

**IDENTITIES FOR DIRICHLET AND LAMBERT-TYPE  
SERIES ARISING FROM THE NUMBERS  
OF A CERTAIN SPECIAL WORD**

*Dedicated to Academician Professor Gradimir Milovanović  
on the occasion of his 70th birthday.*

*Irem Kucukoglu\* and Yilmaz Simsek*

The goal of this paper is to give several new Dirichlet-type series associated with the Riemann zeta function, the polylogarithm function, and also the numbers of necklaces and Lyndon words. By applying Dirichlet convolution formula to number-theoretic functions related to these series, various novel identities and relations are derived. Moreover, some new formulas related to Bernoulli-type numbers and polynomials obtain from generating functions and these Dirichlet-type series. Finally, several relations among the Fourier expansion of Eisenstein series, the Lambert series and the number-theoretic functions are given.

**1. INTRODUCTION**

A Dirichlet series, associated with a complex sequence  $x_n$ , is given by the following form:

$$\sum_{n=1}^{\infty} \frac{x_n}{n^s}$$

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\*Corresponding author. Irem Kucukoglu

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(*cf.* [3]). This series, which plays a central role for deriving identities and formulae, have recently studied by many authors due to their applications in various branches of pure and applied mathematics, and also other related areas. This series is also closely related to the family of zeta functions such as the Riemann zeta function, the Hurwitz zeta function, the polylogarithm functions, the Dirichlet  $L$ -series, the Lambert series, the number-theoretic functions (or, arithmetic functions) and also other special functions and series. It is also known that the number-theoretic functions build a bridge between the family of zeta-type functions and Dirichlet series.

In [16], the authors studied and investigated Dirichlet series associated with a number-theoretic function related to the Lyndon words. They also put forward the following question: Which numbers or polynomials are interpolated by the functions derived from the Dirichlet-type series including some number-theoretic functions and Lambert series?

By this motivation, the first aim of this paper is to define some Dirichlet-type series associated with new number-theoretic functions involving the Möbius function, the Euler's totient function, and also the other number-theoretic functions that count necklaces and Lyndon words. Thus, this paper not only answers the aforementioned question, but also includes new results. By applying Dirichlet convolution formula to number-theoretic functions related to our new Dirichlet series, our second aim is to derive some new identities and relations. By using well-known analytic continuation of the Riemann zeta function, our third aim is to give various identities and relations including the Bernoulli numbers and the Apostol-Bernoulli numbers. Finally, many identities and relations among Dirichlet-type series, the Fourier expansion of Eisenstein series, the Lambert series and the number-theoretic functions are given.

## 2. PRELIMINARIES

Before proceeding with our results, let us recall some required tools to obtain the results of the present paper, by the following notations, definitions and relations.

Throughout the present paper,  $\mathbb{Z}$ ,  $\mathbb{R}$  and  $\mathbb{C}$  as usual denote the set of integers, the set of real numbers and the set of complex numbers, respectively. Within this context, let  $\mathbb{Z}^- = \{-1, -2, -3, \dots\}$ ,  $\mathbb{Z}_0^- = \mathbb{Z}^- \cup \{0\}$ ,  $\mathbb{N} = \{1, 2, \dots\}$  and  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ . Let  $\text{Re}(s)$  denote the real part of  $s \in \mathbb{C}$ .

We now recall two interesting number-theoretic functions which are intensively used in this paper. Let  $k, n \in \mathbb{N}$ . The first of the aforementioned two number-theoretic functions is denoted by  $N_k(n)$ , which counts the number of necklaces consisting of  $n$ -coloured beads with  $k$ -distinct colours. The function  $N_k(n)$  is defined by the following formula (*cf.* [5]):

$$(1) \quad N_k(n) = \frac{1}{n} \sum_{d|n} \phi\left(\frac{n}{d}\right) k^d$$

where  $\phi(n)$  denotes the number-theoretic Euler's totient function defined by

$$\phi(n) = \sum_{\substack{m=1 \\ (m,n)=1}}^n 1$$

where the sum is over all positive integers  $m \leq n$  that are relatively prime to  $n$  (cf. [3]).

The latter of the aforementioned two number-theoretic functions is denoted by  $L_k(n)$ , which counts the numbers of the Lyndon words. The function  $L_k(n)$  is defined by the following formula (cf. [5], [15], [19], [22]):

$$(2) \quad L_k(n) = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) k^d$$

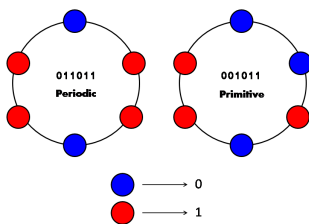
with  $\mu$  being the number-theoretic Möbius function defined by (cf. [3]):

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^r & \text{if } n \text{ is a product of } r \text{ distinct primes,} \\ 0 & \text{if } n \text{ is not square-free.} \end{cases}$$

We observe that the function  $L_k(n)$  is known to emerge naturally in wide variety combinatorial and algebraic problems. For example, according to Betten et al. [6], this function is equal to the number of orbits of maximal length of an any  $n$ -th order cyclic group on  $k^n$  with  $k, n \in \mathbb{N}$ . Equation (2) also enumerates the  $n$ -th degree monic irreducible polynomials over  $F_k = \mathbb{Z}/k\mathbb{Z}$  which is the finite field with  $k$  elements (see, for details, cf. [6], [8], [22]).

