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MEIR-KEELER TYPE AND CARISTI TYPE FIXED POINT THEOREMS

Dedicated to Academician Professor Gradimir Milovanović
on the occasion of his 70th birthday.

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Agarwal et al [**1**] have proved some interesting local and global fixed point theorems for Meir-Keeler [**7**] type and Caristi [**2**] type maps. We obtain analogues of the main results of Agarwal et al [**1**] under weaker conditions so as to include continuous as well as discontinuous maps. Our results provide new answers to Rhoades' problem ([**15**], p. 242) on existence of contractive definitions which admit discontinuity at the fixed point. Several examples are given to illustrate our results.

1. INTRODUCTION

Agarwal et al [**1**] have proved some interesting local and global fixed point theorems for Meir-Keeler [**7**] type and Caristi [**2**] type maps. We obtain analogues of the main results of Agarwal et al [**1**] under weaker conditions so as to include continuous as well as discontinuous maps. Our results provide new answers to Rhoades' problem ([**15**], p. 242) on existence of contractive definitions which admit discontinuity at the fixed point. Let us point out that in 1999, Pant [**13**] proved the following fixed point theorem and got the first result for the contractive map which is discontinuous at the fixed point.

Theorem 1.1. *Let f be a self-mapping of a complete metric space (X, d) such that for all x, y in X we have*

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(a) Given $\epsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$\epsilon < \max\{d(x, fx), d(y, fy)\} < \epsilon + \delta \implies d(fx, fy) \le \epsilon,$$

(b) $d(fx, fy) \le \phi(\max\{d(x, fx), d(y, fy)\}), \phi : \mathbb{R}_+ \to \mathbb{R}_+$ *is such that* $\phi(t) < t$ *for each* $t > 0$.

Then f *has a unique fixed point, say* z. *Moreover,* f *is continuous at* z *if and only if* $\lim_{x \to z} \max\{d(x, fx), d(y, fy)\} = 0$.

Fixed point theorems for discontinuous mappings have found wide applications, for example application of such theorems in the study of neural networks with discontinuous activation functions is a very active area of research (e. g. Ding et al [**5**], Forti and Nistri [**6**], Nie and Zheng [**8, 9, 10**], Wu and Shan [**16**]). Recently Ozgur and Tas [**11, 12**] have obtained application of the results on discontinuity at the fixed point in neural networks with discontinuous activation functions.

In 1971 Ćirić [**3**] (see also [**4**]) introduced the notion of orbital continuity. If f is a self-mapping of a metric space (X, d) then the set $O(x, f) = \{f^n x : n = 0, 1, 2, \ldots\}$ is called the orbit of f at x and f is called orbitally continuous if $u = \lim_i f^{m_i} x$ implies $fu = \lim_i f f^{m_i} x$.
Continuity of f obviously implies orbital continuity but not conversely [**3**]. The following definition gives another weaker form of continuity.

Definition 1.2 ([**14**]). *A self-mapping* f *of a metric space* X *is called* k*-continuous,* $k = 1, 2, 3, \ldots$, *if* $f^k x_n \to ft$ *whenever* $\{x_n\}$ *is a sequence in* X *such that* $f^{k-1} x_n \to t$.

Example 1.3. *Let* $X = [0, 2]$ *equipped with the usual metric and* $f : X \to X$ *be defined by*

$$fx = 1 \text{ if } 0 \le x \le 1, \qquad fx = 0 \text{ if } x > 1.$$

Then $fx_n \to t \Rightarrow f^2 x_n \to t$ *since* $fx_n \to t$ *implies* $t = 0$ *or* $t = 1$ *and* $f^2 x_n = 1$ *for all* n, *that is,* $f^2 x_n \to 1 = ft$. *Hence* f *is* 2*-continuous. However* f *is discontinuous at* $x = 1$.

