

## A- STATISTICAL CONVERGENCE OF ORDER $\alpha$ VIA $\varphi$ - FUNCTION

*Dedicated to Academician Professor Gradimir Milovanović  
on the occasion of his 70th birthday.*

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In this paper, we introduce and examine some properties of  $A$ - statistical convergence of order  $\alpha$  by using  $\varphi$ - function, modulus function and generalized three parametric real matrix  $A$ .

### Introduction and Background

Let  $w$  denote the set of all real and complex sequences  $x = (x_k)$ . By  $l_\infty$  and  $c$ , we denote the Banach spaces of bounded and convergent sequences  $x = (x_k)$  normed by  $\|x\| = \sup_{nk} |x_k|$ , respectively. In summability theory, the concept of almost convergence was first introduced by G.G. Lorentz in 1948. Let us observe the outline of it. A linear functional  $L$  on  $l_\infty$  is said to be a Banach limit [2] if it has the following properties:

1.  $L(x) \geq 0$  if  $x_n \geq 0$  (i.e.  $x_n \geq 0$  for all  $n$ ),
2.  $L(e) = 1$  where  $e = (1, 1, \dots)$ ,
3.  $L(Dx) = L(x)$ , where the shift operator  $D$  is defined by  $D(x_n) = \{x_{n+1}\}$ .

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2010 Mathematics Subject Classification. 40H05, 40C05  
Modulus function, Almost convergence, Lacunary sequence,  $\varphi$  function, order  $\alpha$ .

Let  $B$  be the set of all Banach limits on  $l_\infty$ . A sequence  $x \in l_\infty$  is said to be almost convergent if all Banach limits of  $x$  coincide. It is easy to verify that if  $x$  is a convergent sequence, then  $L(x) = \lim_n x_n$  for any Banach limits  $L$ . In the other words,  $L(x)$  takes the same value for any Banach limits  $L$ . It is notable that this condition is meaningful not only for convergent sequences, but also for a certain type of bounded sequences. Let  $\hat{c}$  denote the space of almost convergent sequences. Lorentz [11] has shown that

$$\hat{c} = \left\{ x \in l_\infty : \lim_m t_{m,n}(x) \text{ exists uniformly in } n \right\}$$

where

$$t_{m,n}(x) = \frac{x_n + x_{n+1} + x_{n+2} + \dots + x_{n+m}}{m + 1}.$$

Before defining  $(\mathbf{A}, \varphi)$ - statistical convergence, let us first introduce the notion of lacunary sequence and modulus function.

Recall that the standard notation  $\theta = (k_r)$  denotes lacunary sequence, where  $(k_r)$  is a sequence of the positive integers such that  $k_0 = 0$ ,  $0 < k_r < k_{r+1}$ , and  $k_r - k_{r-1} \rightarrow \infty$  as  $r \rightarrow \infty$ , throughout the article, the intervals determined by  $\theta$  will be denoted by  $I_r = (k_{r-1}, k_r]$  and  $h_r = k_r - k_{r-1}$ . The ratio  $\frac{k_r}{k_{r-1}}$  will be denoted by  $q_r$ . The space of lacunary strongly convergent sequences  $N_\theta$  was defined by Freedman et al [9] as follows:

$$N_\theta = \left\{ x = (x_k) : \lim_r \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| = 0, \text{ for some } L \right\}.$$

There is a strong connection between  $N_\theta$  and the space  $w$  of strongly Cesàro summable sequences which is defined by Maddox [12] as follows:

$$w = \left\{ x = (x_k) : \lim_n \frac{1}{n} \sum_{k=0}^n |x_k - L| = 0, \text{ for some } L \right\}.$$

In the special case where  $\theta = (2^r)$ , we have  $N_\theta = w$ .

Following Ruckle [16], a modulus function  $f$  is a function from  $[0, \infty)$  to  $[0, \infty)$  such that

- (i)  $f(x) = 0$  if and only if  $x = 0$ ,
- (ii)  $f(x + y) \leq f(x) + f(y)$  for all  $x, y \geq 0$ ,
- (iii)  $f$  increasing,
- (iv)  $f$  is continuous from the right at zero.

Since  $|f(x) - f(y)| \leq f(|x - y|)$ , it follows from condition (iv) that  $f$  is continuous on  $[0, \infty)$ .

A modulus may be bounded or unbounded. For example,  $f(x) = x^p$ , for  $0 < p \leq 1$  is unbounded, but  $f(x) = \frac{x}{1+x}$  is bounded.

Ruckle used the idea of a modulus function  $f$  to construct a class of FK spaces

$$L(f) = \left\{ x = (x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty \right\}.$$

The space  $L(f)$  is closely related to the space  $l_1$  which is an  $L(f)$  space with  $f(x) = x$  for all real  $x \geq 0$ .

