

ON SHARPENING OF AN INEQUALITY OF TURÁN

*Dedicated to Academician Professor Gradimir Milovanović
on the occasion of his 70th birthday.*

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Let $P(z) = \sum_{\nu=0}^n a_{\nu} z^{\nu}$ be a polynomial of degree n . Then as a generalization of a well-known result of Turán [18], it was proved by Govil [5] that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq K$, $K \geq 1$, then

$$(0.1) \quad \max_{|z|=1} |P'(z)| \geq \frac{n}{1+K^n} \max_{|z|=1} |P(z)|.$$

In this paper, we prove a polar derivative generalization of this inequality, which as a corollary gives a sharpening of this inequality (0.1).

1. INTRODUCTION AND STATEMENT OF RESULTS

If $P(z) = \sum_{\nu=0}^n a_{\nu} z^{\nu}$ is a polynomial of degree n then from a well-known inequality due to Bernstein [3], we have

$$(1.2) \quad \max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|.$$

The Inequality (1.2) is best possible and equality holds, if $P(z)$ has all its zeros at the origin. In case the polynomial $P(z)$ has no zeros in $|z| < 1$, then Erdős conjectured and later Lax [12] proved that

$$(1.3) \quad \max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|.$$

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The Inequality (1.3) is also best possible and equality holds for $P(z) = a + bz^n$, where $|a| = |b|$. The generalization of the Inequality (1.3) to class of polynomials having no zeros in $|z| < K, K \geq 1$ was done by Malik [14] (See also Govil and Rahman [10, Theorem 4]), who proved that, if a polynomial $P(z)$ of degree n has no zeros in $|z| < K, K \geq 1$, then

$$(1.4) \quad \max_{|z|=1} |P'(z)| \leq \frac{n}{1+K} \max_{|z|=1} |P(z)|.$$

In the other direction, it was proved by Turán [18] that if $P(z)$ is a polynomial of degree n , having all its zeros in $|z| \leq 1$, then

$$(1.5) \quad \max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|.$$

The Inequality (1.5) is also best possible and equality holds if the polynomial $P(z)$ has all its zeros on $|z| = 1$.

The Inequality (1.5) of Turán [18] has been of considerable interest and applications, and so it would obviously be of interest to generalize it for polynomials having all their zeros in $|z| \leq K, K > 0$. The case when $0 < K \leq 1$ was settled by Malik [14], while the case when $K \geq 1$ by Govil [5], who proved that if $P(z) = \sum_{\nu=0}^n a_{\nu} z^{\nu}$ is a polynomial of degree n having all its zeros in $|z| \leq K, K \geq 1$, then

$$(1.6) \quad \max_{|z|=1} |P'(z)| \geq \frac{n}{1+K^n} \max_{|z|=1} |P(z)|, \text{ for } K \geq 1.$$

As is easy to see that the Inequality (1.6) is sharp and becomes equality if $P(z) = z^n + K^n$. From this, one would expect that if we exclude the class of polynomials having zeros on $|z| = K$, then it may be possible to improve upon the above Inequality (1.6). In this direction, it was proved by Govil [6], that if $P(z) = \sum_{\nu=0}^n a_{\nu} z^{\nu}$ is a polynomial of degree n having all its zeros in $|z| \leq K, K \geq 1$, then

$$(1.7) \quad \max_{|z|=1} |P'(z)| \geq \frac{n}{1+K^n} \left(\max_{|z|=1} |P(z)| + \min_{|z|=K} |P(z)| \right).$$

Govil [7] also proved that, if $P(z) = \sum_{\nu=0}^n a_{\nu} z^{\nu}$, is a polynomial of degree $n > 2$, having all its zeros in $|z| \leq K, K \geq 1$, then

$$(1.8) \quad \begin{aligned} \max_{|z|=1} |P'(z)| &\geq \frac{n}{1+K^n} \max_{|z|=1} |P(z)| \\ &+ \frac{n|a_{n-1}|}{(1+K^n)K} \left(\frac{K^n-1}{n} - \frac{K^{n-2}-1}{n-2} \right) + |a_1|(1-1/K^2). \end{aligned}$$