Example 1.4. *Let* $X = [0, 4]$ *equipped with the usual metric. Define* $f : X \to X$ *by*

$$fx = 1 \text{ if } 0 \le x \le 1, \qquad fx = 0 \text{ if } 1 < x \le 3, \qquad fx = \frac{x}{3} \text{ if } 3 < x \le 4.$$

Then $f^2 x_n \to t \Rightarrow f^3 x_n \to ft$ *since* $f^2 x_n \to t$ *implies* $t = 0$ *or* $t = 1$ *and* $f^3 x_n = 1 = ft$ *for each* n. *Hence* f *is* 3*-continuous. However,* $fx_n \to t$ *does not imply* $f^2 x_n \to ft$, *that is,* f *is not* 2*-continuous.*

Example 1.5. *Let $X = [0, 2]$ and d be the usual metric. Define $f : X \to X$ by*

$$fx = \frac{(1+x)}{2} \text{ if } 0 \le x \le 1, \qquad fx = 0 \text{ if } x > 1.$$

Then it can be verified that f is 2-continuous but not continuous. It is also easy to see that f^k is discontinuous for each positive integer k. Thus 2-continuity of f does not imply continuity of f^2. In general, k-continuity of f does not imply continuity of f^k.

Example 1.6. *Let $X = [0, 3] \cup (4, 5)$ equipped with usual metric and let $f : X \to X$ be defined by*

$$fx = 1 \text{ if } 0 \le x \le 1, \qquad fx = 0 \text{ if } 1 < x \le 3, \qquad fx = \frac{x}{4} \text{ if } 4 < x < 5.$$

Then f^2 is continuous but f is not 2-continuous. If we consider the sequence $\{x_n\}$ given by $x_n = 4 + \frac{1}{n}$ then $fx_n \to 1$ but $f^2 x_n \to 0 \ne f1$. Hence f is not 2-continuous.

The above examples show that continuity of f^k and k-continuity of f are independent conditions when $k > 1$. It is easy to see that 1-continuity is equivalent to continuity and

$$\text{continuity } \Rightarrow 2-\text{continuity } \Rightarrow 3-\text{continuity } \Rightarrow \ldots, \text{ but not conversely.}$$

2. RESULTS

Let (X, d) be a complete metric space, $x_0 \in X$ and $r > 0$. Define $B(x_0, r) = \{x \in X : d(x, x_0) < r\}$ and let $\overline{B(x_0, r)}$ denote the closure of $B(x_0, r)$. If $f : \overline{B(x_0, r)} \to X$ is a map and $x, y \in \overline{B(x_0, r)}$, let us denote $M(x, y) = \max\{d(x, fx), d(y, fy)\}$.

Theorem 2.7. *Let (X, d) be a complete metric space, $x_0 \in X$ and $r > 0$. Suppose $f : \overline{B(x_0, r)} \to X$ is a map such that*

(i) $d(fx, fy) < M(x, y)$ *whenever* $M(x, y) > 0$,

(ii) given $\epsilon > 0$ *there exists a* $\delta > 0$ *such that* $M(x, y) < \epsilon + \delta \Rightarrow d(fx, fy) \le \epsilon$,

(iii) $d(x_0, f^n x_0) < r, n = 1, 2, \ldots$.

If f^k is continuous or if f is k-continuous for some $k \ge 1$ or if f is orbitally continuous then f possesses a unique fixed point, say, $t \in \overline{B(x_0, r)}$. Moreover, f is continuous at the fixed point t if and only if $\lim_{x \to t} M(x, t) = 0$.

Proof. We observe that, under condition (i), condition (ii) is equivalent to

$$\epsilon < M(x, y) < \epsilon + \delta \Rightarrow d(fx, fy) \le \epsilon. \tag{2.1}$$

If (ii) is satisfied then condition (2.1) is obviously satisfied. On the other hand suppose (i) and (2.1) are satisfied. If $0 < M(x,y) \leq \epsilon$ then by (i) we get $d(fx, fy) < M(x,y) \leq \epsilon$; and if $\epsilon < M(x,y) < \epsilon + \delta$ then (2.1) implies $d(fx, fy) \leq \epsilon$. Thus $M(x,y) < \epsilon + \delta \Rightarrow d(fx, fy) \leq \epsilon$ and (ii) is satisfied.

