \mathbb{S}(x) = 0$, that

$$\lim_{T \rightarrow \infty} \psi_{n,T}(x) = \lambda_n^{-ix}$$

and that $\psi_{n,T}$ is an infinitely differentiable function of T .

One of the motivations for our study is the celebrated Montgomery-Vaughan inequality and its ramifications. In one form it asserts that [14, p. 74, Corollary 2], [13, p. 128, Thm. 1]

$$(8) \quad \int_0^T \left| \sum_{j=1}^n a_j \lambda_j^{-it} \right|^2 dt = (T + 2\pi\epsilon\delta^{-1}) \sum_{j=1}^n |a_j|^2,$$

where

$$\delta = \min \{ |\log \lambda_j - \log \lambda_k| : 1 \leq j, k \leq n \text{ and } j \neq k \},$$

while $|\varepsilon| \leq 1$. We hope that a theory of orthogonal Dirichlet polynomials might contribute to this circle of ideas and to estimates involving Dirichlet polynomials. We begin with a simple result related to the Montgomery-Vaughan inequality: write for $j \geq 1$, $T > 0$,

$$(9) \quad \lambda_j^{-it} = \sum_{k=1}^j c_{T,j,k} \psi_{k,T}(t).$$

Also write

$$(10) \quad \psi_{n,T}(t) = \sum_{j=1}^n d_{T,n,j} \lambda_j^{-it}.$$

Let

$$(11) \quad C_{T,n} = \begin{bmatrix} c_{T,1,1} & c_{T,2,1} & c_{T,3,1} & \cdots & c_{T,n,1} \\ 0 & c_{T,2,2} & c_{T,3,2} & \cdots & c_{T,n,2} \\ 0 & 0 & c_{T,3,3} & \cdots & c_{T,n,3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_{T,n,n} \end{bmatrix}.$$

Theorem 1. (a) For any complex numbers $\{a_j\}_{j=1}^n$,

$$(12) \quad \frac{1}{2T} \int_{-T}^T \left| \sum_{j=1}^n a_j \lambda_j^{-it} \right|^2 dt = \|C_{T,n} \mathbf{a}\|^2$$

where $\mathbf{a} = [a_1 \ a_2 \ \dots \ a_n]^T$ and the norm is the usual Euclidean norm. In particular,

$$(13) \quad \sup_{\{a_j\}} \frac{1}{2T} \int_{-T}^T \left| \sum_{j=1}^n a_j \lambda_j^{-it} \right|^2 dt / \sum_{j=1}^n |a_j|^2 = \|C_{T,n}\|^2,$$

where the norm is the usual matrix norm induced by the Euclidean norm.

(b) The coefficients $\{c_{T,j,k}\}$ and $\{d_{T,n,k}\}$ are real.

(c) For $j, k \geq 1$,

$$(14) \quad \sum_{\ell=1}^{\min\{j,k\}} c_{T,k,\ell} c_{T,j,\ell} = \mathbb{S}(T \log \lambda_j / \lambda_k).$$

Next, we consider $\psi'_{n,T}$:

Theorem 2.

(a)

$$\begin{aligned}
 \psi'_{n,T}(t) &= (-i \log \lambda_n) \psi_{n,T}(t) \\
 &\quad + \frac{1}{2T} \left(\psi_{n,T}(t) K_{n-1,T}(t, T) - \overline{\psi_{n,T}(t)} K_{n-1,T}(t, -T) \right) \\
 &= (-i \log \lambda_n) \psi_{n,T}(t) \\
 (15) \quad &\quad + \frac{i}{T} \sum_{j=1}^{n-1} \psi_{j,T}(t) \operatorname{Im} \left(\psi_{n,T}(T) \overline{\psi_{j,T}(T)} \right).
 \end{aligned}$$