Although the Inequality (1.7) sharpens Inequality (1.6) but it has a drawback that if there is even one zero on $|z| = K$, then $\min_{|z|=K} |P(z)| = 0$, and

so the Inequality (1.7) fails to give any improvement over (1.6). Similarly, if $|a_1| = |a_{n-1}| = 0$, then Inequality (1.8) also fails to give any improvement over the Inequality (1.6). Therefore it is quite natural to ask now; is there any way to overcome these deficiencies, for example, can the bound be expressed in terms of coefficients of the polynomial under consideration, which is more informative than the one given in (1.6)? By intuition, one may try to answer this question in different ways at different levels. In this paper we approach this problem by using the information on the leading and constant coefficients of the underlying polynomial, and our result thus obtained sharpens Inequality (1.6) even when Inequalities (1.7), and (1.8) fail.

We will do this by first proving a polar derivative generalization of (1.6) and then use the result so proved to obtain the desired sharpening of (1.6).

If $P(z)$ is a polynomial of degree n , then the *polar derivative of $P(z)$ with respect to a complex number α* is defined as

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

Note that $D_\alpha\{P(z)\}$ is a polynomial of degree at most $n - 1$, and it is a 'generalization' of the ordinary derivative in the sense that

$$\lim_{\alpha \rightarrow \infty} \frac{D_\alpha\{P(z)\}}{\alpha} = P'(z),$$

uniformly with respect to z for $|z| \leq R$, $R > 0$. For more information on polar derivatives of polynomials, one can refer to monographs by Marden [14], Rahman and Schmeisser [17], or Milovanović et al. [15].

Inequalities have been extended widely in the literature from 'ordinary derivative' to 'polar derivative' of complex polynomials, and for some of the papers in this direction, we refer to a recently published book chapter by Govil and Kumar [8] (see also Govil and Kumar [9], and Kumar [11]).

We begin, by presenting the following result involving polar derivative of a polynomial having all its zeros in $|z| \leq K$, $K \geq 1$. As will be shown, this result generalizes Inequality (1.6), and will be used to obtain a sharpening of it.

Theorem 1.1. *If $P(z) = z^m(a_0 + a_1z + \dots + a_{n-m}z^{n-m})$, $0 \leq m \leq n$, is a polynomial of degree n having all its zeros in $|z| \leq K, K \geq 1$, then for any complex number α with $|\alpha| \geq K$,*

$$(1.9) \quad \max_{|z|=1} |D_\alpha\{P(z)\}| \geq (|\alpha| - K) \left(\frac{n + m}{1 + K^n} + \frac{(|a_{n-m}|K^{n-m} - |a_0|)}{(1 + K^n)(|a_{n-m}|K^{n-m} + |a_0|)} \right) \max_{|z|=1} |P(z)|.$$

If we divide (1.9) by $|\alpha|$ and make $|\alpha| \rightarrow \infty$, we easily get

Corollary 1.2. *If $P(z) = z^m(a_0 + a_1z + \dots + a_{n-m}z^{n-m})$, $0 \leq m \leq n$, is a polynomial of degree n which has all its zeros in the disk $|z| \leq K, K \geq 1$, then*

$$(1.10) \quad \max_{|z|=1} |P'(z)| \geq \left(\frac{n + m}{1 + K^n} + \frac{(|a_{n-m}|K^{n-m} - |a_0|)}{(1 + K^n)(|a_{n-m}|K^{n-m} + |a_0|)} \right) \max_{|z|=1} |P(z)|.$$

The result is best possible and equality in (1.10) holds for the polynomial $P(z) = z^n + K^n$.

The case $m = 0$ of the above Corollary 1.2 is also of some interest and the same is presented below.

