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On cubic Thue equations and the indices of algebraic integers in cubic fields

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Let $F(x,y) = ax^3 + bx^2y + cxy^2 + dy^3 \in \mathbb{Z}[x,y]$ be an irreducible cubic form. In this paper, we investigate arithmetic properties of the common indices of algebraic integers in cubic fields. For each integer k such that $v_2(k) \not\equiv 0 \pmod 3$ and $2v_2(-2b^3 - 27a^2d + 9abc) = 3v_2(b^2 - 3ac)$, we prove that the cubic Thue equation $F(x,y) = k$ has no solution $(x,y) \in \mathbb{Z}^2$. As application, we construct parametrized families of twisted elliptic curves $E : ax^3 + bx^2 + cx + d = ey^2$ without integer points (x,y).

1. INTRODUCTION

Let $f(x,y) \in \mathbb{Z}[x,y]$ be a homogeneous irreducible polynomial of degree $n \geq 3$ and k be a non-zero integer. In 1909, Thue proved the following fundamental result.

Theorem 1 ([**37**])**.** *The diophantine equation*

$$f(x,y) = k \tag{1}$$

has only finitely many solutions $(x,y) \in \mathbb{Z}^2$.

However, Thue's proof is not effective. The problem of estimating the number of solutions of (1) has rich history, see for example Siegel [**34**], Mahler [**24**], Erdös-Mahler [**12**], Davenport-Roth [**11**] and Lewis-Mahler [**23**].

For each integer k, $w(k)$ denote the number of distinct prime factors of k. In 1933, Mahler [**24**] proved that if f is irreducible then (1) has at most $C^{1+w(k)}$

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primitive solutions where C depending only on f.

Studying linear forms in logarithms of algebraic numbers, Baker could give an effective upper bound for the solutions of the equation (1). Precisely, in 1968 [**3**] Baker proved the following result: let $\kappa > n+1$ and $(x,y) \in \mathbb{Z}^2$ be a solution of (1). Then

$$\max\{|x|,|y|\} < Ce^{\log^{\kappa} k},$$

where $C = C(n,\kappa,f)$ is an effectively computable number.

Since that time, these bounds have been improved by Bugeaud and Györy [**7**]. Note that the bounds obtained by Baker's method are rather large, thus the solutions practically cannot be found by simple enumeration. Baker and Davenport [**11**] proposed for a similar problem a method to reduce drastically the bound by using continued fraction reduction. Pethö and Schulenberg [**32**] replaced the continued fraction reduction by the LLL-algorithm and gave a general method to solve equation (1) in the totally real case for $k=1$ and arbitrary n. Tzanakis and De Weger [**41**] describe the general case. Finally, Bilu and Hanort [**6**] observed that Thue equations imply not only one, but $r-1$ independent linear forms in logarithms of algebraic number in the same very small size. They were able to replace the LLL-algorithm by much faster continued fraction method and solve Thue equations up to degree 1000.

If $f(x,y)$ is an irreducible binary cubic form with negative discriminant, Delauney [**10**] and Nagell [**30**] showed that the equation $f(x,y)=1$ has at most five integer solutions (x,y). Now if its discriminant is positive, then Evertse [**13**] showed that the equation $f(x,y)=1$ has at most twelve integer solutions (x,y). Recently, Bennett [**4**] refined Delauney-Nagell-Evertse result as follows: if $f(X,1)$ has at least two distinct complex roots, then the equation $f(x,y)=1$ possess at most 10 solutions in integers x and y. Again, let $f(x,y)$ be an irreducible binary cubic form, the general equation $f(x,y)=k$ with $|D(f)| > \gamma k^{33}$, k non-negative integer, and γ certain positive constant, Siegel proved there are at most 18 solutions if $D(f)$ is positive (resp. Fjellstedt proved there is at most 14 solutions when $D(f)$ is negative). See [**30**, p.208].