On the one hand the Lyndon word, which is one of the considerable concept of combinatorics on words, is defined as the lexicographically (or, in a dictionary order) smallest element of the set of conjugate classes which are the result of cyclic shifts of the letters in a primitive word. The function  $L_k(n)$  is related rather closely to the function  $N_k(n)$ , since the Lyndon words coincide with aperiodic necklace class representatives (see, for details, [5], [15], [19], [22]). But on the other hand, in the work of Duval [13], a generation algorithm method, which formed this special word according to a certain rule, was given. Additionally, Kucukoglu et al. [15] presented ordinary generating functions for the numbers of these special words. For further information related to these special words, refer to the works (among others) (cf. [5], [15], [19], [22]).

We now illustrate how a special binary word is represented by the periodic and primitive necklaces consisting of 6-coloured beads with 2-distinct colours by Fig. 1.



**Figure 1:** Representing special binary word by a periodic or a primitive necklace (*cf.* [15]).

With the aid of Fig. 1, one may observed that the word 001011 is primitive and smallest in its conjugacy class. Hence this leads that the word 001011 is precisely a Lyndon word.

## 2.1. Some algebraic properties of number-theoretic functions

Here, we give few fundamental properties of number-theoretic functions.

Let  $f(n)$  and  $g(n)$  be number-theoretic functions. The Dirichlet convolution of the functions  $f(n)$  and  $g(n)$ , which is denoted by  $f * g$ , is given by (*cf.* [3]):

$$(f * g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right).$$

It is worth mentioning that the set of all number-theoretic functions is a commutative ring with unity under pointwise addition and Dirichlet convolution (see, for details, [12]).

Convergence property of Dirichlet series associated with the number-theoretic functions  $f(n)$  and  $g(n)$  is given as follows:

$$(3) \quad F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \quad \text{and} \quad G(z) = \sum_{n=1}^{\infty} \frac{g(n)}{n^z};$$

assuming that two Dirichlet series in (3) converge absolutely for the half-plane  $\text{Re}(s) > a$  and  $\text{Re}(z) > b$ , respectively, where  $a, b \in \mathbb{R}$  denote the abscissas of absolute convergence (*cf.* [3]). Moreover, by further recalling that, for  $s$  is an element of the half-plane where both of the series in (3) converge absolutely, the multiplication of  $F(s)$  and  $G(s)$  is given as follows (*cf.* [3, Theorem 11.5, p. 228]):

$$(4) \quad F(s) G(s) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s}$$

where  $h(n)$  denotes

$$h(n) = (f * g)(n).$$

The Lambert series is defined by

$$(5) \quad \sum_{n=1}^{\infty} f(n) \frac{x^n}{1-x^n}$$

where  $f(n)$  is a number-theoretic function. Assuming that (5) converges absolutely, then we have

$$(6) \quad \sum_{n=1}^{\infty} f(n) \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} H(n) x^n$$

where

$$H(n) = \sum_{d|n} f(d)$$

(cf. [2, p. 24]).

By substituting  $f(n) = L_k(n)$  into (6), one easily has the following well-known identity:

$$(7) \quad \sum_{n=1}^{\infty} nL_k(n) \frac{x^n}{1-x^n} = \frac{kx}{1-kx},$$

where  $|kx| < 1$ .

Considering that  $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ , then the Eisenstein series  $G(z, k, r, h)$  is defined as follows:

$$G(z, k, r, h) = \sum_{r \neq (m, n) \in \mathbb{Z}^2} \frac{e^{2\pi i(mh_1 + nh_2)}}{((m + r_1)z + n + r_2)^s}$$

where  $z \in \mathbb{H}$  and  $\text{Re}(s) > 2$  (cf. [2], [18], [23, Eq. (1.3)]).

The Fourier expansion of Eisenstein series is given by

$$(8) \quad G(z, k, r, h) = 2\mathcal{Z}(k, h) + \frac{2(-2\pi i)^k}{(k-1)!} \sum_{a, n=1}^{\infty} a^{k-1} e^{2\pi i(n+r)az},$$

where  $k \in \mathbb{N} \setminus \{1\}$ ,  $r$  and  $h$  are rational numbers,  $z \in \mathbb{H}$ , and the function  $\mathcal{Z}(k, h)$  is given as

$$\mathcal{Z}(k, h) = \sum_{n > -a} (n+a)^{-s},$$

where  $a \in \mathbb{R}$  and  $\text{Re}(s) > 1$  (cf. [18], [23, Corollary 1]).