(b)

$$\begin{aligned}
 \frac{1}{2T} \int_{-T}^T |\psi'_{n,T}|^2 &= (\log \lambda_n)^2 + \frac{1}{T^2} \sum_{j=1}^{n-1} \left| \operatorname{Im} \left(\psi_{n,T}(T) \overline{\psi_{j,T}(T)} \right) \right|^2 \\
 (16) \quad &= (\log \lambda_n)^2 + \frac{1}{T} \operatorname{Re} \left(\psi_{n,T} \overline{\psi'_{n,T}} \right) (T).
 \end{aligned}$$

Next, we compare the orthonormal polynomials $\psi_{n,T}$ and $\psi_{n,S}$ for different S, T :

Theorem 3. *Let $S > T$.*

(a)

$$(17) \quad \Delta_{n,T} = \frac{1}{2T} \int_{-T}^T \left| \psi_{n,S}(t) - \frac{\gamma_{n,S}}{\gamma_{n,T}} \psi_{n,T}(t) \right|^2 dt \leq \frac{S}{T} - \left(\frac{\gamma_{n,S}}{\gamma_{n,T}} \right)^2.$$

(b)

$$(18) \quad \frac{\gamma_{n,S}}{\gamma_{n,T}} \leq \left(\frac{S}{T} \right)^{1/2}.$$

(c)

$$(19) \quad K_{n,T}(x, x) + \left(\frac{S}{T} - 2 \right) K_{n,S}(x, x) \geq 0.$$

Finally, we consider the rate of change of several quantities w.r.t. T :

Theorem 4.

(a)

$$(20) \quad \frac{\partial}{\partial T} K_{n,T}(x, x) = \frac{1}{T} K_{n,T}(x, x) - \frac{1}{2T} \left(|K_n(x, T)|^2 + |K_n(x, -T)|^2 \right).$$

(b)

$$(21) \quad \frac{\partial (\ln \gamma_{n,T})}{\partial T} = \frac{1}{2T} (1 - |\psi_{n,T}(T)|^2).$$

(c)

$$(22) \quad \frac{\partial}{\partial T} \ln A_{n,T} = -\frac{1}{T} (n - K_{n,T}(T, T)).$$

(d) Let $c_{T,j,k}$ be the connection coefficient as in (9). Then

$$(23) \quad \begin{aligned} \frac{\partial}{\partial T} c_{T,j,k} + \frac{1}{T} c_{T,j,k} &= \frac{1}{2T} \left[\lambda_j^{-iT} \overline{\psi_{k,T}(T)} + \lambda_j^{iT} \psi_{k,T}(T) \right] \\ &+ \frac{1}{2T} \int_{-T}^T \lambda_j^{-it} \frac{\partial}{\partial T} \overline{\psi_{k,T}(t)} dt. \end{aligned}$$

We prove Theorems 1 and 2 in Section 2, and Theorems 3 and 4 in Section 3.

2. Proof of Theorems 1 and 2

Proof of Theorem 1

(a)

$$(24) \quad \begin{aligned} \frac{1}{2T} \int_{-T}^T \left| \sum_{k=1}^n a_k \lambda_k^{-it} \right|^2 dt &= \frac{1}{2T} \int_{-T}^T \left| \sum_{k=1}^n a_k \left(\sum_{j=1}^k c_{T,k,j} \psi_{j,T}(t) \right) \right|^2 dt \\ &= \frac{1}{2T} \int_{-T}^T \left| \sum_{j=1}^n \psi_{j,T}(t) \left\{ \sum_{k=j}^n a_k c_{T,k,j} \right\} \right|^2 dt \\ &= \sum_{j=1}^n \left| \sum_{k=j}^n a_k c_{T,k,j} \right|^2 \\ &= \sum_{j=1}^n |(C_{T,n\mathbf{a}})_j|^2 = \|C_{T,n\mathbf{a}}\|^2. \end{aligned}$$

(b) First, if $j < k$,

$$\begin{aligned} &\frac{1}{2T} \int_{-T}^T \lambda_j^{-it} \overline{\psi_{k,T}(-t)} dt \\ &= \frac{1}{2T} \int_{-T}^T \lambda_j^{is} \psi_{k,T}(s) ds \\ &= \frac{1}{2T} \int_{-T}^T \lambda_j^{-is} \overline{\psi_{k,T}(s)} ds = 0. \end{aligned}$$

