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ON DEGENERATE CENTRAL COMPLETE BELL POLYNOMIALS

Dedicated to Academician Professor Gradimir Milovanović on the occasion of his 70th birthday.

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In this paper, we consider of generalized central complete and incomplete Bell polynomials called degenerate central complete and incomplete Bell polynomials. These polynomials are generalizations of the recently introduced central complete Bell polynomials and 'degenerate' analogues for the central complete and incomplete Bell polynomials. We investigate some properties and identities for these polynomials. Especially, we give explicit formulas for the degenerate central complete and incomplete Bell polynomials related to degenerate central factorial numbers of the second kind.

1. Introduction and preliminaries

The Stirling numbers of the second kind are given by

$$\frac{1}{k!}(e^t-1)^k=\sum_{n=k}^{\infty}S_2(n,k)\frac{t^n}{n!},\quad \text{(see [4, 5, 7, 8, 13, 14])}. \tag{1}$$

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It is well known that the Bell polynomials (also called Tochard polynomials or exponential polynomials) are defined by

$$e^{x(e^t-1)}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\quad \text{(see [1, 5, 9, 10, 12, 16])}. \tag{2}$$

From (1) and (2), we note that

$$\begin{aligned} B_n(x) &= e^{-x}\sum_{k=0}^{\infty}\frac{k^n}{k!}x^k \\ &= \sum_{k=0}^{n}x^k S_2(n,k),\quad (n\geq 0),\quad \text{(see [2, 4, 5])}. \end{aligned} \tag{3}$$

When $x=1$, $B_n=B_n(1)$ are called Bell numbers.

The (exponential) incomplete Bell polynomials (also called (exponential) partial Bell polynomials) are defined by the generating function

$$\frac{1}{k!}\Big(\sum_{m=1}^{\infty}x_m\frac{t^m}{m!}\Big)^k=\sum_{n=k}^{\infty}B_{n,k}(x_1,\cdots,x_{n-k+1})\frac{t^n}{n!},\quad (k\geq 0),\quad \text{(see [12, 16])}. \tag{4}$$

Thus, by (4), we get

$$\begin{aligned} B_{n,k}(x_1,\cdots,x_{n-k+1}) &= \sum\frac{n!}{i_1!i_2!\cdots i_{n-k+1}!}\Big(\frac{x_1}{1!}\Big)^{i_1}\Big(\frac{x_2}{2!}\Big)^{i_2}\times\cdots \\ &\quad\times\Big(\frac{x_{n-k+1}}{(n-k+1)!}\Big)^{i_{n-k+1}}, \end{aligned} \tag{5}$$

where the summation is over all integers $i_1,\cdots,i_{n-k+1}\geq 0$ such that $i_1+i_2+\cdots+i_{n-k+1}=k$ and $i_1+2i_2+\cdots+(n-k+1)i_{n-k+1}=n$.

From (1) and (4), we note that

$$B_{n,k}\underbrace{(1,1,\cdots,1)}_{n-k+1-times}=S_2(n,k),\quad (n,k\geq 0). \tag{6}$$

By (5), we easily get

$$B_{n,k}(\alpha x_1,\alpha x_2,\cdots,\alpha x_{n-k+1})=\alpha^k B_{n,k}(x_1,x_2,\cdots,x_{n-k+1}) \tag{7}$$

and

$$B_{n,k}(\alpha x_1,\alpha^2 x_2,\cdots,\alpha^{n-k+1}x_{n-k+1})=\alpha^n B_{n,k}(x_1,x_2,\cdots,x_{n-k+1}), \tag{8}$$

where $\alpha \in \mathbb{R}$ (see [12, 14]).

It is known that the central factorial numbers of the second kind are given by

$$\frac{1}{k!}\big(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\big)^k = \sum_{n=k}^{\infty} T(n,k)\frac{t^n}{n!}, \quad \text{(see [3, 9, 11])}, \tag{9}$$

where $k \geq 0$.

From (9), we can derive the following equation

$$T(n,k) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\big(j - \frac{k}{2}\big)^n, \tag{10}$$

where $n, k \in \mathbb{Z}$ with $n \geq k \geq 0$, (see [8, 9, 11]).