## 2.2. Apostol-type numbers and polynomials with their interpolation functions

The Apostol-Bernoulli polynomials  $\mathcal{B}_n(x; \lambda)$  are defined by the following generating function:

$$(9) \quad \frac{te^{xt}}{\lambda e^t - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(x; \lambda) \frac{t^n}{n!}$$

where  $\lambda \in \mathbb{C}$ ;  $|t| < 2\pi$  when  $\lambda = 1$ ;  $|t| < |\log \lambda|$  when  $\lambda \neq 1$  (cf. [1]).

Substituting  $\lambda = 1$  into (9), Apostol-Bernoulli polynomials are reduced to the generating function for the Bernoulli polynomials  $B_n(x)$  as follows:

$$(10) \quad \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!},$$

(cf. [1], [7], [9], [10], [15], [21], [24], [25], [26]).

Upon setting  $x = 0$  in (9), the Apostol-Bernoulli polynomials are reduced to the Apostol-Bernoulli numbers  $\mathcal{B}_n(\lambda)$ , given by

$$(11) \quad \frac{t}{\lambda e^t - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(\lambda) \frac{t^n}{n!}$$

which, for  $\lambda = 1$ , yields the Bernoulli numbers  $B_n$ , given by the following generating function:

$$(12) \quad \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!},$$

whose first few values are given as follows:

$$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42},$$

with

$$B_{2m+1} = 0, \quad (m \in \mathbb{N})$$

and also it follows from (9) and (11) that

$$(13) \quad \lambda \mathcal{B}_m(1; \lambda) = \mathcal{B}_m(\lambda)$$

for  $m \geq 2$  (cf. [1], [7], [9], [10], [15], [21], [24], [25], [26]).

The Lerch transcendent function (or, the Hurwitz-Lerch zeta function) is given by

$$\Phi(\lambda, s, a) = \sum_{n=0}^{\infty} \frac{\lambda^n}{(n+a)^s},$$

which converges for the case of  $a \in \mathbb{C} \setminus \mathbb{Z}_0^-$ ,  $s \in \mathbb{C}$  when  $|\lambda| < 1$  and the case of  $\text{Re}(s) > 1$  when  $|\lambda| = 1$  (cf. [1], [7], [9], [10], [15], [21], [24], [25], [26]).

For  $m \in \mathbb{N}$ , a relation between these functions and the Apostol-Bernoulli numbers is given by (cf. [1]):

$$(14) \quad \Phi(\lambda, -m, a) = -\frac{\mathcal{B}_{m+1}(a; \lambda)}{m+1}.$$

In the special case of  $a = 1$  and  $\lambda = 1$ , the function  $\Phi(\lambda, s, a)$  reduces to the Riemann zeta function given by

$$\Phi(1, s, 1) = \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \text{Re}(s) > 1$$

and the following relation holds true between the Riemann zeta function and the Bernoulli numbers

$$(15) \quad \zeta(-m) = -\frac{B_{m+1}}{m+1}$$

where  $m \in \mathbb{N}$  (cf. [1], [7], [9], [10], [15], [21], [24], [25], [26]).

The polylogarithm function  $Li_s(z)$  is given by

$$(16) \quad Li_s(\lambda) = \sum_{n=1}^{\infty} \frac{\lambda^n}{n^s}$$

which absolutely converges for  $s \in \mathbb{C}$  and  $|\lambda| < 1$  (cf. [4], [24], [25]).

Observe that a relation between the Lerch transcendent function and the polylogarithm function is given by (cf. [4], [24], [25]):

$$(17) \quad Li_s(\lambda) = \lambda \Phi(\lambda, s, 1)$$

which is interpolation function of both the Eulerian numbers  $A(m, j)$  and the Apostol-Bernoulli numbers at negative integers as follows:

$$(18) \quad Li_{-m}(\lambda) = \frac{1}{(1-\lambda)^{m+1}} \sum_{j=0}^m A(m, j) \lambda^{m-j}, \quad \text{for } m \in \mathbb{N}$$

and

$$(19) \quad Li_{-m}(\lambda) = -\frac{\mathcal{B}_{m+1}(\lambda)}{m+1}, \quad \text{for } m \in \mathbb{N}$$

as a result of the equations (13), (14) and (17) (cf. [1], [4], [11], [24], [25]).