Also,

$$\overline{\psi_{k,T}(-t)} = \sum_{j=1}^k \overline{d_{T,k,j}} \lambda_j^{-it}$$

so is also an orthonormal polynomial (recall the leading coefficient is positive). By uniqueness,

$$(25) \quad \overline{\psi_{k,T}(-t)} = \psi_{k,T}(t),$$

and hence the $\{d_{T,k,j}\}$ are real. Next, by orthogonality,

$$\begin{aligned} \overline{c_{T,j,k}} &= \frac{1}{2T} \int_{-T}^T \lambda_j^{-it} \overline{\psi_{k,T}(t)} dt \\ &= \frac{1}{2T} \int_{-T}^T \lambda_j^{it} \psi_{k,T}(t) dt \\ &= \frac{1}{2T} \int_{-T}^T \lambda_j^{-it} \psi_{k,T}(-t) dt \\ &= \frac{1}{2T} \int_{-T}^T \lambda_j^{-it} \overline{\psi_{k,T}(t)} dt = c_{T,j,k}. \end{aligned}$$

Thus the $\{c_{T,k,j}\}$ are real.

(c) From (4), (24), and (a),

$$\begin{aligned} &\sum_{1 \leq j,k \leq n} a_j \overline{a_k} \mathfrak{S}(T \log \lambda_j / \lambda_k) \\ &= \sum_{\ell=1}^n \left| \sum_{k=\ell}^n a_k c_{T,k,\ell} \right|^2 \\ &= \sum_{1 \leq k,j \leq n} a_j \overline{a_k} \sum_{\ell=1}^{\min\{j,k\}} c_{T,k,\ell} c_{T,j,\ell}. \end{aligned}$$

Choosing some $a_j = 1$ and all remaining $a's = 0$ gives

$$\sum_{\ell=1}^j c_{T,j,\ell}^2 = 1.$$

Next choose distinct j, k and $a_j = a_k = 1$ with all remaining $a's = 0$. Then we obtain

$$2\mathfrak{S}(T \log \lambda_j / \lambda_k) + 1 = 2 \sum_{\ell=1}^{\min\{j,k\}} c_{T,k,\ell} c_{T,j,\ell} + \sum_{\ell=1}^j c_{T,j,\ell}^2.$$

Thus we obtain (14) in full generality. □

Proof of Theorem 2

(a) Write

$$\psi'_{n,T}(t) = (-i \log \lambda_n) \psi_{n,T}(t) + \sum_{j=1}^{n-1} \beta_j \psi_{j,T}(t).$$

Here, integrating by parts, for $j \leq n-1$,

$$\begin{aligned} \beta_j &= \frac{1}{2T} \int_{-T}^T \psi'_{n,T}(t) \overline{\psi_{j,T}(t)} dt \\ &= \frac{1}{2T} \left\{ \psi_{n,T}(T) \overline{\psi_{j,T}(T)} - \psi_{n,T}(-T) \overline{\psi_{j,T}(-T)} \right\} \\ &\quad - \frac{1}{2T} \int_{-T}^T \psi_{n,T}(t) \overline{\psi'_{j,T}(t)} dt \\ &= \frac{1}{2T} \left\{ \psi_{n,T}(T) \overline{\psi_{j,T}(T)} - \overline{\psi_{n,T}(T)} \psi_{j,T}(T) \right\} \\ &= \frac{i}{T} \operatorname{Im} \left(\psi_{n,T}(T) \overline{\psi_{j,T}(T)} \right). \end{aligned}$$