In [12], the central Bell polynomials $B_n^{(c)}(x)$ are defined by

$$B_n^{(c)}(x) = \sum_{k=0}^{n} T(n,k)x^k, \quad (n \geq 0). \tag{11}$$

When $x = 1$, $B_n^{(c)} = B_n^{(c)}(1)$ are called the central Bell numbers.

For $n \geq 0$, the Stirling numbers of the first kind are given by

$$\frac{1}{k!}\big(\log(1+t)\big)^k = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!}, \quad \text{(see } [1-16, 19]). \tag{12}$$

For $\lambda \in \mathbb{R}$, we define the degenerate exponential function as follows:

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}}, \quad \text{(see } [2, 14]). \tag{13}$$

Note that $\lim_{\lambda\to 0} e_\lambda^x(t) = \lim_{\lambda\to 0}(1+\lambda t)^{\frac{x}{\lambda}} = e^{xt}$. It is known that the degenerate Bell polynomials (also called degenerate Tochard polynomials or degenerate exponential polynomials) are defined by

$$e^{x\left(e_\lambda(t)-1\right)} = \sum_{n=0}^{\infty} B_{n,\lambda}(x)\frac{t^n}{n!}, \quad \text{(see } [15]), \tag{14}$$

where $e_\lambda(t) = e_\lambda^1(t)$.
When $x = 1$, $B_{n,\lambda} = B_{n,\lambda}(1)$ are called the degenerate Bell numbers.
The degenerate incomplete Bell polynomials (also called degenerate partial Bell polynomials) are defined by the generating function (see [9])

$$\frac{1}{k!}\Big(\sum_{m=1}^{\infty} x_m(1)_{m,\lambda}\frac{t^m}{m!}\Big)^k = \sum_{n=k}^{\infty} B_{n,k}(x_1, \cdots, x_{n-k+1}|\lambda)\frac{t^n}{n!}, \tag{15}$$

where k is a non-negative integer and $(x)_{m,\lambda}$ is the degenerate falling factorial sequence given by

$$(x)_{0,\lambda}=1,\quad (x)_{n,\lambda}=x(x-\lambda)\cdots\big(x-(n-1)\lambda\big),\quad (n\geq 1). \tag{16}$$

Now, we define the degenerate rising factorial sequence as follows:

$$<x>_{0,\lambda}=1,\quad <x>_{n,\lambda}=x(x+\lambda)\cdots\big(x+(n-1)\lambda\big),\quad (n\geq 1). \tag{17}$$

Note that $(-x)_{m,\lambda}=(-1)^m <x>_{m,\lambda}$.
From (15), we note that

$$\begin{aligned} B_{n,k}(x_1,x_2,\cdots,x_{n-k+1}|\lambda) &= \sum \frac{n!}{i_1!i_2!\cdots i_{n-k+1}!}\Big(\frac{x_1(1)_{1,\lambda}}{1!}\Big)^{i_1} \\ &\times \Big(\frac{x_2(1)_{2,\lambda}}{2!}\Big)^{i_2}\times\cdots\times\Big(\frac{x_{n-k+1}(1)_{n-k+1,\lambda}}{(n-k+1)!}\Big)^{i_{n-k+1}}, \end{aligned} \tag{18}$$

where the summation is over all integers $i_1,i_2,\cdots,i_{n-k+1}\geq 0$, such that $i_1+i_2+\cdots+i_{n-k+1}=k$ and $i_1+2i_2+\cdots+(n-k+1)i_{n-k+1}=n$.
It is known that the degenerate Stirling numbers of the second kind are defined by

$$\frac{1}{k!}\big(e_\lambda(t)-1\big)^k=\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!},\quad (k\geq 0),\quad (\text{see [8]}). \tag{19}$$

From (15) and (19), we note that

$$B_{n,k}(\underbrace{1,1,\cdots,1}_{n-k+1-times}|\lambda)=S_{2,\lambda}(n,k),\quad (n,k\geq 0). \tag{20}$$

By (18), we easily get

$$B_{n,k}(\alpha x_1,\alpha x_2,\cdots,\alpha x_{n-k+1}|\lambda)=\alpha^k B_{n,k}(x_1,x_2,\cdots,x_{n-k+1}|\lambda),$$

and

$$B_{n,k}(\alpha x_1,\alpha^2 x_2,\cdots,\alpha^{n-k+1}x_{n-k+1}|\lambda)=\alpha^n B_{n,k}(x_1,x_2,\cdots,x_{n-k+1}|\lambda),$$

where $\alpha\in\mathbb{R}$.
The degenerate complete Bell polynomials are defined by

$$\exp\Big(\sum_{i=1}^{\infty}x_i(1)_{i,\lambda}\frac{t^i}{i!}\Big)=\sum_{n=0}^{\infty}B_n(x_1,x_2,\cdots,x_n|\lambda)\frac{t^n}{n!}. \tag{21}$$