Maddox [13] introduced and examined some properties of the sequence spaces  $w_0(f)$ ,  $w(f)$  and  $w_\infty(f)$  defined using a modulus  $f$ , which generalized the well-known spaces  $w_0$ ,  $w$  and  $w_\infty$  of strongly summable sequences.

Recently E. Savas [18] generalized the concept of strong almost convergence by using a modulus  $f$  and examined some properties of the corresponding new sequence spaces.

In 1999, E. Savas [19] defined the class of sequences, which are strongly almost Cesàro summable with respect to modulus, as follows:

$$[\hat{c}(f, p)] = \left\{ x : \lim_n \frac{1}{n} \sum_{k=1}^n f(|x_{k+m} - L|)^{p_k} = 0, \text{ for some } L, \text{ uniformly in } m \right\}$$

and

$$[\hat{c}(f, p)]_0 = \left\{ x : \lim_n \frac{1}{n} \sum_{k=1}^n f(|x_{k+m}|)^{p_k} = 0, \text{ uniformly in } m \right\}.$$

where  $p = (p_k)$  is a sequence of strictly positive real numbers and  $f$  be a modulus.

By a  $\varphi$ -function we understood a continuous non-decreasing function  $\varphi(u)$  defined for  $u \geq 0$  and such that  $\varphi(0) = 0$ ,  $\varphi(u) > 0$ , for  $u > 0$  and  $\varphi(u) \rightarrow \infty$  as  $u \rightarrow \infty$ .

A  $\varphi$ -function  $\varphi$  is called non weaker than a  $\varphi$ -function  $\psi$  if there are constants  $c, b, k, l > 0$  such that  $c\psi(lu) \leq b\varphi(ku)$ , ( for all large  $u$ ) and we write  $\psi \prec \varphi$ , (see, [21], [24]).

On the other hand in [5] a different direction was given to the study of Cesàro-type summability spaces of order  $\alpha$ ,  $0 < \alpha \leq 1$  and lacunary statistical convergence of order  $\alpha$  where the notion of lacunary statistical convergence was introduced by replacing  $h_r$  by  $h_r^\alpha$  in the denominator in the definition of lacunary statistical convergence. It was observed in [5] that the behavior of this new convergence was not exactly parallel to that of lacunary statistical convergence and some basic properties were obtained.

In the present paper, we introduce and study **A**- statistical convergence of order  $\alpha$  by using  $\varphi$ - function, modulus function and generalized three parametric real matrix **A**.

### Main Results

In this section, we shall define  $(\mathbf{A}, \varphi)$ - statistical convergence and we shall establish some inclusion theorems.

Let  $\varphi$  and  $f$  be given  $\varphi$ -function and modulus function, respectively and  $p = (p_k)$  be a sequence of positive real numbers. Moreover, let  $\mathbf{A} = (\mathbf{A}_i)$  be the generalized three parametric real matrix with  $A_i = (a_{n,k}(i))$ , a lacunary sequence  $\theta = (k_r)$  and  $0 < \alpha \leq 1$  be given. Then we define the following sequence spaces,

$$N_{\theta}^{\alpha}(\mathbf{A}, \varphi, f, p)_0 = \left\{ x = (x_k) : \lim_r \frac{1}{h_r^{\alpha}} \sum_{n \in I_r} f \left( \left| \sum_{k=1}^{\infty} a_{nk}(i) \varphi(|x_k|) \right| \right)^{p_k} = 0, \text{ uniformly in } i \right\},$$

where  $h_r^{\alpha}$  denote the  $\alpha$ th power  $(h_r)^{\alpha}$  of  $h_r$ , that is  $h^{\alpha} = (h_r^{\alpha}) = (h_1^{\alpha}, h_2^{\alpha}, h_3^{\alpha}, \dots)$ .

If  $x \in N_{\theta}^{\alpha}(\mathbf{A}, \varphi, f, p)_0$ , the sequence  $x$  is said to be lacunary strong  $(\mathbf{A}, \varphi)$ -convergent to zero with respect to a modulus  $f$ .

The idea of statistical convergence was given by Zygmund [23] in the first edition of his monograph published in Warsaw in 1935. The notion of statistical convergence was introduced by Fast [6] and Schoenberg [22] independently. Over the years and under different names, statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory, and number theory. Later on, it was further investigated from the sequence space point of view and linked with summability theory by Fridy [7], Connor [4], Šalát [17], Mursaleen [15], Savas[20], Maddox [14] and many others.