So

$$\begin{aligned} \psi'_{n,T}(t) &= (-i \log \lambda_n) \psi_{n,T}(t) \\ &\quad + \frac{1}{2T} \left\{ \psi_{n,T}(T) \sum_{j=1}^{n-1} \overline{\psi_{j,T}(T)} \psi_{j,T}(t) - \psi_{n,T}(-T) \sum_{j=1}^{n-1} \overline{\psi_{j,T}(-T)} \psi_{j,T}(t) \right\} \\ &= (-i \log \lambda_n) \psi_{n,T}(t) + \frac{1}{2T} \left\{ \psi_{n,T}(T) K_{n-1}(t, T) - \overline{\psi_{n,T}(T)} K_{n-1}(t, -T) \right\}. \end{aligned}$$

Also,

$$\psi'_{n,T}(t) = (-i \log \lambda_n) \psi_{n,T}(t) + \frac{i}{T} \sum_{j=1}^{n-1} \operatorname{Im} \left(\psi_{n,T}(T) \overline{\psi_{j,T}(T)} \right) \psi_{j,T}(t).$$

(b) The first identity in (16) follows from the second identity in (15). Next, integrating by parts gives

$$\begin{aligned} &\frac{1}{2T} \int_{-T}^T |\psi'_{n,T}|^2 \\ &= \frac{1}{2T} \int_{-T}^T \psi'_{n,T} \overline{\psi'_{n,T}} \\ (26) \quad &= \frac{1}{2T} \left\{ \left(\psi_{n,T} \overline{\psi'_{n,T}} \right) (T) - \left(\psi_{n,T} \overline{\psi'_{n,T}} \right) (-T) \right\} - \frac{1}{2T} \int_{-T}^T \psi_{n,T} \overline{\psi''_{n,T}}. \end{aligned}$$

Here using (a) twice,

$$\psi''_{n,T} = -(\log \lambda_n)^2 \psi_{n,T} + P,$$

where $P \in \mathcal{L}_{n-1}$, so

$$(27) \quad \frac{1}{2T} \int_{-T}^T \psi_{n,T} \overline{\psi''_{n,T}} = -(\log \lambda_n)^2.$$

Next,

$$\psi'_{n,T}(t) = -i \sum_{j=1}^n f_j (\log \lambda_j) \lambda_j^{-it},$$

where all f_j are real, so

$$\begin{aligned} \overline{\psi'_{n,T}(-T)} &= -i \sum_{j=1}^n f_j (\log \lambda_j) \lambda_j^{iT} \\ &= i \sum_{j=1}^n f_j (\log \lambda_j) \lambda_j^{-iT} = -\psi'_{n,T}(T). \end{aligned}$$

Substituting this and (25) into (26), gives

$$\begin{aligned} &\frac{1}{2T} \int_{-T}^T |\psi'_{n,T}|^2 \\ &= \frac{1}{2T} \left\{ (\psi_{n,T} \overline{\psi'_{n,T}})(T) + (\overline{\psi_{n,T}} \psi'_{n,T})(T) \right\} + (\log \lambda_n)^2 \\ &= \frac{1}{T} \operatorname{Re} \left(\psi_{n,T} \overline{\psi'_{n,T}} \right)(T) + (\log \lambda_n)^2. \end{aligned}$$

□

3. Proof of Theorems 3 and 4n

Proof of Theorem 3

(a)

$$\begin{aligned} \Delta_{n,T} &:= \frac{1}{2T} \int_{-T}^T \left| \psi_{n,S}(t) - \frac{\gamma_{n,S}}{\gamma_{n,T}} \psi_{n,T}(t) \right|^2 dt \\ &= \frac{1}{2T} \int_{-T}^T \left(\psi_{n,S}(t) - \frac{\gamma_{n,S}}{\gamma_{n,T}} \psi_{n,T}(t) \right) \overline{\left(\psi_{n,S}(t) - \frac{\gamma_{n,S}}{\gamma_{n,T}} \psi_{n,T}(t) \right)} dt \\ &= \frac{1}{2T} \int_{-T}^T \left(\psi_{n,S}(t) - \frac{\gamma_{n,S}}{\gamma_{n,T}} \psi_{n,T}(t) \right) \overline{\psi_{n,S}(t)} dt \\ &= \frac{1}{2T} \int_{-T}^T |\psi_{n,S}(t)|^2 dt - \left(\frac{\gamma_{n,S}}{\gamma_{n,T}} \right)^2 \leq \frac{S}{T} - \left(\frac{\gamma_{n,S}}{\gamma_{n,T}} \right)^2. \end{aligned}$$