Then, by (15) and (21), we get

$$B_n(x_1,x_2,\cdots,x_n|\lambda)=\sum_{k=0}^{n}B_{n,k}(x_1,x_2,\cdots,x_{n-k+1}|\lambda). \tag{22}$$

From (22), we note that

$$\begin{aligned} B_n(x,x,\cdots,x|\lambda) &= \sum_{k=0}^{n} x^k B_{n,k}(1,1,\cdots,1|\lambda) \\ &= \sum_{k=0}^{n} x^k S_{2,\lambda}(n,k) = B_{n,\lambda}(x), \quad (n \geq 0). \end{aligned} \tag{23}$$

Recently, the degenerate central factorial numbers of the second kind are defined by

$$\frac{1}{k!}\big(e_\lambda^{\frac{1}{2}}(t) - e_\lambda^{-\frac{1}{2}}(t)\big)^k = \sum_{n=k}^{\infty} T_{2,\lambda}(n,k)\frac{t^n}{n!}, \quad (\text{see [11]}), \tag{24}$$

where k is a non-negative integer.
From (24), we have

$$T_{2,\lambda}(n,k) = \sum_{m=0}^{n}\Big(\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\big(l-\frac{k}{2}\big)^m\Big)\lambda^{n-m}S_1(n,m), \tag{25}$$

where $n,k \in \mathbb{Z}$ with $n \geq k \geq 0$, (see [11]).
The degenerate central Bell polynomials are given by

$$e^{x\left(e_\lambda^{\frac{1}{2}}(t) - e_\lambda^{-\frac{1}{2}}(t)\right)} = \sum_{n=0}^{\infty} B_{n,\lambda}^{(c)}(x)\frac{t^n}{n!}. \tag{26}$$

Thus, by (24) and (26), we get

$$B_{n,\lambda}^{(c)}(x) = \sum_{k=0}^{n} T_{2,\lambda}(n,k)x^k, \quad (n \geq 0). \tag{27}$$

When $x = 1$, $B_{n,\lambda}^{(c)} = B_{n,\lambda}^{(c)}(1)$ are called the degenerate central Bell numbers.

Recently, *central incomplete Bell polynomials* which are given by

$$\frac{1}{k!}\Big(\sum_{m=1}^{\infty}\frac{1}{2^m}(x_m - (-1)^m x_m)\frac{t^m}{m!}\Big)^k = \sum_{n=k}^{\infty} T_{n,k}(x_1,x_2,\cdots,x_{n-k+1})\frac{t^n}{n!}, \tag{28}$$

where $k = 0,1,2,3,\cdots$.

For $n,k \geq 0$ with $n-k \equiv 0 \pmod 2$, by (4) and (5), we get

$$\begin{aligned} T_{n,k}(x_1,x_2,\cdots,x_{n-k+1}) &= \sum \frac{n!}{i_1!i_2!\cdots i_{n-k+1}!}\Big(\frac{x_1}{1!}\Big)^{i_1}\Big(\frac{0}{2\cdot 2!}\Big)^{i_2} \\ &\times \Big(\frac{x_3}{2^2\cdot 3!}\Big)^{i_3}\cdots\Big(\frac{x_{n-k+1}}{2^{n-k}(n-k+1)!}\Big)^{i_{n-k+1}}, \end{aligned} \tag{29}$$

where the summation is over all integers $i_1, i_2, \cdots, i_{n-k+1} \geq 0$ such that $i_1 + \cdots + i_{n-k+1} = k$ and $i_1 + 2i_2 + \cdots + (n-k+1)i_{n-k+1} = n$.

From (5) and (29), we note that

$$T_{n,k}(x_1, x_2, \cdots, x_{n-k+1}) = B_{n,k}\big(x_1, 0, \frac{x_3}{2^2}, 0, \cdots, \frac{x_{n-k+1}}{2^{n-k}}\big), \tag{30}$$

where $n, k \geq 0$ with $n - k \equiv 0 \pmod 2$ and $n \geq k$.