Define a sequence $\{x_n\}$ by $x_1 = fx_0, x_2 = fx_1, \ldots, x_n = fx_{n-1}, \ldots$ Then (iii) implies that $d(x_1, x_0) < r, d(x_n, x_0) = d(f^n x_0, x_0) < r$, that is, $x_n \in B(x_0, r), n = 1, 2, 3, \ldots$. Now

$$\begin{aligned} d(x_n, x_{n+1}) &= d(fx_{n-1}, fx_n) < \max\{d(x_{n-1}, fx_{n-1}), d(x_n, fx_n)\} \\ &= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n). \end{aligned}$$

Thus $\{d(x_n, x_{n+1})\}$ is a strictly decreasing sequence of positive real numbers and, hence, tends to a limit $r \geq 0$. Suppose $r > 0$. Then there exists a positive integer N such that

$$n \geq N \Rightarrow r < d(x_n, x_{n+1}) \leq r + \delta(r). \tag{2.2}$$

This yields $r < \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} = \max\{d(x_n, fx_n), d(x_{n+1}, fx_{n+1})\} \leq r + \delta(r)$ which by virtue of (ii) yields $d(fx_n, fx_{n+1}) = d(x_{n+1}, x_{n+2}) \leq r$. This contradicts (2.2). Hence $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. Now if p is any positive integer then

$$\begin{aligned} d(x_n, x_{n+p}) &= d(fx_{n-1}, fx_{n+p-1}) \\ &< \max\{d(x_{n-1}, fx_{n-1}), d(x_{n+p-1}, fx_{n+p-1})\} \\ &= \max\{d(x_{n-1}, x_n), d(x_{n+p-1}, x_{n+p})\} = d(x_{n-1}, x_n). \end{aligned}$$

This implies that $d(x_n, x_{n+p}) \to 0$ since $d(x_{n-1}, x_n) \to 0$. Therefore, $\{x_n\}$ is a Cauchy sequence. Since X is complete and $x_n \in B(x_0, r)$, there exists t in $\overline{B(x_0, r)}$ such that $x_n \to t$ and $f^k x_n \to t$ for each $k \geq 1$.

Suppose that f^k is continuous for some positive integer k. Then, $\lim_{n\to\infty} f^k x_n = f^k t$. This yields $f^k t = t$ as $f^k x_n \to t$. If $t \neq ft$, using (i) we get

$$\begin{aligned} d(t, ft) &= d(f^k t, f^{k+1} t) < \max\{d(f^{k-1} t, f^k t), d(f^k t, f^{k+1} t)\} \\ &= d(f^{k-1} t, f^k t) < d(f^{k-2} t, f^{k-1} t) < \ldots < d(t, ft), \end{aligned}$$

a contradiction. Hence $t = ft$ and t is a fixed point of f.

Next suppose that f is k-continuous. Since $f^{k-1} x_n \to t$, k-continuity of f implies that $f^k x_n \to ft$. Hence $t = ft$ as $f^k x_n \to t$. Therefore, t is fixed point of f. Finally, suppose that f is orbitally continuous. Since $x_n \to t$, orbital continuity implies that $fx_n \to ft$. This gives $t = ft$ as $fx_n \to t$. Thus t is a fixed point of f. The remaining part of the theorem follows easily. □

Theorem 2.8. *Let (X, d) be a complete metric space, $x_0 \in X$ and $r > 0$. Suppose $f : \overline{B(x_0, r)} \to X$ is a map such that* *(iv)* $d(fx, fy) \leq \phi(M(x, y))$ *for all* x, y *in* $\overline{B(x_0, r)}$*, where the function* $\phi : R_+ \to R_+$ *is such that* $\phi(t) < t$ *for each* $t > 0$*.*

(v) *given* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that* $M(x, y) < \epsilon + \delta \Rightarrow d(fx, fy) \leq \epsilon$*.*