(b) Then also,

$$\frac{\gamma_{n,S}}{\gamma_{n,T}} \leq \left(\frac{S}{T} \right)^{1/2}.$$

(c)

$$\begin{aligned}
 & \frac{1}{2T} \int_{-T}^T |K_{n,S}(x,t) - K_{n,T}(x,t)|^2 dt \\
 (28) \quad & = \frac{1}{2T} \int_{-T}^T |K_{n,S}(x,t)|^2 dt - 2K_{n,S}(x,x) + K_{n,T}(x,x).
 \end{aligned}$$

Also,

$$\begin{aligned}
 & \frac{1}{2T} \int_{-T}^T |K_{n,S}(x,t)|^2 dt \\
 & = \frac{S}{T} \left[\frac{1}{2S} \int_{-S}^S |K_{n,S}(x,t)|^2 dt - \frac{1}{2S} \int_{T \leq |t| \leq S} |K_{n,S}(x,t)|^2 dt \right] \\
 & = \frac{S}{T} \left[K_{n,S}(x,x) - \frac{1}{2S} \int_{T \leq |t| \leq S} |K_{n,S}(x,t)|^2 dt \right]
 \end{aligned}$$

So substituting in (28) above,

$$\begin{aligned}
 0 & \leq \frac{S}{T} \left[K_{n,S}(x,x) - \frac{1}{2S} \int_{T \leq |t| \leq S} |K_{n,S}(x,t)|^2 dt \right] \\
 & \quad - 2K_{n,S}(x,x) + K_{n,T}(x,x) \\
 & = K_{n,T}(x,x) + \left(\frac{S}{T} - 2 \right) K_{n,S}(x,x) \\
 (29) \quad & \quad - \frac{1}{2T} \int_{T \leq |t| \leq S} |K_{n,S}(x,t)|^2 dt.
 \end{aligned}$$

In particular, (19) follows. □**Proof of Theorem 4**(a) Let $S > T$. Now from (29) above,

$$\begin{aligned}
 K_{n,T}(x,x) - K_{n,S}(x,x) & \geq \left(1 - \frac{S}{T} \right) K_{n,S}(x,x) \\
 & \quad + \frac{1}{2T} \int_{T \leq |t| \leq S} |K_{n,S}(x,t)|^2 dt
 \end{aligned}$$

so

$$\begin{aligned}
 \frac{K_{n,T}(x,x) - K_{n,S}(x,x)}{T - S} & \leq \frac{1}{T} K_{n,S}(x,x) \\
 (30) \quad & \quad + \frac{1}{T - S} \frac{1}{2T} \int_{T \leq |t| \leq S} |K_{n,S}(x,t)|^2 dt.
 \end{aligned}$$

Next,

$$\begin{aligned}
 0 &\leq \frac{1}{2S} \int_{-S}^S |K_{n,S}(x,t) - K_{n,T}(x,t)|^2 dt \\
 &= K_{n,S}(x,x) - 2K_{n,T}(x,x) + \frac{1}{2S} \int_{-S}^S |K_{n,T}(x,t)|^2 dt \\
 &= K_{n,S}(x,x) - 2K_{n,T}(x,x) \\
 &\quad + \frac{T}{S} \left[K_{n,T}(x,x) + \frac{1}{2T} \int_{T \leq |t| \leq S} |K_{n,T}(x,t)|^2 dt \right] \\
 &= K_{n,S}(x,x) + \left(\frac{T}{S} - 2 \right) K_{n,T}(x,x) + \frac{1}{2S} \int_{T \leq |t| \leq S} |K_{n,T}(x,t)|^2 dt,
 \end{aligned}$$

so

$$\begin{aligned}
 K_{n,S}(x,x) - K_{n,T}(x,x) &\geq \left(1 - \frac{T}{S} \right) K_{n,T}(x,x) \\
 &\quad - \frac{1}{2S} \int_{T \leq |t| \leq S} |K_{n,T}(x,t)|^2 dt.
 \end{aligned}$$