A real number sequence  $x$  is said to be statistically convergent to the number  $L$  if for every  $\epsilon > 0$

$$\lim_n \frac{1}{n} |\{k < n : |x_k - L| \geq \epsilon\}| = 0,$$

where by  $k < n$  we mean that  $k = 0, 1, 2, \dots, n$  and the vertical bars indicate the number of elements in the enclosed set. In this case, we write  $st_1 - \lim x = L$  or  $x_k \rightarrow L(st_1)$ . Statistical convergence is a generalization of the usual notion of convergence for real valued sequences that parallels the usual theory of convergence.

In another direction, a new type of convergence called lacunary statistical convergence was introduced in [8] as follows.

A sequence  $(x_k)$  of real numbers is said to be lacunary statistically convergent to  $L$  ( or,  $S_{\theta}$ -convergent to  $L$  ) if for any  $\epsilon > 0$ ,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \epsilon\}| = 0$$

where  $|A|$  denotes the cardinality of  $A \subset \mathbb{N}$ . In [8] the relation between lacunary statistical convergence and statistical convergence was established among other things.

In the recent past, Et and Hacer [5] presented lacunary statistical convergence of order  $\alpha$  as follows: Let  $\theta$  be a lacunary sequence and  $0 < \alpha \leq 1$  be given. The sequence  $(x_k)$  of real numbers is said to be lacunary statistically convergent of order  $\alpha$  to  $L$  ( or,  $S_\theta$ -convergent of order  $\alpha$  to  $L$  ) if for any  $\varepsilon > 0$ ,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r^\alpha} |\{k \in I_r : |x_k - L| \geq \varepsilon\}| = 0.$$

In this case we write  $S_\theta^\alpha - \lim x_k = L$ .

The definition of  $A$ -statistically convergence is given by Kolk [10] as follows:

Assume that  $A$  is a non-negative regular summability matrix. Then the sequence  $x = (x_k)$  is called  $A$ -statistically convergent to  $L$  provided that, for every  $\varepsilon > 0$ ,

$$\lim_k \sum_{n: |x_k - L| \geq \varepsilon} a_{jn} = 0$$

We denote this by  $st_A - \lim_k x_k = L$ .

Let  $\theta$  be a lacunary sequence, and let  $\mathbf{A} = (a_{nk}(i))$  be the generalized three parametric real matrix and the sequence  $x = (x_k)$ , the  $\varphi$ - function  $\varphi(u)$  and a positive number  $\varepsilon > 0$  be given. We write, for all  $i$

$$K_\theta^r((A, \varphi), \varepsilon, i) = \left\{ n \in I_r : \sum_{k=1}^{\infty} a_{nk}(i) \varphi(|x_k|) \geq \varepsilon \right\}.$$

The sequence  $x$  is said to be  $(\mathbf{A}, \varphi)$ - lacunary statistically convergent of order  $\alpha$  to a number zero if for every  $\varepsilon > 0$

$$\lim_r \frac{1}{h_r^\alpha} \mu(K_\theta^r((A, \varphi), \varepsilon, i)) = 0, \text{ uniformly in } i$$

where  $\mu(K_\theta^r((A, \varphi), \varepsilon, i))$  denotes the number of elements belonging to  $K_\theta^r((\mathbf{A}, \varphi), \varepsilon, i)$ . We denote by  $S_\theta^\alpha(\mathbf{A}, \varphi)_0$ , the set of sequences  $x = (x_k)$  which are lacunary  $(\mathbf{A}, \varphi)$ -statistical convergent of order  $\alpha$  to zero and we write

$$S_\theta^\alpha(\mathbf{A}, \varphi)_0 = \left\{ x = (x_k) : \lim_r \frac{1}{h_r^\alpha} \mu(K_\theta^r((A, \varphi), \varepsilon, i)) = 0, \text{ uniformly in } i \right\}.$$

If we take  $A = I$  and  $\varphi(x) = x$  respectively, then  $S_\theta^\alpha(\mathbf{A}, \varphi)_0$  reduce to  $(S_\theta^\alpha)_0$  which was defined as follows:

$$(S_\theta^\alpha)_0 = \left\{ x = (x_k) : \frac{1}{h_r^\alpha} |\{k \in I_r : |x_k| \geq \varepsilon\}| = 0 \right\}.$$

**Remark 1.** (i) If for all  $i$ ,

$$a_{nk}(i) := \begin{cases} \frac{1}{n}, & \text{if } n \geq k, \\ 0, & \text{otherwise.} \end{cases}$$

then  $S_\theta^\alpha(\mathbf{A}, \varphi)_0$  reduce to  $S_\theta^\alpha(C, \varphi)_0$ , i.e., uniform  $(C, \varphi)$ - statistical convergence.