(vi) $d(x_0, f^n x_0) < r, n = 1, 2, \ldots$

Then f *has a unique fixed point, say,* t*. Moreover,* f *is continuous at the fixed point* t *if and only if* $\lim_{x \to t} M(x, t) = 0$*.*

Proof. Condition (iv) implies condition (i) and, hence, under condition (iv) condition (v) is equivalent to (2.1). As in Theorem 2.7, define a sequence $\{x_n\}$ by $x_1 = fx_0, x_2 = fx_1, \ldots, x_n = fx_{n-1}, \ldots$. Then, as in Theorem 2.7, it follows that $\{x_n\}$ is a Cauchy sequence. Completeness of X implies that there exists t in $\overline{B(x_0, r)}$ such that $x_n \to t$ and $fx_n \to t$. We assert that $t = ft$. If not, then using (iv) we get

$$d(fx_n, ft) \leq \phi(\max\{d(x_n, fx_n), d(t, ft)\}.$$

On taking limit as $n \to \infty$, this yields $d(t, ft) \leq \phi(d(t, ft)) < d(t, ft)$, a contradiction. Hence $t = ft$ and t is a fixed point of f . Uniqueness of the fixed point follows from (iv). □

Example 2.9. *Let* $X = [0, 2], x_0 = \frac{3}{4}$ *and* $r = \frac{1}{2}$*. Define* $f : \overline{B(x_0, r)} \to X$ *by*

$$fx = 1 \text{ if } \frac{1}{4} \leq x \leq 1, \qquad fx = 0 \text{ if } 1 < x \leq \frac{5}{4}.$$

Then f *satisfies all the conditions of Theorems 2.7 and 2.8 and has a unique fixed point* $x = 1$ *at which* f *is discontinuous. It may be seen in this example that* f *satisfies conditions (ii) and (v) with* $\delta(\epsilon) = 1$ *when* $\epsilon \geq 1$ *and* $\delta(\epsilon) = 1 - \epsilon$ *when* $\epsilon < 1$*. The mapping* f *satisfies condition (iv) with* $\phi(t) = \frac{1+t}{2}$ *if* $t > 1$ *and* $\phi(t) = \frac{t}{2}$ *if* $t \leq 1$*. It can also be verified that* $\lim_{x \to 1} M(x, 1) = \lim_{x \to 1} \max\{d(x, fx), d(1, f1)\} = \lim_{x \to 1} d(x, fx)$ *does not exist since* $fx = 1$ *if* $x \leq 1$ *while* $fx = 0$ *when* $x > 1$*.*

Example 2.10. *Let* $X = [0, 2], x_0 = \frac{4}{5}$ *and* $r = \frac{3}{5}$*. Define* $f : \overline{B(x_0, r)} \to X$ *by*

$$fx = 1 \text{ if } \frac{1}{5} \leq x \leq \frac{6}{5}, \qquad fx = 0 \text{ if } \frac{6}{5} < x \leq \frac{7}{5}$$

Then f *satisfies all the conditions of Theorems 2.7 and 2.8 and has a unique fixed point* $x = 1$ *at which* f *is continuous. It is easy to verify in this example that* $\lim_{x \to 1} M(x, 1) = 0$*.*

The next two theorems respectively give the global versions of Theorems 2.7 and 2.8.

Theorem 2.11. *Let* f *be a self-mapping of a complete metric space* (X, d) *such that*

(vii) $d(fx, fy) < M(x, y)$ *whenever* $M(x, y) > 0$*,*

(viii) given $\epsilon > 0$ there exists a $\delta > 0$ such that $M(x, y) < \epsilon + \delta \Rightarrow d(fx, fy) \leq \epsilon$.

If f^k is continuous or if f is k-continuous for some $k \geq 1$ then f possesses a unique fixed point, say, t. Moreover, f is continuous at the fixed point t if and only if $\lim_{x \to t} M(x, t) = 0$.