Then

$$\begin{aligned}
 \frac{K_{n,S}(x,x) - K_{n,T}(x,x)}{S - T} &\geq \frac{1}{S} K_{n,T}(x,x) \\
 &\quad - \frac{1}{S - T} \frac{1}{2S} \int_{T \leq |t| \leq S} |K_{n,T}(x,t)|^2 dt.
 \end{aligned}$$

Together with (30), this establishes

$$\begin{aligned}
 &\frac{1}{S} K_{n,T}(x,x) - \frac{1}{S - T} \frac{1}{2S} \int_{T \leq |t| \leq S} |K_{n,T}(x,t)|^2 dt \\
 &\leq \frac{K_{n,S}(x,x) - K_{n,T}(x,x)}{S - T} \\
 (31) \quad &\leq \frac{1}{T} K_{n,S}(x,x) + \frac{1}{T - S} \frac{1}{2T} \int_{T \leq |t| \leq S} |K_{n,S}(x,t)|^2 dt.
 \end{aligned}$$

Inasmuch as $K_{n,T}(x,x) - K_{n,S}(x,x) \rightarrow 0$ as $|S - T| \rightarrow 0$, (indeed, the representation (5) shows that $\psi_{n,T}$ and hence $K_{n,T}$ are infinitely differentiable in T), this last inequality yields that

$$\frac{\partial}{\partial T} K_{n,T}(x,x) = \frac{1}{T} K_{n,T}(x,x) - \frac{1}{2T} \left(|K_n(x,T)|^2 + |K_n(x,-T)|^2 \right).$$

(b)

$$\begin{aligned}
& \frac{1}{2T} \int_{-T}^T |\psi_{n,S}(t) - \psi_{n,T}(t)|^2 dt \\
&= \frac{1}{2T} \int_{-T}^T |\psi_{n,S}(t)|^2 dt - 2 \frac{\gamma_{n,S}}{\gamma_{n,T}} + 1 \\
&= \frac{S}{T} \left(1 - \frac{1}{2S} \int_{T \leq |t| \leq S} |\psi_{n,S}(t)|^2 dt \right) - 2 \frac{\gamma_{n,S}}{\gamma_{n,T}} + 1,
\end{aligned}$$

So

$$2 \frac{\gamma_{n,T} - \gamma_{n,S}}{\gamma_{n,T}} \geq 1 - \frac{S}{T} + \frac{1}{2T} \int_{T \leq |t| \leq S} |\psi_{n,S}(t)|^2 dt.$$

Then recalling that $\psi_{n,S}(-t) = \overline{\psi_{n,S}(t)}$,

$$(32) \quad \frac{\gamma_{n,T} - \gamma_{n,S}}{T - S} \leq \frac{1}{2T} \gamma_{n,T} + \frac{1}{T - S} \gamma_{n,T} \frac{1}{2T} \int_T^S |\psi_{n,S}(t)|^2 dt.$$

In the other direction,

$$\begin{aligned}
& \frac{1}{2S} \int_{-S}^S |\psi_{n,S}(t) - \psi_{n,T}(t)|^2 dt \\
&= 1 - 2 \frac{\gamma_{n,T}}{\gamma_{n,S}} + \frac{1}{2S} \int_{-S}^S |\psi_{n,T}(t)|^2 dt \\
&= 1 - 2 \frac{\gamma_{n,T}}{\gamma_{n,S}} + \frac{T}{S} + \frac{1}{2S} \int_{T \leq |t| \leq S} |\psi_{n,T}(t)|^2 dt,
\end{aligned}$$

so

$$2 \frac{\gamma_{n,S} - \gamma_{n,T}}{\gamma_{n,S}} \geq 1 - \frac{T}{S} - \frac{1}{2S} \int_{T \leq |t| \leq S} |\psi_{n,T}(t)|^2 dt,$$

and hence

$$\frac{\gamma_{n,S} - \gamma_{n,T}}{S - T} \geq \frac{\gamma_{n,S}}{2S} - \frac{\gamma_{n,S}}{2S} \frac{1}{S - T} \int_T^S |\psi_{n,T}(t)|^2 dt.$$