In this article, we consider the degenerate central incomplete Bell polynomials $T_{n,k}(x_1, x_2, \cdots, x_{n-k+1}|\lambda)$ given by

$$\frac{1}{k!}\Big(\sum_{m=1}^{\infty} x_m\big((\frac{1}{2})_{m,\lambda} - (-1)^m < \frac{1}{2} >_{m,\lambda}\big)\frac{t^m}{m!}\Big)^k = \sum_{n=k}^{\infty} T_{n,k}(x_1, x_2, \cdots, x_{n-k+1}|\lambda)\frac{t^n}{n!}, \quad \text{(see (16), (17))},$$

and the degenerate central complete Bell polynomials $B_n^{(c)}(x_1, x_2, \cdots, x_n|\lambda)$ given by

$$\exp\Big(\sum_{i=1}^{\infty} x_i\big((\frac{1}{2})_{i,\lambda} - (-1)^i < \frac{1}{2} >_{i,\lambda}\big)\frac{t^i}{i!}\Big) = \sum_{n=0}^{\infty} B_n^{(c)}(x_1, x_2, \cdots, x_n|\lambda)\frac{t^n}{n!}.$$

and investigate some properties and identities for these polynomials.

Note that $\lim_{\lambda\to 0} T_{n,k}(x_1, x_2, \cdots, x_{n-k+1}|\lambda) = T_{n,k}(x_1, x_2, \cdots, x_{n-k+1})$ and $\lim_{\lambda\to 0} B_n^{(c)}(x_1, x_2, \cdots, x_n|\lambda) = B_n^{(c)}(x_1, x_2, \cdots, x_n)$.

So, they are degenerate versions of the central complete and incomplete Bell polynomials. They are also viewed as 'central' analogues for degenerate complete and incomplete Bell polynomials (see [9]), which are motivated by (15) and (21). Finally, we would like to mention the recent related papers [17,18] for the interested readers.

2. On degenerate central complete and incomplete Bell polynomials

In view of (15), we consider the degenerate central incomplete Bell polynomials given by

$$\frac{1}{k!}\Big(\sum_{m=1}^{\infty} x_m\big((\frac{1}{2})_{m,\lambda} - (-1)^m < \frac{1}{2} >_{m,\lambda}\big)\frac{t^m}{m!}\Big)^k = \sum_{n=k}^{\infty} T_{n,k}(x_1, x_2, \cdots, x_{n-k+1}|\lambda)\frac{t^n}{n!}, \tag{31}$$

where k is a non-negative integer.
For $n,k\geq 0$ with $n-k\equiv 0$ $(mod\ 2)$, by (31), we get

$$
\begin{aligned}
T_{n,k}(x_1,x_2,\cdots,x_{n-k+1}|\lambda) &= \sum \frac{n!}{i_1!i_2!\cdots i_{n-k+1}!} \\
&\times\Big(\frac{x_1\big((\frac{1}{2})_{1,\lambda}+<\frac{1}{2}>_{1,\lambda}\big)}{1!}\Big)^{i_1}\Big(\frac{x_2\big((\frac{1}{2})_{2,\lambda}-<\frac{1}{2}>_{2,\lambda}\big)}{2!}\Big)^{i_2}\times\cdots \\
&\times\Big(\frac{x_{n-k+1}\big((\frac{1}{2})_{n-k+1,\lambda}+<\frac{1}{2}>_{n-k+1,\lambda}\big)}{(n-k+1)!}\Big)^{i_{n-k+1}},
\end{aligned} \tag{32}
$$

where the summation is over all integers $i_1,i_2,\cdots,i_{n-k+1}\geq 0$ such that $i_1+i_2+\cdots+i_{n-k+1}=k$ and $i_1+2i_2+\cdots+(n-k+1)i_{n-k+1}=n$.
From (32), we can derive the following equation (33).
For $n,k\geq 0$ with $n-k\equiv 0$ $(mod\ 2)$, we have

$$
\begin{aligned}
&T_{n,k}(x_1,x_2,\cdots,x_{n-k+1}|\lambda) \\
&= B_{n,k}\Big(x_1\big((\frac{1}{2})_{1,\lambda}+<\frac{1}{2}>_{1,\lambda}\big),x_2\big((\frac{1}{2})_{2,\lambda}-<\frac{1}{2}>_{2,\lambda}\big),\cdots,x_{n-k+1} \\
&\times\big((\frac{1}{2})_{n-k+1,\lambda}+<\frac{1}{2}>_{n-k+1,\lambda}\big)\Big).
\end{aligned} \tag{33}
$$