Theorem 2.12. *Let f be a self-mapping of a complete metric space (X, d) such that for all x, y in X*

(ix) $d(fx, fy) \leq \phi(M(x, y))$, where the function $\phi : R_+ \to R_+$ is such that $\phi(t) < t$ for each $t > 0$.

(x) given $\epsilon > 0$ there exists a $\delta > 0$ such that $M(x, y) < \epsilon + \delta \Rightarrow d(fx, fy) \leq \epsilon$.

Then f has a unique fixed point, say, t. Moreover, f is continuous at the fixed point t if and only if $\lim_{x \to t} M(x, t) = 0$.

Remark 2.13. *Theorem 2.7 proved above generalizes Theorem 2.1 of Agarwal et al [**1**] since our theorem applies to discontinuous mappings as well. Theorem 2.7 provides a new solution to the question of existence of contractive mappings which admit discontinuity at the fixed point [[**15**], p. 242]. The Meir Keeler type contractive condition (2.1) of Agarwal et al [**1**] implies conditions (i) and (ii) of Theorem 2.7 but not conversely. For example the mapping f in Example 2.9 satisfies conditions (i) and (ii) of Theorem 2.7 above but not condition (2.7) of Agarwal et al [**1**].*

The next theorem gives an analogue of Theorem 2.7 of Agarwal et al [**1**] under weaker continuity conditions which are either equivalent to or imply condition (2.8) of Agarwal et al. As a corollary of the next theorem we shall obtain a new solution, in the form of a Caristi type fixed point theorem, to the question of existence of contractive mappings which admit discontinuity at the fixed point.

Theorem 2.14. *Let (X, d) be a complete metric space, $x_0 \in X, r > 0$ and $f : \overline{B(x_0, r)} \to X$. Suppose there exists a function $\phi : X \to [0, \infty)$ such that for each $x \in \overline{B(x_0, r)}$ we have*

(xi) $d(x, fx) \leq \phi(x) - \phi(fx)$

(xii) $\phi(x_0) < r$.

If f is orbitally continuous or if f^k is continuous or if f is k-continuous for some $k \geq 1$ then f possesses a fixed point, say $t \in \overline{B(x_0, r)}$.

Proof. Define a sequence $\{x_n\}$ by $x_1 = fx_0, x_2 = fx_1, \ldots, x_n = fx_{n-1}, \ldots$. Then

$$d(x_0, x_1) = d(x_0, fx_0) \leq \phi(x_0) - \phi(fx_0) \leq \phi(x_0) < r,$$

and

$$\begin{aligned} d(x_0,x_2) &\le d(x_0,x_1)+d(x_1,x_2) \\ &= d(x_0,fx_0)+d(x_1,fx_1) \\ &\le \phi(x_0)-\phi(fx_0)+\phi(x_1)-\phi(fx_1) \\ &= \phi(x_0)-\phi(x_2)\le \phi(x_0)<r. \end{aligned}$$

Similarly for each n we get

$$\begin{aligned} d(x_0,x_n) &\le d(x_0,x_1)+d(x_1,x_2)+\ldots+d(x_{n-1},x_n) \\ &= d(x_0,fx_0)+d(x_1,fx_1)+\ldots+d(x_{n-1},fx_{n-1}) \\ &\le \phi(x_0)-\phi(fx_0)+\phi(x_1)-\phi(fx_1)+\ldots+\phi(x_{n-1})-\phi(fx_{n-1}) \\ &= \phi(x_0)-\phi(x_n)\le \phi(x_0)<r. \end{aligned}$$

This shows that $d(x_0,x_n)\le \phi(x_0)-\phi(x_n)$ and $x_n\in \overline{B(x_0,r)}$ for each $n\ge 1$. Also, for each $n\ge 1$

$$\begin{aligned} d(x_0,x_1)+d(x_1,x_2)+ \ldots &+d(x_{n-1},x_n) = d(x_0,fx_0)+d(x_1,fx_1)+\ldots+d(x_{n-1},fx_{n-1}) \\ &\le \phi(x_0)-\phi(fx_0)+\phi(x_1)-\phi(fx_1)+\ldots+\phi(x_{n-1})-\phi(fx_{n-1}) \\ &= \phi(x_0)-\phi(x_n)\le \phi(x_0)<r. \end{aligned}$$