### 3. DIRICHLET-TYPE SERIES ASSOCIATED WITH SOME CERTAIN NUMBER-THEORETIC FUNCTIONS

In this section, we define some Dirichlet-type series associated with number-theoretic functions involving the Möbius function, the Euler's totient function, and also the numbers of necklaces and Lyndon words. Moreover, applying the Dirichlet convolution formula to number-theoretic functions related to these series, we derive new identities and relations. In the light of (15) and (19), we investigate fundamental properties of these Dirichlet-type series including the Apostol-Bernoulli numbers and the Bernoulli numbers.

#### 3.1. New number-theoretic function related to the function $L_k(n)$ and its Dirichlet-type series

By using (2), we set the following presumably new number-theoretic function:

$$(20) \quad L_k(x : n) = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) k^d x^d.$$

It will be shown in the following that the Dirichlet-type series defined through the function  $L_k(x : n)$  interpolates some certain numbers formed of the Apostol-Bernoulli numbers and the Bernoulli numbers.

**Remark 1.** *The function (20) immediately yields (2) when  $x = 1$ . That is*

$$L_k(n) = L_k(1 : n).$$

Substituting  $n = 6$ , which is the smallest number that can be written as the multiplication of two distinct prime numbers, into (20) few values of the function  $L_k(x : n)$  are given as follows:

$$\begin{aligned} L_1(x : 6) &= \frac{x^6 - x^3 - x^2 + x}{6}, \\ L_2(x : 6) &= \frac{32x^6 - 4x^3 - 2x^2 + x}{3}. \end{aligned}$$

By (20), we now establish Dirichlet-type series  $\zeta_1(x : k, s)$  associated with the function  $L_k(x : n)$  by the following definition:

**Definition 1.** *Let  $k \in \mathbb{N}$ ,  $x \in \mathbb{R}$  and  $s \in \mathbb{C}$ . We define*

$$\zeta_1(x : k, s) = \sum_{n=1}^{\infty} \frac{nL_k(x : n)}{n^s}.$$

**Theorem 1.** *Let  $k \in \mathbb{N}$  and  $x \in \mathbb{R}$  such that  $|kx| < 1$ . Let  $s \in \mathbb{C}$  with  $\operatorname{Re}(s) > 1$ . Then we have*

$$(21) \quad \zeta(s)\zeta_1(x : k, s) = Li_s(kx).$$



*Proof.* Let

$$F_\mu(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)},$$

where  $\operatorname{Re}(s) > 1$  (cf. [3, p. 228, Example 1]) and

$$G(x : k, s) = \sum_{n=1}^{\infty} \frac{(kx)^n}{n^s} = Li_s(kx),$$

where  $|kx| < 1$  and  $\operatorname{Re}(s) > 1$ . Replacing  $F(s)$  by  $F_\mu(s)$  and  $G(s)$  by  $G(x : k, s)$  in (4) yields the assertion of Theorem 1.  $\square$

By substituting  $s = -m$ , with  $m \in \mathbb{N}$ , into (21) and by using (15) and (19), we get the following theorem:

**Theorem 2.** *Let  $k, m \in \mathbb{N}$ . Then we have*

$$(22) \quad \zeta_1(x : k, -m) B_{m+1} = \mathcal{B}_{m+1}(kx).$$

By making use of generating function method for the Bernoulli numbers and the Apostol-Bernoulli numbers, we now present another proof of Theorem 2 as follows:

In precisely the same manner as the proof that of (7), upon substituting  $f(n) = L_k(y : n)$  into (6) yields the following Lambert series:

$$(23) \quad \sum_{n=1}^{\infty} n L_k(y : n) \frac{x^n}{1 - x^n} = \frac{k y x}{1 - k y x}$$

for  $|k y x| < 1$ . Replacing  $x$  by  $e^z$  in (23), we get

$$\sum_{n=1}^{\infty} n L_k(y : n) \frac{e^{zn}}{1 - e^{zn}} = \frac{k y e^z}{1 - k y e^z}.$$

By using (11) and (12) in the above equation, we obtain

$$\sum_{n=1}^{\infty} n L_k(y : n) \sum_{m=0}^{\infty} \left( \frac{n^m}{m+1} \sum_{j=0}^{m+1} \binom{m+1}{j} B_j \right) \frac{z^m}{m!} = \frac{k y}{z} \sum_{m=0}^{\infty} \mathcal{B}_m(1; k y) \frac{z^m}{m!}.$$

Therefore, we have

$$\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{n^{m+1} L_k(y : n) B_{m+1}(1)}{m+1} \frac{z^m}{m!} = k y \sum_{m=0}^{\infty} \frac{\mathcal{B}_{m+1}(1; k y)}{m+1} \frac{z^m}{m!}.$$

Thus, comparing the coefficients of  $\frac{z^m}{m!}$  on both sides of the above equation and using (13), we arrive at the assertion of Theorem 2.