Corollary 1.3. *If $P(z) = a_0 + a_1z + \cdots + a_{n-1}z^{n-1} + a_nz^n$ is a polynomial of degree n which has all its zeros in the disk $|z| \leq K, K \geq 1$, then*

$$(1.11) \quad \max_{|z|=1} |P'(z)| \geq \left(\frac{n}{1+K^n} + \frac{(|a_n|K^n - |a_0|)}{(1+K^n)(|a_n|K^n + |a_0|)} \right) \max_{|z|=1} |P(z)|.$$

The result is best possible and equality in (1.11) holds for the polynomial $P(z) = z^n + K^n$.

Remark 1.4. It may be remarked that the coefficient a_n cannot be zero, because otherwise the polynomial $P(z)$ will not be of degree n . It is clear that, in general for any polynomial the Inequalities (1.10) and (1.11) would give an improvement over the bound obtained from Inequality (1.6), and for the class of polynomials having a zero on $|z| = K$, the Inequalities (1.10) and (1.11) will give bound that is sharper than obtainable from Inequality (1.7). Also, for all polynomials $P(z)$ with $|a_1| = |a_{n-1}| = 0$ and $|a_n|K^n - |a_0| \neq 0$, the Inequality (1.11) gives a bound that is better than obtainable from the Inequality (1.8). One can also observe that for larger values of $m > 0$, Inequality (1.10) improves inequality (1.8) considerably when $|a_1| = |a_{n-1}| = 0$ and $|a_{n-m}|K^{n-m} - |a_0| \neq 0$.

Example 1.5. Let $P(z) = z^2(z^2 - 4)$. Then $P(z)$ is a polynomial of degree $n = 4$ having all its zeros in $|z| \leq 2$. For this polynomial $P(z)$, we have $|a_0| = 0$, $|a_n| = |a_4| = 1$, $\max_{|z|=1} |P(z)| = 5$, and $\min_{|z|=2} |P(z)| = 0$. Then it is easy to see that by Inequalities (1.6), (1.7), and (1.8), we have $\max_{|z|=1} |P'(z)| \geq 20/17$, while our Inequality (1.10) gives $\max_{|z|=1} |P'(z)| \geq 35/17$, an improvement of 75% over the bounds obtained from inequalities (1.6), (1.7) and (1.8). Also, the Inequality (1.11) gives $\max_{|z|=1} |P'(z)| \geq 25/17$, an improvement of 25% over the bounds obtained from inequalities (1.6), (1.7) and (1.8).

If we take $K = 1$ in Corollary 1.3, we get

Corollary 1.6. *If $P(z) = a_0 + a_1z + \cdots + a_{n-1}z^{n-1} + a_nz^n$ is a polynomial of degree n which has all its zeros in the disk $|z| \leq 1$, then*

$$(1.12) \quad \max_{|z|=1} |P'(z)| \geq \frac{n}{2} \left(1 + \frac{(|a_n| - |a_0|)}{n(|a_n| + |a_0|)} \right) \max_{|z|=1} |P(z)|.$$

The result is best possible and equality in (1.12) holds for polynomials having all their zeros on $|z| = 1$.

Clearly, the above Corollary 1.6 sharpens Inequality (1.5) due to Turán [18] in all cases excepting when $P(z)$ has all its zeros on $|z| = 1$.

Also, it may be remarked that by using Lemma 2.8, we can, in fact, obtain the following inequality which holds for every z on $|z| = 1$, and so is more general than (1.12).

$$(1.13) \quad |P'(z)| \geq \frac{n}{2} \left(1 + \frac{(|a_n| - |a_0|)}{n(|a_n| + |a_0|)} \right) |P(z)|, \text{ for every } z \text{ on } |z| = 1.$$

2. LEMMAS

For the proof of our theorem, we will need the following lemmas.