Here $B_{n,k}(x_1,x_2,\cdots,x_{n-k+1})$ are the incomplete Bell polynomials which are defined by

$$
\begin{aligned}
B_{n,k}(x_1,x_2,\cdots,x_{n-k+1}) &= \sum\frac{n!}{i_1!i_2!\cdots i_{n-k+1}!}\Big(\frac{x_1}{1!}\Big)^{i_1} \\
&\times\Big(\frac{x_2}{2!}\Big)^{i_2}\times\cdots\times\Big(\frac{x_{n-k+1}}{(n-k+1)!}\Big)^{i_{n-k+1}},
\end{aligned} \tag{34}
$$

where the summation is over all integers $i_1,i_2,\cdots,i_{n-k+1}\geq 0$ such that $i_1+i_2+\cdots+i_{n-k+1}=k$ and $i_1+2i_2+\cdots+(n-k+1)i_{n-k+1}=n$.
Therefore, by (33) and (34), we obtain the following lemma.
Lemma 1. For $n,k\geq 0$ with $n\geq k$ and $n-k\equiv 0$ $(mod\ 2)$, we have

$$
\begin{aligned}
T_{n,k}(x_1,x_2,\cdots,x_{n-k+1}|\lambda) &= B_{n,k}\Big(x_1\big((\frac{1}{2})_{1,\lambda}+<\frac{1}{2}>_{1,\lambda}\big),x_2\big((\frac{1}{2})_{2,\lambda} \\
&-<\frac{1}{2}>_{2,\lambda}\big),\cdots,x_{n-k+1}\big((\frac{1}{2})_{n-k+1,\lambda}+<\frac{1}{2}>_{n-k+1,\lambda}\big)\Big).
\end{aligned}
$$

Let $n,k \geq 0$ with $n \geq k$ with $n-k \equiv 0$ $(mod\ 2)$. Then, by (31), we get

$$\begin{aligned}
&\sum_{n=k}^{\infty} T_{n,k}(x,x^2,\cdots,x^{n-k+1}|\lambda)\frac{t^n}{n!} \\
&= \frac{1}{k!}\Big(x\big((\tfrac{1}{2})_{1,\lambda}+<\tfrac{1}{2}>_{1,\lambda}\big)t + x^2\big((\tfrac{1}{2})_{2,\lambda}-<\tfrac{1}{2}>_{2,\lambda}\big)\frac{t^2}{2!}+\cdots\Big)^k \\
&= \frac{1}{k!}\big(e_\lambda^{\frac{1}{2}}(xt)-e_\lambda^{-\frac{1}{2}}(xt)\big)^k = \frac{1}{k!}e_\lambda^{-\frac{k}{2}}(xt)\big(e_\lambda(xt)-1\big)^k \qquad (35)\\
&= \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}e_\lambda^{(l-\frac{k}{2})}(xt) = \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l} \\
&\times e^{\frac{1}{\lambda}(l-\frac{k}{2})\log(1+\lambda xt)} \\
&= \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\sum_{m=0}^{\infty}\lambda^{-m}\big(l-\frac{k}{2}\big)^m\frac{1}{m!}\big(\log(1+\lambda xt)\big)^m \\
&= \sum_{n=0}^{\infty}\Bigg(\sum_{m=0}^{n}\Big(\frac{x^n}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\big(l-\frac{k}{2}\big)^m\Big)\lambda^{n-m}S_1(n,m)\Bigg)\frac{t^n}{n!}.
\end{aligned}$$

Therefore, by comparing the coefficients on both sides of (35), we obtain the following theorem.