Combined with (32), this gives

$$\begin{aligned}
\frac{\gamma_{n,S}}{2S} - \frac{\gamma_{n,S}}{2S} \frac{1}{S - T} \int_T^S |\psi_{n,T}(t)|^2 dt &\leq \frac{\gamma_{n,S} - \gamma_{n,T}}{S - T} \\
&\leq \frac{\gamma_{n,T}}{2T} - \frac{1}{S - T} \frac{\gamma_{n,T}}{2T} \int_T^S |\psi_{n,S}(t)|^2 dt.
\end{aligned}$$

This gives

$$\frac{\partial \gamma_{n,T}}{\partial T} = \frac{\gamma_{n,T}}{2T} - \frac{\gamma_{n,T}}{2T} |\psi_{n,T}(T)|^2,$$

and hence the result.

(c) Recall that

$$\gamma_{n,T} = \sqrt{\frac{A_{n-1,T}}{A_{n,T}}},$$

so

$$\gamma_{2,T} \cdots \gamma_{n,T} = \sqrt{\frac{A_{1,T}}{A_{n,T}}} = \sqrt{\frac{1}{A_{n,T}}}.$$

Thus

$$\begin{aligned} \frac{\partial}{\partial T} \ln \sqrt{\frac{1}{A_{n,T}}} &= \sum_{j=2}^n \frac{\partial (\ln \gamma_{j,T})}{\partial T} = \sum_{j=2}^n \frac{1}{2T} (1 - |\psi_{j,T}(T)|^2) \\ &\Rightarrow \frac{\partial}{\partial T} \ln A_{n,T} = -\frac{1}{T} \sum_{j=2}^n (1 - |\psi_{j,T}(T)|^2) \\ &= -\frac{1}{T} (n - K_{n,T}(T, T)), \end{aligned}$$

recall that $\psi_{1,T}(x) = \lambda_1^{-ix}$ so $|\psi_{1,T}(x)| = 1$.

(d)

$$\begin{aligned} &c_{S,j,k} - c_{T,j,k} \\ &= \frac{1}{2S} \int_{-S}^S \lambda_j^{-it} \overline{\psi_{k,S}(t)} dt - \frac{1}{2T} \int_{-T}^T \lambda_j^{-it} \overline{\psi_{k,T}(t)} dt \\ &= \frac{1}{2S} \int_{T \leq |t| \leq S} \lambda_j^{-it} \overline{\psi_{k,S}(t)} dt + \frac{1}{2} \left(\frac{1}{S} - \frac{1}{T} \right) \int_{-T}^T \lambda_j^{-it} \overline{\psi_{k,S}(t)} dt \\ &\quad + \frac{1}{2T} \int_{-T}^T \lambda_j^{-it} [\overline{\psi_{k,S}(t)} - \overline{\psi_{k,T}(t)}] dt \end{aligned}$$

so

$$\begin{aligned} &\frac{c_{S,j,k} - c_{T,j,k}}{S - T} \\ &= \frac{1}{S - T} \frac{1}{2S} \int_{T \leq |t| \leq S} \lambda_j^{-it} \overline{\psi_{k,S}(t)} dt - \frac{1}{2} \frac{1}{ST} \int_{-T}^T \lambda_j^{-it} \overline{\psi_{k,S}(t)} dt \\ &\quad + \frac{1}{2T} \int_{-T}^T \lambda_j^{-it} \left[\frac{\overline{\psi_{k,S}(t)} - \overline{\psi_{k,T}(t)}}{S - T} \right] dt \end{aligned}$$

Now let $S \rightarrow T$.

□

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