In 1984, Ayad [**2**] proved that if $f(x,y)$ is a binary form of degree 3 with coefficients in $\mathbb{Z}$, $Aut(f)$ its automorphisms group and $H(f)$ its Hessian, then $Aut(f)$ is trivial except when $H(f) = \lambda g(x,y)$, $\lambda \in \mathbb{Z}^*$ and $g(x,y)$ is equivalent to x^2-xy-y^2. In this last case, $Aut(f)$ is cyclic of order 3 and f is equivalent to one binary form of type :

$$f_{m,n}(x,y) = mx^3 - nx^2y - (n+3m)xy^2 - my^3, \quad m,n \in \mathbb{Z},$$

so, the number of representations of integer k by $f(x,y)$ is divisible by 3. Note that the case $m=k=1$ is proved by Avanesov [**1**].
Recently, in [**43**] Wakabayashi proved that for $k=1$, the Thue equation $f_{m,n}(x,y) = 1$ has at most three integer solutions except for a few known cases. Using the Padé

approximation method he obtained a upper bound for the size of solutions. However, only few results are known when the group of automorphisms is trivial. The family of cubic Thue equations $f_{1,n}(x,y) = \pm 1$, with $n \geq -1$, was studied by Thomas in [**35**]. Using Baker's method, Thomas proved that it has only trivial solutions except for a finite number of values of the parameter n, explicitly for $n < 10^8$. Later, Mignotte solved the remaining cases for this family of equations in [**25**].

Let n be a rational integer and $\mathbb{K}_n = \mathbb{Q}(\theta)$ be a cyclic cubic number field generated by a root θ of $f_{1,n}(X,1) = X^3 - nX^2 - (n+3)X - 1$ and let $\mathbb{O}_{\mathbb{K}_n}$ be its ring of integers. The polynomial $f_{1,n}(X,1)$ has discriminant $(n^2+3n+9)^2$. If n^2+3n+9 is square-free, then we have the discriminant of $\mathbb{K}_n$, $D(\mathbb{K}_n) = (n^2+3n+9)^2$ and $\mathbb{O}_{\mathbb{K}_n} = \mathbb{Z}[\theta]$ (there exists infinitely many such n, cf. Cusick [**9**, pp. 63-73]).

For θ a root of $f_{1,n}(X,1)$, Pëtho and Lemmermeyer in [**20**] proved that for all $\alpha \in \mathbb{Z}[\theta]$ either $|N(\alpha)| \geq 2n+3$, or α is associated to an integer. Moreover, if $|N(\alpha)| = 2n+3$, then α is associated to one of the conjugates of $\alpha - 1$. In [**26**], Mignotte, Pëtho and Lemmermeyer, by Baker's method and used the results of [**20**] and [**9**, pp.63-73] solved completely the case $m = 1$, $n \geq -1$ and $1 \leq k \leq 2n+3$. Lettl, Pëtho and Voutier [**21**], following an idea of Chudnovsky [**8**], improved the usual estimate of the Padé approximation and gave a good upper bound for the size of solutions for $f_{1,n}(x,y) = k$, $n \geq 30$ and k is arbitrary. Wakabayashi [**44**] studied Thue inequality $|f_{m,n}(x,y)| \leq k$ with two parameters m, n.

In 2011, A. Hoshi [**15**] studied the case when k is a positive divisor of $n^2 + 3n + 9$, and gave a correspondence between integer solutions to the parametric family of cubic Thue equations

$$x^3 - nx^2y - (n+3)xy^2 - y^3 = k$$

and isomorphism classes of the simplest cubic fields. For more details on the study of simplest cubic fields see [**33**].

Lee [**19**] and independently Mignotte and Tzanakis [**27**] proved for $n \geq 3.33 \cdot 10^{23}$, the family of Thue equations

$$x^3 - nx^2y - (n+1)xy^2 - y^3 = 1$$

has only trivial solutions $(x,y) = (1,0), (0,-1), (1,-1), (-n-1,-1), (1,-n)$. Recently, Mignotte [**28**] could solve this equation completely.