Lemma 2.7. *If $0 \leq x \leq 1, 0 \leq y \leq 1$, then*

$$\frac{1}{1+x} + \frac{1}{1+y} - \frac{1}{1+xy} - \frac{1}{2} \geq 0.$$

Proof. For any x, y such that $0 \leq x \leq 1, 0 \leq y \leq 1$, a simple verification yields

$$\frac{1}{1+x} + \frac{1}{1+y} - \frac{1}{1+xy} - \frac{1}{2} = \frac{(1-x)(1-y)(1-xy)}{2(1+x)(1+y)(1+xy)},$$

which is clearly ≥ 0 , and the proof of the lemma is complete. □

Lemma 2.8. *If $P(z) = a_0 + a_1z + \dots + a_{n-1}z^{n-1} + a_nz^n$ is a polynomial of degree $n \geq 1$ having all its zeros in $|z| \leq 1$, then for all z on $|z| = 1$ for which $P(z) \neq 0$*

$$(2.14) \quad \operatorname{Re} \left(\frac{zP'(z)}{P(z)} \right) \geq \frac{n}{2} + \frac{|a_n| - |a_0|}{2(|a_n| + |a_0|)}.$$

Proof. It may be remarked that this result is also mentioned in a paper of Dubinin [4], where it has been proved by using the Boundary Schwarz Lemma [16, Lemma 1, p. 3514]. Here we present a proof which we believe is new, direct and does not make use of the Boundary Schwarz Lemma [16, Lemma 1, p. 3514]. It only makes use of the principle of mathematical induction.

To prove (2.14), it suffices to establish its equivalent form

$$(2.15) \quad \operatorname{Re} \left(\frac{zP'(z)}{P(z)} \right) \geq \frac{n-1}{2} + \frac{1}{1 + \frac{|a_0|}{|a_n|}}.$$

Clearly, without loss of generality, we can assume $a_n = 1$. We will prove the above Inequality (2.15) with the assumption $a_n = 1$, by the use of the principle of mathematical induction on the degree n , and for this, we first verify the result for $n = 1$.

If $n = 1$, then $P(z) = z - w$ with $|w| \leq 1$, and therefore for $|z| = 1$ and $z \neq w$, we have

$$\operatorname{Re} \left(\frac{zP'(z)}{P(z)} \right) = \operatorname{Re} \left(\frac{z}{z-w} \right) \geq \frac{1}{1 + |w|},$$

which is nothing but (2.15) when $n = 1$.

Let $Q(z) := (z - w)P(z)$ with $|w| \leq 1$, where $P(z) = \sum_{\gamma=0}^n a_\gamma z^\gamma$ is a polynomial of degree n having all its zeros in $|z| \leq 1$. Then for all z on $|z| = 1$ where $Q(z) \neq 0$, we get by using induction hypothesis

$$\begin{aligned} \operatorname{Re} \left(\frac{zQ'(z)}{Q(z)} \right) &= \operatorname{Re} \left(\frac{z}{z-w} \right) + \operatorname{Re} \left(\frac{zP'(z)}{P(z)} \right) \\ &\geq \frac{1}{1+|w|} + \frac{n-1}{2} + \frac{1}{1+|a_0|}. \end{aligned}$$

To complete the induction principle, we need to show that on $|z| = 1$,

$$(2.16) \quad \operatorname{Re} \left(\frac{zQ'(z)}{Q(z)} \right) \geq \frac{n}{2} + \frac{1}{1+|w||a_0|}.$$

Clearly, the Inequality (2.16) holds if

$$\frac{1}{1+|w|} + \frac{n-1}{2} + \frac{1}{1+|a_0|} \geq \frac{n}{2} + \frac{1}{1+|w||a_0|},$$

which is equivalent to

$$(2.17) \quad \frac{1}{1+|w|} - \frac{1}{2} + \frac{1}{1+|a_0|} - \frac{1}{1+|w||a_0|} \geq 0.$$

But Inequality (2.17) follows from Lemma 2.7, since $0 \leq |w| \leq 1$, and $0 \leq |a_0| \leq 1$. Hence (2.16) is also true, and with this, the proof becomes complete on using induction hypothesis. \square

Our next result is a generalization of Lemma 2.8. However, it was necessary for us to prove Lemma 2.8 because the same will be needed to prove Lemma 2.9, given below.