This implies that $\Sigma_{n=0}^{\infty} d(x_n,x_{n+1})<r$, that is, $\{x_n\}$ is a Cauchy sequence. Since X is complete and $x_n\in B(x_0,r)$ for each n, there exists t in $\overline{B(x_0,r)}$ such that $x_n\to t$. Moreover, for each $k\ge 1$ we get $fx_n\to t$ and $f^kx_n\to t$.

Suppose that f^k is continuous for some positive integer k. Then, $\lim_{n\to\infty} f^kx_n = f^kt$. This yields $f^kt=t$ as $f^kx_n\to t$. If $t\ne ft$, using (i) we get

$$\begin{aligned} d(t,ft)=d(ft,f^kt) &\le d(ft,f^2t)+d(f^2t,f^3t)+\ldots+d(f^{k-1}t,f^kt) \\ &\le \phi(ft)-\phi(f^2t)+\phi(f^2t)-\phi(f^3t)+\ldots+\phi(f^{k-1}t)-\phi(f^kt) \\ &= \phi(ft)-\phi(f^kt)=\phi(ft)-\phi(t), \end{aligned}$$

a contradiction since $d(t,ft)\le\phi(t)-\phi(ft)$. Hence $t=ft$ and t is a fixed point of f.

Next suppose that f is k-continuous. Since $f^{k-1}x_n\to t$, k-continuity of f implies that $f^kx_n\to ft$. Hence $t=ft$ as $f^kx_n\to t$. Therefore, t is fixed point of f.

Finally, suppose that f is orbitally continuous. Since $x_n\to t$, orbital continuity implies that $fx_n\to ft$. This gives $t=ft$ as $fx_n\to t$. Thus t is a fixed point of f. It may be observed that condition (2.8) of Agarwal et al [**1**] is equivalent to the notion of orbital continuity.

The next result gives the global version of Theorem 2.8

Theorem 2.15. *Let f be a self-mapping of a complete metric space (X, d). Suppose $\phi : X \to [0, \infty)$ is a function such that for each x in X we have*

(xiii) $d(x, fx) \leq \phi(x) - \phi(fx)$.

If f is orbitally continuous or if f^k is continuous or if f is k-continuous for some $k \geq 1$ then f possesses a fixed point.

The next result is a particular case of the above theorem for contractive self-mappings and is also applicable to contractive mappings which admit discontinuity at the fixed point.

Theorem 2.16. *Let f be a contractive type self-mapping of a complete metric space (X, d). Suppose $\phi : X \to [0, \infty)$ is a function such that for each x in X we have*

(xiv) $d(x, fx) \leq \phi(x) - \phi(fx)$.

If f is orbitally continuous or if f^k is continuous or if f is k-continuous for some $k \geq 1$ then f possesses a unique fixed point.

□

The next example illustrates Theorem 2.16.

Example 2.17. *Let $X = (-\infty, \infty)$ equipped with Euclidean metric. Define $f : X \to X$ by*

$$fx = 1 \text{ if } x \leq 1, \qquad fx = 0 \text{ if } x > 1.$$

Then f satisfies the conditions of Theorem 2.16 and has a unique fixed point $x = 1$ at which f is discontinuous. The mapping f satisfies the contractive condition $d(fx, fy) < \max\{d(x, fx), d(y, fy)\}$ and satisfies condition (xiv) with $\phi : X \to [0, \infty)$ defined by

$$\phi(x) = 1 - x \text{ if } x \leq 1, \qquad \phi(x) = 1 + x \text{ if } x > 1.$$

Remark 2.18. *Theorem 2.16 is the first Caristi type fixed point theorem to provide an answer to the question of existence of contractive mappings which admit discontinuity at the fixed point.*

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