Thomas [**36**] proved that for $0 < a < b$ and $n \geq (2 \cdot 10^6 \cdot (a+2b))^{4.85/(b-a)}$ the family of Thue equations

$$x(x - n^a y)(x - n^b y) \pm y^3 = 1$$

has only trivial solutions $(x, y) = (1, 0), (0, \pm 1), (\pm n^a, \pm 1), (\pm n^b, \pm 1)$. He also investigated this family with n^a and n^b replaced by polynomials in n of degree a and b, respectively.

Recently, Wakabayashi [**42**], using Baker's method, proved that for any integer $n \geq 1.35 \cdot 10^{14}$, the family of parametrized Thue equations

$$x^3 - n^2xy^2 + y^3 = 1$$

has only trivial solutions $(x, y) = (0, 1), (1, 0), (1, n^2), (n, 1), (-n, 1)$.

A. Togbé [**38**], using Baker's method and the results obtained by L. C. Washington [**45**] and O. Lecacheux [**18**], solved the family of parametrized Thue equations

$$x^3 - (n^3 - 2n^2 + 3n - 3)x^2y - n^2xy^2 - y^3 = \pm 1, \quad \text{when } n \geq 1.$$

A. Togbé [**40, 39**] using Baker's method and the results obtained by Y. Kishi [**17**], solved the two families of parametrized Thue equations

$$\begin{aligned} &x^3 - n(n^2 + n + 3)(n^2 + 2)x^2y - (n^3 + 2n^2 + 3n + 3)xy^2 - y^3 = \pm 1, \\ &x^3 + (n^8 + 2n^6 - 3n^5 + 3n^4 - 4n^3 + 5n^2 - 3n + 3)x^2y \\ &\qquad\qquad -(n^3 - 2)n^2xy^2 - y^3 = \pm 1, \quad \text{when } n \geq 0. \end{aligned}$$

Very recently, Bennett and Ghadermarzi [**5**] using lower bounds for linear forms in logarithms and lattice-basis reduction, solved completely the family of Thue equations

$$x^3 - (n^4 - n)x^2y + (n^5 - 2n^2)xy^2 + y^3 = 1.$$

Let a, b, c, d, e be integers. The equation $ax^3 + bx^2 + cx + d = ey^2$ was studied by Mordell [**29**, pp.255-261]. He proved the following important result: if the polynomial $ax^3 + bx^2 + cx + d$ has no squared linear factor in x, then the equation $ax^3 + bx^2 + cx + d = ey^2$ has only a *finite number of integer solutions.*

Now we state our main result.

2. MAIN RESULTS

Let N be any integer. We denote by $v_2(N)$ the greatest exponent s such that 2^s divides N.

The discriminant of the form

$$F(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$$

is the invariant

$$D = 18abcd + b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2. \tag{2}$$

The binary form F has the quadratic and cubic covariants

$$H(x,y) = A_0x^2 + B_0xy + C_0y^2, \tag{3}$$

$$G(x,y) = A_1x^3 + B_1x^2y + C_1xy^2 + D_1y^3. \tag{4}$$

where

$$\begin{aligned} A_0 &:= b^2 - 3ac, B_0 := bc - 9ad, C_0 := c^2 - 3bd; \\ A_1 &:= -2b^3 - 27a^2d + 9abc, B_1 = 3(b^2c - 9abd + 6b^2d), \\ C_1 &= 3(bc^2 + 9acd - 6b^2d), D_1 = (2c^3 + 27ad^2 - 9bcd). \end{aligned} \tag{5}$$

the quadratic form H is the Hessian and G is the gradient of F, see [**29**, pp.213].

Throught this paper, we assume that $D \neq 0$ and $\gcd(a,b,c,d) = 1$. Now we state our main result.