By combining (16) and (18) with (21), we can readily put Theorem 2 in the alternative form by the following theorem:

**Theorem 3.** Let  $k, m \in \mathbb{N}$ . Then we have

$$(24) \quad \sum_{j=0}^m A(m, j) (kx)^{m-j} = (1 - kx)^{m+1} \zeta_1(x : k, -m) \zeta(-m).$$

**Remark 2.** By comparing (22) and (24), a relation between the Eulerian numbers and the Apostol-Bernoulli numbers is obtained as follows:

$$\mathcal{B}_{m+1}(kx) = -\frac{m+1}{(1-kx)^{m+1}} \sum_{j=0}^m A(m, j) (kx)^{m-j}$$

whose alternative form was also given in [7, Eq. (4.6)].

Moreover, we establish another Dirichlet-type series  $\zeta_{1,\text{odd}}(x : k, s)$  as follows:

$$(25) \quad \zeta_{1,\text{odd}}(x : k, s) = \sum_{n=1}^{\infty} \frac{L_k(x : 2n-1)}{(2n-1)^{s-1}}.$$

Here, we show that the function  $\zeta_{1,\text{odd}}(x : k, s)$  interpolates the Apostol-Bernoulli numbers and the Bernoulli numbers.

**Theorem 4.** Let  $k \in \mathbb{N}$  and  $x \in \mathbb{R}$  such that  $|kx| < 1$ . Let  $s \in \mathbb{C}$  with  $s \neq 0$ . Then we have

$$(26) \quad \zeta(s) \zeta_{1,\text{odd}}(x : k, s) = \frac{2^s Li_s(kx) - Li_s(k^2 x^2)}{2^s - 1}.$$

In order to give proof of Theorem 4, we need the following lemmas:

**Lemma 1.** Let  $m \in \mathbb{N}$  and  $p$ 's be prime numbers. Then the following equality holds true:

$$(27) \quad \sum_{n=1}^{\infty} \frac{\mu(mn)}{n^s} = \frac{\mu(m)}{\zeta(s)} \prod_{p|m} \frac{1}{1-p^{-s}},$$

(cf. [17, Eq. (2.31)], [20, p. 30]).

**Lemma 2.** Let  $k \in \mathbb{N}$  and  $x \in \mathbb{R}$  such that  $|kx| < 1$ . Then we have

$$(28) \quad \zeta_{1,\text{odd}}(x : k, s) = \sum_{n \neq 0(2)} \frac{\mu(n)}{n^s} \sum_{m \neq 0(2)} \frac{(kx)^m}{m^s}.$$

*Proof.* Before starting the proof of Lemma 2, in order to multiply the following series

$$(29) \quad \sum_{n \neq 0(2)} \frac{\mu(n)}{n^z}$$

and

$$(30) \quad \sum_{m \neq 0(2)} \frac{(kx)^m}{m^s},$$

we assume that  $\operatorname{Re}(z) > b_1$ ; and also  $|kx| < 1$  and  $\operatorname{Re}(s) > b_2$ , respectively. In the light of proof given for Theorem 11.5 in [3], for the half-plane in which both series (29) and (30) are absolute convergence, we have

$$\sum_{n \neq 0(2)} \frac{\mu(n)}{n^s} \sum_{m \neq 0(2)} \frac{(kx)^m}{m^s} = \sum_{n \neq 0(2)} \sum_{m \neq 0(2)} \mu(n) (kx)^m (nm)^{-s}.$$

Setting  $nm = a$ ;  $a = 1, 3, 5, \dots$  in the above equation and using the Dirichlet convolution formula, we deduce that

$$\begin{aligned} \sum_{n \neq 0(2)} \frac{\mu(n)}{n^s} \sum_{m \neq 0(2)} \frac{(kx)^m}{m^s} &= \sum_{a \neq 0(2)} \sum_{nm=a} \mu(n) (kx)^m a^{-s} \\ &= \sum_{a \neq 0(2)} \frac{\sum_{d|a} \mu\left(\frac{a}{d}\right) (kx)^d}{a^s}, \end{aligned}$$

which yields the assertion of Lemma 2. □

*Proof of Theorem 4.* Substituting  $m = 2$  into (27), we have

$$\sum_{n=1}^{\infty} \frac{\mu(2n)}{n^s} = -\frac{1}{\zeta(s)} \left( \frac{1}{1-2^{-s}} \right).$$

Thus, we have

$$F_{\mu, \text{odd}}(s) = \sum_{n \neq 0(2)} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} \left( \frac{2^s}{2^s - 1} \right).$$

Hence

$$\begin{aligned} G_{\text{odd}}(kx, s) &= \sum_{n \neq 0(2)} \frac{(kx)^n}{n^s} \\ &= Li_s(kx) - \sum_{n \equiv 0(2)} \frac{(kx)^n}{n^s} \\ &= Li_s(kx) - \frac{1}{2^s} Li_s(k^2 x^2). \end{aligned}$$