Theorem 2. For $n,k \geq 0$ with $n-k \equiv 0$ $(mod\ 2)$, we have

$$\begin{aligned}
&\sum_{m=0}^{n}\Big(\frac{x^n}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\big(l-\frac{k}{2}\big)^m\Big)\lambda^{n-m}S_1(n,m) \\
&\qquad = \begin{cases} T_{n,k}(x,x^2,\cdots,x^{n-k+1}|\lambda), & \text{if} \quad n \geq k, \\ 0, & \text{if} \quad n<k. \end{cases}
\end{aligned}$$

In particular,

$$\begin{aligned}
&\sum_{m=0}^{n}\Big(\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\big(l-\frac{k}{2}\big)^m\Big)\lambda^{n-m}S_1(n,m) \\
&\qquad = \begin{cases} T_{n,k}(1,1,\cdots,1|\lambda), & \text{if} \quad n \geq k, \\ 0, & \text{if} \quad n<k. \end{cases}
\end{aligned}$$

For $n,k \geq 0$ with $n-k \equiv 0$ $(mod\ 2)$ and $n \geq k$, by (24) and (31), we get

$$T_{n,k}(1,1,\cdots,1|\lambda) = T_{2,\lambda}(n,k). \qquad (36)$$

Therefore, by (36), we obtain the following corollary.

Corollary 3. For $n,k \geq 0$ with $n-k \equiv 0$ $(mod\ 2)$ and $n \geq k$, we have

$$T_{n,k}(x,x^2,\cdots,x^{n-k+1}|\lambda) = x^nT_{n,k}(1,1,\cdots,1|\lambda) = x^nT_{2,\lambda}(n,k),$$

and

$$T_{2,\lambda}(n,k) = T_{n,k}(1,1,\cdots,1|\lambda) = \sum \frac{n!}{i_1!i_2!\cdots i_{n-k+1}!}$$
$$\times \Big(\frac{(\frac{1}{2})_{1,\lambda} + <\frac{1}{2}>_{1,\lambda}}{1!}\Big)^{i_1}\Big(\frac{(\frac{1}{2})_{2,\lambda} - <\frac{1}{2}>_{2,\lambda}}{2!}\Big)^{i_2} \times \cdots$$
$$\times \Big(\frac{(\frac{1}{2})_{n-k+1,\lambda} + <\frac{1}{2}>_{n-k+1,\lambda}}{(n-k+1)!}\Big)^{i_{n-k+1}},$$

where the summation is over all integers $i_1, i_2, \cdots, i_{n-k+1} \geq 0$ such that $i_1 + i_2 + \cdots + i_{n-k+1} = k$ and $i_1 + 2i_2 + \cdots + (n-k+1)i_{n-k+1} = n$.

For $n, k \geq 0$ with $n \geq k$ and $n - k \equiv 0$ $(mod\ 2)$, we note from (35) that

$$\sum_{n=k}^{\infty} T_{n,k}(x,0,0,\cdots,0|\lambda)\frac{t^n}{n!} = \frac{1}{k!}(xt)^k. \tag{37}$$

Thus, by (37), we get

$$T_{n,k}(x,0,0,\cdots,0|\lambda) = x^k \binom{0}{n-k}.$$

From (32), we have

$$T_{n,k}(x,x,\cdots,x|\lambda) = x^k T_{n,k}(1,1,\cdots,1|\lambda), \tag{38}$$

and

$$T_{n,k}(\alpha x_1, \alpha x_2, \cdots, \alpha x_{n-k+1}|\lambda) = \alpha^k T_{n,k}(x_1,x_2,\cdots,x_{n-k+1}|\lambda), \tag{39}$$

where $n, k \geq 0$ with $n - k \equiv 0$ $(mod\ 2)$ and $n \geq k$.
Now, we observe that

$$\begin{aligned}
&\exp\Big(\sum_{i=1}^{\infty} x_i\big((\frac{1}{2})_{i,\lambda} - (-1)^i <\frac{1}{2}>_{i,\lambda}\big)\frac{t^i}{i!}\Big) \\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\Big(\sum_{i=1}^{\infty} x_i\big((\frac{1}{2})_{i,\lambda} - (-1)^i <\frac{1}{2}>_{i,\lambda}\big)\frac{t^i}{i!}\Big)^k \\
&= 1 + \sum_{k=1}^{\infty}\frac{1}{k!}\Big(\sum_{i=1}^{\infty} x_i\big((\frac{1}{2})_{i,\lambda} - (-1)^i <\frac{1}{2}>_{i,\lambda}\big)\frac{t^i}{i!}\Big)^k \qquad (40)\\
&= 1 + \sum_{k=1}^{\infty}\sum_{n=k}^{\infty} T_{n,k}(x_1,x_2,\cdots,x_{n-k+1}|\lambda)\frac{t^n}{n!} \\
&= 1 + \sum_{n=1}^{\infty}\Big(\sum_{k=1}^{n} T_{n,k}(x_1,x_2,\cdots,x_{n-k+1}|\lambda)\Big)\frac{t^n}{n!}.
\end{aligned}$$