Lemma 2.9. *If $P(z) = z^m(a_0 + a_1z + \cdots + a_{n-m}z^{n-m})$, $0 \leq m \leq n$, is a polynomial of degree $n \geq 1$ having all its zeros in $|z| \leq 1$, then for all z on $|z| = 1$ for which $P(z) \neq 0$,*

$$(2.18) \quad \operatorname{Re} \left(\frac{zP'(z)}{P(z)} \right) \geq \frac{n+m}{2} + \frac{|a_{n-m}| - |a_0|}{2(|a_{n-m}| + |a_0|)}.$$

Proof. Let $P(z) = z^mQ(z)$ where $Q(z) := a_0 + a_1z + \cdots + a_{n-m}z^{n-m}$. Then on $|z| = 1$,

$$(2.19) \quad \operatorname{Re} \left(\frac{zP'(z)}{P(z)} \right) = m + \operatorname{Re} \left(\frac{zQ'(z)}{Q(z)} \right).$$

Therefore by applying Lemma 2.8 to the polynomial $Q(z)$ in (2.19), and noting that $Q(z)$ is of degree $(n - m)$, we get on $|z| = 1$,

$$\operatorname{Re} \left(\frac{zP'(z)}{P(z)} \right) \geq m + \frac{n-m}{2} + \frac{|a_{n-m}| - |a_0|}{2(|a_{n-m}| + |a_0|)},$$

which is equivalent to the desired inequality. \square

Lemma 2.10. *If $P(z)$ is a polynomial of degree n which has all its zeros in the disk $|z| \leq K, K \geq 1$, then*

$$(2.20) \quad \max_{|z|=K} |P(z)| \geq \frac{2K^n}{1 + K^n} \max_{|z|=1} |P(z)|.$$

Proof. This result appears in Aziz [2], however for the sake of completeness we present here its proof. It may be noted that our proof is different from the one given in Aziz [2], and is much shorter.

Since $P(z)$ is a polynomial of degree n having all its zeros in the disk $|z| \leq K, K \geq 1$, the polynomial $R(z) = P(Kz)$ is of degree n , and has all its zeros in $|z| \leq 1$. Let $Q(z) := z^n R(1/z)$. Then $Q(z)$ is a polynomial of degree at most n having no zeros in $|z| < 1$ and therefore by an inequality due to Ankeny and Rivlin [1, Theorem 1], we have

$$\max_{|z|=K} |Q(z)| \leq \frac{K^n + 1}{2} \max_{|z|=1} |Q(z)|,$$

which is equivalent to

$$(2.21) \quad \max_{|z|=1} |Q(z)| \geq \frac{2}{K^n + 1} \max_{|z|=K} |Q(z)|.$$

Since $Q(z) = z^n R(1/z) = z^n P(K/z)$, the above Inequality (2.21) clearly gives

$$(2.22) \quad \max_{|z|=K} |P(z)| \geq \frac{2K^n}{K^n + 1} \max_{|z|=1} |P(z)|,$$

which is the Inequality (2.20), and the proof of Lemma 2.10 is thus complete. \square

Lemma 2.11. *If $P(z) = z^m(a_0 + a_1z + \dots + a_{n-m}z^{n-m})$, $0 \leq m \leq n$, is a polynomial of degree n having all its zeros in $|z| \leq 1$, then for any complex number α with $|\alpha| \geq 1$, and on $|z| = 1$,*

$$(2.23) \quad |D_\alpha\{P(z)\}| \geq (|\alpha| - 1) \left(\frac{n + m}{2} + \frac{(|a_{n-m}| - |a_0|)}{2(|a_{n-m}| + |a_0|)} \right) |P(z)|.$$