Replacing  $F(s)$  by  $F_{\mu, \text{odd}}(s)$  and  $G(s)$  by  $G_{\text{odd}}(kx, s)$  in (4) and combining the final equation with (28), we get the assertion of Theorem 4. □

By substituting  $s = -m$ , with  $m \in \mathbb{N}$ , into (26) and using (15) and (19), we get the following theorem:

**Theorem 5.** *Let  $k, m \in \mathbb{N}$ . Then we have*

$$(31) \quad \zeta_{1,odd}(x : k, -m) B_{m+1} = \frac{\mathcal{B}_{m+1}(kx) - 2^m \mathcal{B}_{m+1}(k^2x^2)}{1 - 2^m}.$$

Since the function  $\zeta_1(x : k, s)$  converges absolutely for  $\operatorname{Re}(s) > 1$  and  $|kx| < 1$ , we also set a function denoted by  $\zeta_{1,even}(x : k, s)$  readily follows by

$$(32) \quad \begin{aligned} \zeta_{1,even}(x : k, s) &= \sum_{n=1}^{\infty} \frac{L_k(x : 2n)}{(2n)^{s-1}} \\ &= \zeta_1(x : k, s) - \zeta_{1,odd}(x : k, s). \end{aligned}$$

By combining (21) and (26) with (32), we obtain the following theorem:

**Theorem 6.** *Let  $k \in \mathbb{N}$  and  $x \in \mathbb{R}$  such that  $|kx| < 1$ . Let  $s \in \mathbb{C}$  with  $s \neq 0$ . Then we have*

$$(33) \quad \zeta(s) \zeta_{1,even}(x : k, s) = \frac{Li_s(k^2x^2) - Li_s(kx)}{2^s - 1}.$$

By substituting  $s = -m$ , with  $m \in \mathbb{N}$ , into (26) and using (15) and (19), we get the following theorem:

**Theorem 7.** *Let  $k, m \in \mathbb{N}$ . Then we have*

$$(34) \quad \zeta_{1,even}(x : k, -m) B_{m+1} = \frac{2^m}{1 - 2^m} (\mathcal{B}_{m+1}(k^2x^2) - \mathcal{B}_{m+1}(kx)).$$

### 3.2. New number-theoretic function related to the function $N_k(n)$ and its Dirichlet-type series

By using (1), we set the following presumably new number-theoretic function:

$$(35) \quad N_k(x : n) = \frac{1}{n} \sum_{d|n} \phi\left(\frac{n}{d}\right) k^d x^d.$$

It will be shown in the following that the Dirichlet-type series defined through the function  $N_k(x : n)$  interpolates some certain numbers formed of the Apostol-Bernoulli numbers and the Bernoulli numbers.

**Remark 3.** *For  $x = 1$ , (35) is immediately reduced to (1). That is*

$$N_k(n) = N_k(1 : n).$$

With the help of (35), few values of the function  $N_k(x : n)$  are given as follows:

$$\begin{aligned} N_1(x : 6) &= \frac{x^6 + x^3 + 2x^2 + 2x}{6}, \\ N_2(x : 6) &= \frac{32x^6 + 4x^3 + 4x^2 + 2x}{3}. \end{aligned}$$

By using (35), we now establish Dirichlet-type series  $\zeta_2(x : s, k)$  associated with the function  $N_k(x : n)$  as follows:

**Definition 2.** Let  $k \in \mathbb{N}$ ,  $x \in \mathbb{R}$  and  $s \in \mathbb{C}$ . We define

$$\zeta_2(x : k, s) = \sum_{n=1}^{\infty} \frac{nN_k(x : n)}{n^s}.$$

A relation between the functions  $\zeta(s)$ ,  $Li_s(x)$  and  $\zeta_2(x : k, s)$  is given by the following theorem:

**Theorem 8.** Let  $k \in \mathbb{N}$ ,  $x \in \mathbb{R}$  and  $s \in \mathbb{C}$  such that  $\text{Re}(s) > 2$  and  $|kx| < 1$ . Then we have

$$(36) \quad \zeta(s)\zeta_2(x : k, s) = \zeta(s - 1)Li_s(kx).$$

*Proof.* Let

$$(37) \quad F_\phi(s) = \sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \frac{\zeta(s - 1)}{\zeta(s)},$$

where  $\text{Re}(s) > 2$  (cf. [3, p. 229, Example 4]). Thus, replacing  $F(s)$  by  $F_\phi(s)$  and  $G(s)$  by  $G(x : k, s)$  in (4) yields the assertion of Theorem 8.  $\square$