Theorem 2. *Let a, b, c, d and k be integers such that*

$$2v_2(A_1) = 3v_2(A_0), \ v_2(k) \equiv 1, 2 \pmod 3.$$

Then the cubic Thue Diophantine equations

$$ax^3 + bx^2y + cxy^2 + dy^3 = k$$

has no integer solution (x, y).

Corollary 1. *Let $(n,k) \in \mathbb{Z}^2$ where $3 \nmid v_2(k)$. Then the following two families of cubic Thue equations*

1. $x^3 - (n^3 - 2n^2 + 3n - 3)x^2y - n^2xy^2 - y^3 = k, \ n \not\equiv 1 \pmod 4,$
2. $x^3 - n(n^2 + n + 3)(n^2 + 2)x^2y - (n^3 + 2n^2 + 3n + 3)xy^2 - y^3 = k.$

have no integer solution.

Corollary 2. *Let a, b, c, d, k as in Theorem 2. Then the homogeneous form*

$$ax^3 + bx^2y + cxy^2 + dy^3 = kz^3$$

has only the integer solution $(x, y, z) = (0, 0, 0)$.

Corollary 3. *Let a, b, c, d as in Theorem 2, and e be integers such that $v_2(e) \geq v_2(A_1)$ and $v_2(e) \equiv 1 \pmod 3$. Then the familly of twisted ellitpic curves*

$$E: \ ax^3 + bx^2 + cx + d = ey^2 \tag{6}$$

have no integer points (x, y).

3. PROOF OF THEOREM 2:

Our strategy of proof is different to the previous methods in literature. Our method is based on the index theory of algebraic integers and ramification of prime number p in cubic fields. Before proving Theorem 2 we state some auxiliary lemmas.

Let $\mathbb{K}$ be an algebraic number field of degree n and let $\mathbb{O}_{\mathbb{K}}$ be its ring of integers. Denote by $\widehat{\mathbb{O}}_{\mathbb{K}}$ the set of primitive elements of $\mathbb{O}_{\mathbb{K}}$. For any $\theta \in \mathbb{O}_{\mathbb{K}}$ we denote $F_\theta(x)$ the characteristic polynomial of θ over $\mathbb{Q}$. Let $D(\mathbb{K})$ be the absolute discriminant of $\mathbb{K}$. It is well known that if $\theta \in \mathbb{O}_{\mathbb{K}}$, the discriminant of F_θ has the form

$$D(F_\theta) = D(1, \theta, \cdots, \theta^{n-1}) = I(\theta)^2 \times D(\mathbb{K}), \tag{7}$$

where $I(\theta) = Card(\mathbb{O}_{\mathbb{K}}/\mathbb{Z}[\theta])$ is called the index of θ. Let

$$I(\mathbb{K}) = \gcd_{\theta \in \widehat{\mathbb{O}}_{\mathbb{K}}} I(\theta). \tag{8}$$

$I(\mathbb{K})$ is called the index of $\mathbb{K}$. A prime number p is called a common index divisor of $\mathbb{K}$ if $p \mid I(\mathbb{K})$.
Every cubic field $\mathbb{K}$ can be written in the form $\mathbb{K} = \mathbb{Q}(\theta)$ where θ is a root of an irreducible polynomial of the type

$$f(X) = X^3 + AX + B, \quad A, B \in \mathbb{Z}. \tag{9}$$

The discriminant of $f(X)$ is $D(f) = -4A^3 - 27B^2$. If for any prime number p we have

$$v_p(A) \geq 2 \quad \text{and} \quad v_p(B) \geq 3$$

then θ/p is an algebraic integer whose equation is $x^3 + (A/p^2)x + B/p^3 = 0$. Therefore, we can assume that for any prime number p,

$$v_p(A) < 2 \quad \text{or} \quad v_p(B) < 3. \tag{10}$$

We state the next lemmas which give the index of integers in cubic fields.