In view of (21), we define the degenerate central complete Bell polynomials by

$$\exp\Big(\sum_{i=1}^{\infty} x_i\big((\frac{1}{2})_{i,\lambda}-(-1)^i<\frac{1}{2}>_{i,\lambda}\big)\frac{t^i}{i!}\Big) = \sum_{n=0}^{\infty} B_n^{(c)}(x_1,x_2,\cdots,x_n|\lambda)\frac{t^n}{n!}. \tag{41}$$

From (40) and (41), we have

$$B_n^{(c)}(x_1,x_2,\cdots,x_n|\lambda)=\sum_{k=0}^{n} T_{n,k}(x_1,x_2,\cdots,x_{n-k+1}|\lambda), \tag{42}$$

By (36) and (42), we get

$$\begin{aligned} B_n^{(c)}(x,x,\cdots,x|\lambda) &= \sum_{k=0}^{n} T_{n,k}(\underbrace{x,x,\cdots,x}_{n-k+1-times}|\lambda) \\ &= \sum_{k=0}^{n} x^k T_{n,k}(1,1,\cdots,1|\lambda)=\sum_{k=0}^{n} x^k T_{2,\lambda}(n,k) \\ &= B_{n,\lambda}^{(c)}(x). \end{aligned} \tag{43}$$

From (40), we note that

$$\begin{aligned} &\exp\Big(\sum_{i=1}^{\infty} x_i\big((\frac{1}{2})_{i,\lambda}-(-1)^i<\frac{1}{2}>_{i,\lambda}\big)\frac{t^i}{i!}\Big) \\ &= 1+\sum_{n=1}^{\infty}\frac{1}{n!}\Big(\sum_{i=1}^{\infty} x_i\big((\frac{1}{2})_{i,\lambda}-(-1)^i<\frac{1}{2}>_{i,\lambda}\big)\frac{t^i}{i!}\Big)^n \\ &= 1+\frac{1}{1!}\sum_{i=1}^{\infty} x_i\big((\frac{1}{2})_{i,\lambda}-(-1)^i<\frac{1}{2}>_{i,\lambda}\big)\frac{t^i}{i!} \\ &+\frac{1}{2!}\Big(\sum_{i=1}^{\infty} x_i\big((\frac{1}{2})_{i,\lambda}-(-1)^i<\frac{1}{2}>_{i,\lambda}\big)\frac{t^i}{i!}\Big)^2 \\ &+\frac{1}{3!}\Big(\sum_{i=1}^{\infty} x_i\big((\frac{1}{2})_{i,\lambda}-(-1)^i<\frac{1}{2}>_{i,\lambda}\big)\frac{t^i}{i!}\Big)^3+\cdots \\ &= \sum_{n=0}^{\infty}\Bigg(\sum_{m_1+2m_2+\cdots+nm_n=n}\frac{n!}{m_1!m_2!\cdots m_n!} \\ &\times\Big(\frac{x_1\big((\frac{1}{2})_{1,\lambda}+<\frac{1}{2}>_{1,\lambda}\big)}{1!}\Big)^{m_1}\Big(\frac{x_2\big((\frac{1}{2})_{2,\lambda}-<\frac{1}{2}>_{2,\lambda}\big)}{2!}\Big)^{m_2}\times\cdots \\ &\times\Big(\frac{x_n\big((\frac{1}{2})_{n,\lambda}-(-1)^n<\frac{1}{2}>_{n,\lambda}\big)}{n!}\Big)^{m_n}\Bigg)\frac{t^n}{n!}, \end{aligned} \tag{44}$$

where the sum is over all nonnegative integers $m_1, m_2, \cdots, m_n$ such that $m_1 + 2m_2 + \cdots + nm_n = n$.