By substituting  $s = -m$ , with  $m \in \mathbb{N}$ , into (36) and using (15) and (19), we get the following theorem:

**Theorem 9.** For suitable values of  $k$ ,  $m$  and  $x$ , the following assertion holds true:

$$(38) \quad (m + 2)\zeta_2(x : k, -m)B_{m+1} = -B_{m+2}\mathcal{B}_{m+1}(kx).$$

Now, we give another Dirichlet-type series  $\zeta_{2,\text{odd}}(x : k, s)$ , which also interpolates the Apostol-Bernoulli numbers and the Bernoulli numbers, as follows:

$$(39) \quad \zeta_{2,\text{odd}}(x : k, s) = \sum_{n=1}^{\infty} \frac{N_k(x : 2n - 1)}{(2n - 1)^{s-1}},$$

where  $k \in \mathbb{N}$ ,  $x \in \mathbb{R}$  and  $s \in \mathbb{C}$ .

**Theorem 10.** Let  $k \in \mathbb{N}$ ,  $x \in \mathbb{R}$  and  $s \in \mathbb{C}$  such that  $\operatorname{Re}(s) > 2$  and  $|kx| < 1$ . Then we have

$$(40) \quad 2^s \zeta(s) \zeta_{2, \text{odd}}(x : k, s) = (2^s - 1) \zeta(s - 1) \left( Li_s(kx) - \frac{1}{2^s} Li_s(k^2 x^2) \right).$$

In order to give proof of Theorem 10, we need the following lemma:

**Lemma 3.** Let  $k \in \mathbb{N}$ ,  $x \in \mathbb{R}$  and  $s \in \mathbb{C}$  such that  $\operatorname{Re}(s) > 2$  and  $|kx| < 1$ . Then we have

$$(41) \quad \zeta_{2, \text{odd}}(x : k, s) = \sum_{n \neq 0(2)} \frac{\phi(n)}{n^s} \sum_{m \neq 0(2)} \frac{(kx)^m}{m^s}.$$

*Proof.* The proof follows precisely along the same lines as in the proof of Lemma 2. Therefore we give sketch proof as follows. In order to multiply (30) with the following series

$$(42) \quad \sum_{n \neq 0(2)} \frac{\phi(n)}{n^s},$$

we assume that  $\operatorname{Re}(s) > b_1$ . By the same technique in Lemma 2, for the half-plane in which both series (30) and (42) are absolute convergence, we have

$$\sum_{n \neq 0(2)} \frac{\phi(n)}{n^s} \sum_{m \neq 0(2)} \frac{(kx)^m}{m^s} = \sum_{n \neq 0(2)} \sum_{m \neq 0(2)} \phi(n) (kx)^m (nm)^{-s}.$$

Setting  $nm = a$ ;  $a = 1, 3, 5, \dots$  in the above equation and using the Dirichlet convolution formula, we deduce that

$$\begin{aligned} \sum_{n \neq 0(2)} \frac{\phi(n)}{n^s} \sum_{m \neq 0(2)} \frac{(kx)^m}{m^s} &= \sum_{a \neq 0(2)} \sum_{mn=a} \phi(n) (kx)^m a^{-s} \\ &= \sum_{a \neq 0(2)} \frac{\sum_{d|a} \phi\left(\frac{a}{d}\right) (kx)^d}{a^s}, \end{aligned}$$

which yields the assertion of Lemma 3.  $\square$

*Proof of Theorem 10.* If we begin with setting

$$F_{\phi, \text{odd}}(s) = \sum_{n \neq 0(2)} \frac{\phi(n)}{n^s}$$

and using the well-known property of the Euler's totient function  $\phi(n)$  for  $n$  even integer  $\phi(2n) = 2\phi(n)$  and  $n$  odd integer  $\phi(2n) = \phi(n)$ , then (37) can be written as follows:

$$\sum_{n=1}^{\infty} \frac{\phi(2n)}{(2n)^s} = \frac{\zeta(s-1)}{2^s \zeta(s)}.$$

Therefore

$$F_{\phi, \text{odd}}(s) = \left(1 - \frac{1}{2^s}\right) \frac{\zeta(s-1)}{\zeta(s)}.$$

Substituting  $F(s) = F_{\phi, \text{odd}}(s)$  and  $G(s) = G_{\text{odd}}(kx, s)$  into (4) and combining the final equation with (41), we get assertion of Theorem 10.  $\square$

By substituting  $s = -m$ , with  $m \in \mathbb{N}$ , into (40) and using (15) and (19), we get the following theorem:

**Theorem 11.** *For suitable values of  $k, m$  and  $x$ , the following assertion holds true:*

$$(m+2)\zeta_{2, \text{odd}}(x : k, -m) B_{m+1} = (1 - 2^m) (2^m \mathcal{B}_{m+1}(k^2 x^2) - \mathcal{B}_{m+1}(kx)) B_{m+2}.$$

We also establish the function  $\zeta_{2, \text{even}}(x : k, s)$  by

$$(43) \quad \zeta_{2, \text{even}}(x : k, s) = \sum_{n=1}^{\infty} \frac{N_k(x : 2n)}{(2n)^{s-1}},$$

where  $k \in \mathbb{N}$ ,  $x \in \mathbb{R}$  and  $s \in \mathbb{C}$ .