To prove the main result we need the following lemmas.

Lemma 1. *Let $\mathbb{K}$ be a cubic field defined by polynomial (9) such that A and B are odd. If $\theta \in \mathbb{O}_{\mathbb{K}}$ has even index then $(\theta + k)/2 \in \mathbb{O}_{\mathbb{K}}$ for some $k \in \mathbb{Z}$.*

Proof. Let $\mathbb{K}$ be a cubic field defined by the polynomial (9) where A and B are odd. Then by [**22**, Theorem 1], 2 is inert. Suppose that there exists $\theta \in \mathbb{O}_{\mathbb{K}}$ with even index, so there exists $a_0, b_0, c_0 \in \mathbb{Z}$ such that θ is a root of $g(X) = X^3 + a_0X^2 + b_0X + c_0$. We then have $3\theta + a_0 \in \mathbb{O}_{\mathbb{K}}$ is a root of $h(X) = X^3 + a_1X + b_1$, where $a_1 = -3a_0^2 + 9b_0$ and $b_1 = 2a_0^3 - 9a_0b_0 + 27c_0$. We note that $D(h) = 3^6 \cdot D(g)$. As $I(\theta) \equiv 0 \pmod 2$, we have $D(g) \equiv 0 \pmod 2$ and $-4a_1^3 - 27b_1^2 = D(h) \equiv 0 \pmod 2$. We obtain b_1 is even. Moreover, if $v_2(a_1) < 2$ or $v_2(b_1) < 3$, by [**22**, Theorem 1], 2 is not inert. So we deduce that $v_2(a_1) \geq 2$ and $v_2(b_1) \geq 3$. Therefore we can write $((3\theta + a_0)/2)^3 + (a_1/4)(3\theta + a_0)/2 + b_1/8 = 0$, which implies that $(3\theta + a_0)/2 \in \mathbb{O}_{\mathbb{K}}$, thus we have $(\theta + a_0)/2 = (3\theta + a_0)/2 - \theta \in \mathbb{O}_{\mathbb{K}}$.

□

Lemma 2. *Let* $\mathbb{K}$ *be a cubic field defined by the polynomial (9) such that* A *and* B *are odd. Let* $\theta \in \mathbb{O}_{\mathbb{K}}$ *be such that* $\mathbb{K} = \mathbb{Q}(\theta)$. *Then*

$$v_2(I(\theta)) \equiv 0 \pmod 3.$$

Proof. Let $\mathbb{K}$ be a cubic field defined by polynomial (9) such that A and B are odd and let $\Omega = \{t \in \mathbb{N} \,/\exists\ \theta \in \mathbb{O}_{\mathbb{K}}$ with $\mathbb{K} = \mathbb{Q}(\theta)$ and $v_2(I(\theta)) = t \not\equiv 0 \pmod 3\} \subseteq \mathbb{N}$. Let t^* the least value of Ω, so there exists $\theta^* \in \mathbb{O}_{\mathbb{K}}$ such that $\mathbb{K} = \mathbb{Q}(\theta^*)$ and $v_2(I(\theta^*)) = t^* \not\equiv 0 \pmod 3$. By Lemma 1, there exists $k \in \mathbb{Z}$ such that $\beta = (\theta^* + k)/2 \in \mathbb{O}_{\mathbb{K}}$, so we get $I(\beta) = 2^{-3}I(\theta^*)$, then we obtain $v_2(I(\beta)) = t^* - 3$, hence $t^* - 3 \geq 0$ and $v_2(I(\beta)) \not\equiv 0 \pmod 3$, moreover we have $v_2(I(\beta)) < t^*$. This contradicts the minimality of t^*. Hence for every $\theta \in \mathbb{O}_{\mathbb{K}}$ with $\mathbb{K} = \mathbb{Q}(\theta)$, we have $v_2(I(\theta)) = t \equiv 0 \pmod 3$. □

Lemma 3. *Let* a, b, c, d *are integers such that* $\gcd(a, b, c, d) = 1$ *and*

$$v_2(3ac - b^2) = 2t \ \text{ and } \ v_2(27a^2d - 9abc + 2b^3) = 3t,$$

where t *is a non-negative integer. Then we have*

$$v_2(a) \equiv 0 \pmod 3.$$

Note that $3ac - b^2$ and $27a^2d - 9abc + 2b^3$ are the leading coefficients of the covariants H and G.