Now, for $n \in \mathbb{N}$ with $n \equiv 1$ $(mod\ 2)$, by (41) and (44), we get

(45)
$$\begin{aligned} B_n^{(c)}(x_1, x_2, \cdots, x_n|\lambda) &= \sum_{k=1}^{n} T_{n,k}(x_1, x_2, \cdots, x_{n-k+1}|\lambda) \\ &= \sum_{m_1+2m_2+\cdots+nm_n=n} \frac{n!}{m_1!m_2!\cdots m_n!} \\ &\times \Big(\frac{x_1\big((\frac{1}{2})_{1,\lambda} + <\frac{1}{2}>_{1,\lambda}\big)}{1!}\Big)^{m_1} \Big(\frac{x_2\big((\frac{1}{2})_{2,\lambda} - <\frac{1}{2}>_{2,\lambda}\big)}{2!}\Big)^{m_2} \times \cdots \\ &\times \Big(\frac{x_n\big((\frac{1}{2})_{n,\lambda} + <\frac{1}{2}>_{n,\lambda}\big)}{n!}\Big)^{m_n}. \end{aligned}$$

Therefore, by (45), we obtain the following theorem.
Theorem 4. For $n \in \mathbb{N}$ with $n \equiv 1$ $(mod\ 2)$, we have

$$\begin{aligned} B_n^{(c)}(x_1, x_2, \cdots, x_n|\lambda) &= \sum_{m_1+2m_2+\cdots+nm_n=n} \frac{n!}{m_1!m_2!\cdots m_n!} \\ &\times \Big(\frac{x_1\big((\frac{1}{2})_{1,\lambda} + <\frac{1}{2}>_{1,\lambda}\big)}{1!}\Big)^{m_1} \Big(\frac{x_2\big((\frac{1}{2})_{2,\lambda} - <\frac{1}{2}>_{2,\lambda}\big)}{2!}\Big)^{m_2} \times \cdots \\ &\times \Big(\frac{x_n\big((\frac{1}{2})_{n,\lambda} + <\frac{1}{2}>_{n,\lambda}\big)}{n!}\Big)^{m_n}, \end{aligned}$$

where the sum is over all nonnegative integers $m_1, m_2, \cdots, m_n$ such that $m_1 + 2m_2 + \cdots + nm_n = n$.

We observe that

(46)
$$\begin{aligned} &\exp\Big(x \sum_{i=1}^{\infty} \big((\frac{1}{2})_{i,\lambda} - (-1)^i <\frac{1}{2}>_{i,\lambda}\big)\frac{t^i}{i!}\Big) \\ &= 1 + \sum_{k=1}^{\infty} \frac{x^k}{k!}\Big(\sum_{i=1}^{\infty} \big((\frac{1}{2})_{i,\lambda} - (-1)^i <\frac{1}{2}>_{i,\lambda}\big)\frac{t^i}{i!}\Big)^k \\ &= 1 + \sum_{k=1}^{\infty} x^k \sum_{n=k}^{\infty} T_{n,k}(1, 1, \cdots, 1|\lambda)\frac{t^n}{n!} \\ &= 1 + \sum_{n=1}^{\infty} \big(\sum_{k=1}^{n} x^k T_{n,k}(1, 1, \cdots, 1|\lambda)\big)\frac{t^n}{n!}. \end{aligned}$$

On the other hand,

$$\exp\Big(x\sum_{i=1}^{\infty}\big((\frac{1}{2})_{i,\lambda}-(-1)^i<\frac{1}{2}>_{i,\lambda}\big)\frac{t^i}{i!}\Big)=e^{x\big(e_{\lambda}^{\frac{1}{2}}(t)-e_{\lambda}^{-\frac{1}{2}}(t)\big)}$$
$$=\sum_{n=0}^{\infty}B_{n,\lambda}^{(c)}(x)\frac{t^n}{n!}. \tag{47}$$

Therefore, by (46) and (47), we obtain the following theorem.

Theorem 5. For $n, k \geq 0$ with $n \geq k$, we have

$$\sum_{k=0}^{n}x^kT_{n,k}(1,1,\cdots,1|\lambda)=B_{n,\lambda}^{(c)}(x).$$

By **Theorem 5,** we easily get

$$\sum_{k=0}^{n}x^kT_{n,k}(1,1,\cdots,1|\lambda)=\sum_{k=0}^{n}T_{n,k}(x,x,\cdots,x|\lambda)=B_n^{(c)}(x,x,\cdots,x|\lambda). \tag{48}$$

Corollary 6. For $n \geq 0$, we have

$$B_n^{(c)}(x,x,\cdots,x|\lambda)=B_{n,\lambda}^{(c)}(x).$$

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