Since the function  $\zeta_2(x : k, s)$  converges absolutely for  $\text{Re}(s) > 2$  and  $|kx| < 1$ , we have

$$(44) \quad \zeta_{2, \text{even}}(x : k, s) = \zeta_2(x : k, s) - \zeta_{2, \text{odd}}(x : k, s).$$

By combining (36) and (40) with (44), we obtain the following theorem:

**Theorem 12.** *Let  $k \in \mathbb{N}$ ,  $x \in \mathbb{R}$  and  $s \in \mathbb{C}$  such that  $\text{Re}(s) > 2$  and  $|kx| < 1$ . Then we have*

$$(45) \quad 2^s \zeta(s) \zeta_{2, \text{even}}(x : k, s) = \zeta(s-1) \left( Li_s(kx) - \frac{1-2^s}{2^s} Li_s(k^2 x^2) \right).$$

By substituting  $s = -m$ , with  $m \in \mathbb{N}$ , into (26) and using (15) and (19), we get the following theorem:

**Theorem 13.** *For suitable values of  $k, m$  and  $x$ , the following assertion holds true:*

$$(m+2)\zeta_{2, \text{even}}(x : k, -m) B_{m+1} = ((2^m - 1) \mathcal{B}_{m+1}(k^2 x^2) - \mathcal{B}_{m+1}(kx)) 2^m B_{m+2}.$$

Combining (21) and (36) readily yields the following corollary:

**Corollary 1.** *Let  $k \in \mathbb{N}$ ,  $x \in \mathbb{R}$  and  $s \in \mathbb{C}$  such that  $\text{Re}(s) > 2$  and  $|kx| < 1$ . Then we have*

$$\zeta(s-1) \zeta_1(x : k, s) = \zeta_2(x : k, s).$$

#### 4. FURTHER RELATIONS INVOLVING LAMBERT SERIES AND EISENSTEIN SERIES

In this section, by using the Fourier expansion of Eisenstein series, we give some further identities for the Lambert series which are essentially associated with the number-theoretic function  $L_k(n)$ .

Let  $n \in \mathbb{N}$ . Setting  $x = e^{2\pi iz}$  into the following Lambert series:

$$(46) \quad H(n, x) = n \sum_{k=1}^{\infty} L_k(n) \frac{x^k}{1-x^k} = n \sum_{k,m=1}^{\infty} L_k(n) x^{km}$$

yields

$$(47) \quad H(n, e^{2\pi iz}) = \sum_{k,m=1}^{\infty} n L_k(n) e^{2\pi i k m z}.$$

Combining the above equation with (2), we arrive at the following theorem:

**Theorem 14.** *Suppose that  $n \in \mathbb{N}$ . Then the following equality holds true:*

$$(48) \quad H(n, e^{2\pi iz}) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \sum_{k,m=1}^{\infty} k^d e^{2\pi i k m z}.$$

If we combine (48) with (8), then we also get the following theorem:

**Theorem 15.**

$$H(n, e^{2\pi iz}) = \sum_{d|n} d! \mu\left(\frac{n}{d}\right) \frac{G(z, d+1, 0, h) - 2\mathcal{Z}(d+1, h)}{2(-2\pi i)^{d+1}}.$$

In its particular case when  $n$  is a prime number  $p$ , Theorem 15 leads us to the following corollary:

**Corollary 2.** *Let  $p$  be a prime number. Then we have*

$$H(p, e^{2\pi iz}) = \frac{p!(G(z, p+1, 0, h) - 2\mathcal{Z}(p+1, h))}{2(-2\pi i)^{p+1}} + \frac{G(z, 2, 0, h) - 2\mathcal{Z}(2, h)}{8\pi^2}.$$

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**Irem Kucukoglu**

Department of Engineering Fundamental Sciences,  
Faculty of Engineering,  
Alanya Alaaddin Keykubat University  
TR-07425 Antalya,  
Turkey  
E-mail: [irem.kucukoglu@alanya.edu.tr](mailto:irem.kucukoglu@alanya.edu.tr)

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**Yilmaz Simsek**

Department of Mathematics,  
Faculty of Science  
University of Akdeniz  
TR-07058 Antalya,  
Turkey  
E-mail: [ysimsek@akdeniz.edu.tr](mailto:ysimsek@akdeniz.edu.tr)