Proof. Let $P(X) = aX^3 + bX^2 + cX + d$ be a polynomial such that $v_2(A) = 2t$ and $v_2(B) = 3t$, $t \in \mathbb{N}$ where $A = 9ac - 3b^2$ and $B = 27a^2d - 9abc + 2b^3$. • *We claim that the* $P(X)$ *is irreducible* over $\mathbb{Q}$. For this we consider the cubic fields $\mathbb{K} = \mathbb{Q}(\theta')$ generated by a root θ' of P. Write $\alpha = (3a\theta' + b)/2^t$, we have $\mathbb{K} = \mathbb{Q}(\alpha)$ and α is a root of the polynomial $X^3 + (A/2^{2t})X + (B/2^{3t})$ where $A/2^{2t}$ and $B/2^{3t}$ are odd. Modulo 2 the polynomial $X^3 + (A/2^{2t})X + (B/2^{3t})$ is irreducible. Then the claim follows.

• **Valuation of quotient of indices.** The number field $\mathbb{K}$ is generated by θ a root of the irreducible polynomial $X^3 + bX^2 + acX + da^2$. Let $\beta = (ax + by)\theta + y\theta^2 \in \mathbb{Z}[\theta]$ be an primitive integer. By Lemma 2, we have

$$v_2(I(\theta)) \equiv 0 \pmod 3 \ \text{ and } \ v_2(I(\beta)) \equiv 0 \pmod 3. \tag{11}$$

Let $\theta_1, \theta_2, \theta_3$ (resp. $\beta_1, \beta_2, \beta_3$) be a conjugates of $\theta = \theta_1$ (resp. $\beta = \beta_1$). We get exactly

$$D\left(1, \beta, \beta^2\right) = I(\beta)^2 \times D\left(\mathbb{K}\right) = -\prod_{i<j}(\beta_i - \beta_j)^2,$$
$$D\left(1, \theta, \theta^2\right) = I(\theta)^2 \times D\left(\mathbb{K}\right) = -\prod_{i<j}(\theta_i - \theta_j)^2,$$

on the other hand,

$$\frac{\beta_i - \beta_j}{\theta_i - \theta_j} = \frac{(ax+by)(\theta_i - \theta_j) + y(\theta_i^2 - \theta_j^2)}{\theta_i - \theta_j} = ax + by + y(\theta_i + \theta_j).$$

Taking the product we obtain the following equation

$$\frac{I(\beta)}{I(\theta)} = \left|\prod_{i<j}(ax + by + y(\theta_i + \theta_j))\right|.$$

Hence we obtain the following equality

$$\begin{aligned}
&(ax + by + y(\theta_1 + \theta_2))(ax + by + y(\theta_1 + \theta_3))(ax + by + y(\theta_2 + \theta_3)) = \\
&\qquad (ax+by)^3 + 2(\theta_1 + \theta_2 + \theta_3)(ax+by)^2 y \\
&\qquad + ((\theta_1 + \theta_2)(\theta_2 + \theta_3) + (\theta_1 + \theta_3)(\theta_2 + \theta_3) + (\theta_1 + \theta_2)(\theta_1 + \theta_3))y^2(ax + by) \\
&\qquad + (\theta_1 + \theta_2)(\theta_1 + \theta_3)(\theta_2 + \theta_3)y^3.
\end{aligned}$$