Proof. Since $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, and $Q(z) := z^n \overline{P\left(\frac{1}{z}\right)}$, then with the fact that $|P'(z)| \geq |Q'(z)|$ on $|z| = 1$, we have for $|\alpha| \geq 1$, and on $|z| = 1$,

$$\begin{aligned} |D_\alpha P(z)| &= |nP(z) + (\alpha - z)P'(z)| \\ &\geq |\alpha||P'(z)| - |nP(z) - zP'(z)| \\ &= |\alpha||P'(z)| - |Q'(z)| \geq (|\alpha| - 1)|P'(z)|. \end{aligned}$$

Therefore

$$(2.24) \quad |D_\alpha P(z)| \geq (|\alpha| - 1)|P'(z)|$$

on $|z| = 1$, and also on using Lemma 2.9, we get that on $|z| = 1$,

$$(2.25) \quad |P'(z)| \geq \left(\frac{n+m}{2} + \frac{|a_{n-m}| - |a_0|}{2(|a_{n-m}| + |a_0|)} \right) |P(z)|.$$

The equations (2.24) and (2.25) gives the Inequality (2.23). \square

3. PROOF OF THEOREM 1.1

Since $P(z)$ has all its zeros in $|z| \leq K$, $K \geq 1$, hence all the zeros of $P(Kz)$ lie in $|z| \leq 1$. Now, noting that by hypotheses we have $|\alpha|/K \geq 1$, hence on taking $P(z) = P(Kz)$ and using Lemma 2.11, we get

$$\max_{|z|=1} |D_{\alpha/K} P(Kz)| \geq \frac{(|\alpha| - K)}{K} \left[\frac{n+m}{2} + \frac{(|a_{n-m}|K^{n-m} - |a_0|)}{2(|a_{n-m}|K^{n-m} + |a_0|)} \right] \max_{|z|=1} |P(Kz)|,$$

which is nothing but

$$(3.26) \quad \begin{aligned} & \max_{|z|=1} |nP(Kz) + (\alpha/K - z)KP'(Kz)| \\ & \geq \frac{(|\alpha| - K)}{K} \left[\frac{n+m}{2} + \frac{(|a_{n-m}|K^{n-m} - |a_0|)}{2(|a_{n-m}|K^{n-m} + |a_0|)} \right] \max_{|z|=1} |P(Kz)|. \end{aligned}$$

From the fact that $\max_{|z|=1} |nP(Kz) + (\alpha/K - z)KP'(Kz)| = \max_{|z|=K} |D_{\alpha} P(z)|$, and using Lemma 2.10, the above expression (3.26) gives

$$(3.27) \quad \begin{aligned} & \max_{|z|=K} |D_{\alpha} P(z)| \\ & \geq \frac{(|\alpha| - K)}{K} \left[\frac{n+m}{2} + \frac{(|a_{n-m}|K^{n-m} - |a_0|)}{2(|a_{n-m}|K^{n-m} + |a_0|)} \right] \left(\frac{2K^n}{1+K^n} \right) \max_{|z|=1} |P(z)|. \end{aligned}$$

Since $D_{\alpha} P(z)$ is a polynomial of degree $n-1$ and $K \geq 1$, hence by Bernstein Inequality, we have $\max_{|z|=K} |D_{\alpha} P(z)| \leq K^{n-1} \max_{|z|=1} |D_{\alpha} P(z)|$. By making use of this,

the above inequality (3.27) clearly gives

$$\begin{aligned} & K^{n-1} \max_{|z|=1} |D_{\alpha} P(z)| \\ & \geq (|\alpha| - K) \left[\frac{n+m}{2} + \frac{(|a_{n-m}|K^{n-m} - |a_0|)}{2(|a_{n-m}|K^{n-m} + |a_0|)} \right] \left(\frac{2K^{n-1}}{1+K^n} \right) \max_{|z|=1} |P(z)|, \end{aligned}$$

which on simplification and rearrangement of terms yields the desired inequality (1.9), and the proof of the Theorem 1.1 is thus complete. \square

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