In the other hand we have the following equations

$$\theta_1 + \theta_2 + \theta_3 = -b, \tag{12}$$
$$\theta_1\theta_2 + \theta_1\theta_3 + \theta_2\theta_3 = ac, \tag{13}$$
$$\theta_1\theta_2\theta_3 = -a^2 d. \tag{14}$$

We now use equations (12), (13) and (14) to compute $(\theta_1 + \theta_2)(\theta_2 + \theta_3) + (\theta_1 + \theta_3)(\theta_2 + \theta_3) + (\theta_1 + \theta_2)(\theta_1 + \theta_3)$ and $(\theta_1 + \theta_2)(\theta_1 + \theta_3)(\theta_2 + \theta_3)$.

First we have

$$\begin{aligned}
(\theta_1 + \theta_2)(\theta_2 + \theta_3) &+ (\theta_1 + \theta_3)(\theta_2 + \theta_3) + (\theta_1 + \theta_2)(\theta_1 + \theta_3) \\
&= 3(\theta_1\theta_2 + \theta_1\theta_3 + \theta_2\theta_3) + \theta_1^2 + \theta_2^2 + \theta_3^2 \\
&= 3(\theta_1\theta_2 + \theta_1\theta_3 + \theta_2\theta_3) + (\theta_1 + \theta_2 + \theta_3)^2 - 2(\theta_1\theta_2 + \theta_1\theta_3 + \theta_2\theta_3) \\
&= (\theta_1\theta_2 + \theta_1\theta_3 + \theta_2\theta_3) + (\theta_1 + \theta_2 + \theta_3)^2 \\
&= ac + b^2.
\end{aligned}$$

Secondly we obtain

$$\begin{aligned}
(\theta_1 + \theta_2)(\theta_1 + \theta_3)(\theta_2 + \theta_3) &= 2\theta_1\theta_2\theta_3 + \theta_1\theta_3(\theta_1 + \theta_3) + \theta_1\theta_2(\theta_1 + \theta_2) + \theta_2\theta_3(\theta_2 + \theta_3) \\
&= 2\theta_1\theta_2\theta_3 + (\theta_1\theta_2 + \theta_1\theta_3 + \theta_2\theta_3)(\theta_1 + \theta_2 + \theta_3) - 3\theta_1\theta_2\theta_3 \\
&= (\theta_1\theta_2 + \theta_1\theta_3 + \theta_2\theta_3)(\theta_1 + \theta_2 + \theta_3) - \theta_1\theta_2\theta_3 \\
&= -abc + a^2 d.
\end{aligned}$$

Hence, we deduce

$$\frac{I(\beta)}{I(\theta)} =\mid a^3x^3 + a^2bx^2y + a^2cxy^2 + a^2dy^3 \mid . \tag{15}$$

To end the proof we write

$$\frac{I(\beta)}{a^2I(\theta)} =\mid ax^3 + abx^2y + a^2cxy^2 + a^3dy^3 \mid . \tag{16}$$

By (11), we get

$$v_2\left(a^3x^3 + a^2bx^2y + a^2cxy^2 + a^2dy^3\right) \equiv 0 \pmod 3.$$

For $t = 0$, we have immediately $v_2(a) = 0$. Now, suppose that $t > 0$ and prove that $v_2(a) \equiv 1, 2 \pmod 3$ can not show up.
Assume that $v_2(a) \equiv 1, 2 \pmod 3$. It's easy to see that b is even. Now, we can replace $(x, y) = (0, 1)$ in (16), we get $v_2(a^2d) \equiv 0 \pmod 3$, this implies that $v_2(d) \equiv 1, 2 \pmod 3$. This gives the following inequalities

$$v_2(a^2b),\ v_2(a^2d) \geq v_2(a^2) + 1. \tag{17}$$

Secondly, replace $(x, y) = (1, 1)$ in (16), we get the inequality

$$v_2(a^3 + a^2b + a^2c + a^2d) \geq v_2(a^2) + 1. \tag{18}$$

Therefore, from the inequalities (17), (18) we obtain that $v_2(c) \geq 1$. This gives contradiction with $\gcd(a, b, c, d) = 1$.
We conclude that, if $v_2(9ac - 3b^2) = 2t$ and $v_2(27a^2b - 9abc + 2b^3) = 3t$, we then have $v_2(a) \equiv 0 \pmod 3$.

□

Now we give the proof of our main Theorem.

Proof of Theorem 2: Let a, b, c, d be integers such that $\gcd(a, b, c, d) = 1$ and

$$v_2(3ac - b^2) = 2t \text{ and } v_2(27a^2d - 9abc + 2b^3) = 3t.$$

Let $\gamma = (ax + by)\theta + y\theta^2$ with $x, y \in \mathbb{Z}$. By Lemmas 2, 3, we have

$$v_2\left(\frac{I(\gamma)}{a^2I(\theta)}\right) = v_2\left(ax^3 + bx^2y + cxy^2 + dy^3\right) \equiv 0 \pmod 3.$$

This implies that for each integer k where $3 \nmid v_2(k)$ the parametric family of Thue equations $ax^3 + bx^2y + cxy^2 + dy^3 = k$ has no integer solution (x, y). This completes the proof of Theorem 2.

Proof of Corollary 1:

1. Let $\phi_n(x,y) = x^3 - (n^3 - 2n^2 + 3n - 3)x^2y - n^2xy^2 - y^3$.

   We have
$$A_0 = (n^3 - 2n^2 + 3n - 3)^2 + 3n^2,$$
$$A_1 = 2(n^3 - 2n^2 + 3n - 3)^3 + 9(n^3 - 2n^2 + 3n - 3)n^2 + 27,$$
   for n even, we get $A_0 \equiv A_1 \equiv 1 \mod 2$ and for $n \equiv 3 \mod 4$ we have $4\|A_0$ and $8\|A_1$. So for any $n \not\equiv 1 \mod 4$ we get $3v_2(A_0) = 2v_2(A_1)$. By Theorem 2, for any $k \in \mathbb{Z}$ where $3 \nmid v_2(k)$ and $n \not\equiv 1 \mod 4$ the parametric family of Thue cubic equations $\phi_n(x,y) = k$, has no integer solution (x,y).

2. Let $\varphi_n(x,y) = x^3 - n(n^2+n+3)(n^2+2)x^2y - (n^3+2n^2+3n+3)xy^2 - y^3$.

   We have
$$A_0 = n^2(n^2+n+3)^2(n^2+2)^2 + 3(n^3+2n^2+3n+3),$$
$$A_1 = 2n^3(n^2+n+3)^3(n^2+2)^3 + 9n(n^2+n+3)(n^2+2)(n^3+2n^2+3n+3) + 27,$$
   for n even we get $A_0 \equiv A_1 \equiv 1 \mod 2$ and for n odd we have $4\|A_0$ and $8\|A_1$. So for any integer n we get $3v_2(A_0) = 2v_2(A_1)$. By Theorem 2, for any $k \in \mathbb{Z}$ where $3 \nmid v_2(k)$ the parametric family of Thue cubic equations $\varphi_n(x,y) = k$, has no integer solution (x,y).

Proof of Corollary 3: Let a, b, c, d as in Theorem 2, and e be integers such that $v_2(e) \geq v_2(A_1)$ and $v_2(e) \equiv 1 \pmod 3$. So we have $v_2(e(y^2+1)) \equiv 1, 2 \pmod 3$ and for $v_2(e) \geq v_2(A_1)$ we get $v_2(27a^2(d+e) - 9abc + 2b^3) = v_2(27a^2d - 9abc + 2b^3)$. By Theorem 2, we have the family of twisted elliptic curves
$$E: \; ax^3 + bx^2 + cx + d + e = e(y^2+1)$$
have no integer points (x,y).

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