(ii) If for all  $i$ ,

$$a_{nk}(i) := \begin{cases} \frac{p_k}{P_n}, & \text{if } n \geq k, \\ 0, & \text{otherwise.} \end{cases}$$

then  $S_\theta^\alpha(\mathbf{A}, \varphi)_0$  reduce to  $S_\theta^\alpha(N, p, \varphi)$ , i.e., uniform  $((N, p), \varphi)$ - statistical convergence, where  $p = p_k$  is a sequence of nonnegative numbers such that  $p_0 > 0$  and

$$P_i = \sum_{k=0}^n p_k \rightarrow \infty (n \rightarrow \infty).$$

**Theorem 1.** If  $0 < \alpha \leq \beta \leq 1$  then  $S_\theta^\alpha(\mathbf{A}, \varphi)_0 \subset S_\theta^\beta(\mathbf{A}, \varphi)_0$ .

*Proof.* Let  $0 < \alpha \leq \beta \leq 1$ . Then

$$\frac{1}{h_r^\beta} \mu(K_\theta^r((A, \varphi), \varepsilon, i)) \leq \frac{1}{h_r^\alpha} \mu(K_\theta^r((A, \varphi), \varepsilon, i))$$

for every  $\varepsilon > 0$  and finally we have that  $S_\theta^\alpha(\mathbf{A}, \varphi)_0 \subset S_\theta^\beta(\mathbf{A}, \varphi)_0$ . This proves the theorem.  $\square$

We now have

**Theorem 2.** If  $\psi \prec \varphi$  then  $S_\theta^\alpha(A, \psi) \subset S_\theta^\alpha(A, \varphi)$ .

*Proof.* By assumption we have  $\psi(|x_k - L|) \leq b\varphi(c|x_k - L|)$  and we have ,

$$\sum_{k=1}^{\infty} a_{nk}(i)\psi(|x_k - L|) \leq b \sum_{k=1}^{\infty} a_{nk}(i)\varphi(c|x_k - L|) \leq M \sum_{k=1}^{\infty} a_{nk}(i)\varphi(|x_k - L|)$$

for  $b, c > 0$ , where the constant  $M$  is connected with properties of  $\varphi$ . Thus, the condition  $\sum_{k=1}^{\infty} a_{nk}(i)\psi(|x_{k,l} - L|) \geq \varepsilon$  implies the condition  $\sum_{k=1}^{\infty} a_{nk}(i)\varphi(|x_{k,l} - L|) \geq \varepsilon$  and in consequence we get  $\mu(K_\theta^\alpha(A, \varphi, \varepsilon)) \subset \mu(K_\theta^\alpha(A, \psi, \varepsilon))$  and

$$\lim_r \frac{1}{h_r^\alpha} \mu(K_\theta(A, \varphi, \varepsilon, i)) \leq \lim_r \frac{1}{h_r^\alpha} \mu(K_\theta(A, \psi, \varepsilon, i)).$$

This completes the proof.  $\square$

Finally we conclude this paper by presenting inclusion relations between  $N_\theta^0(\mathbf{A}, \varphi, f, p)$  and  $S_\theta^0(A, \varphi)$ .

In the following theorem we assume that  $0 < h = \inf p_k \leq p_k \leq \sup p_k \leq H \leq \infty$ .

**Theorem 3.** (a) If the matrix  $A$  and  $\sup p_k = H$ , the sequence  $\theta$  and functions  $f$  and  $\varphi$  be given, then

$$N_{\theta}^{\alpha}((A, \varphi), f, p)_0 \subset S_{\theta}^{\alpha}(A, \varphi)_0.$$

(b) If the  $\varphi$ -function  $\varphi(u)$  and the matrix  $A$  are given, and if the modulus function  $f$  is bounded, then

$$S_{\theta}^{\alpha}(A, \varphi)_0 \subset N_{\theta}^{\alpha}(A, \varphi, f, p)_0.$$

*Proof.* (a) Let  $f$  be a modulus function and let  $\varepsilon$  be a positive number. We write the following inequalities, for all  $i$

$$\begin{aligned} \frac{1}{h_r^{\alpha}} \sum_{n \in I_r} f \left( \left| \sum_{k=1}^{\infty} a_{nk}(i) \varphi(|x_k|) \right| \right)^{p_n} &= \frac{1}{h_r^{\alpha}} \sum_{n \in I_r^1} f \left( \left| \sum_{k=1}^{\infty} a_{nk}(i) \varphi(|x_k|) \right| \right)^{p_n} \\ &\geq \frac{1}{h_r^{\alpha}} \sum_{n \in I_r^1} [f(\varepsilon)]^{p_n} \\ &\geq \frac{1}{h_r^{\alpha}} \sum_{n \in I_r^1} \min \left( [f(\varepsilon)]^{\inf p_n}, [f(\varepsilon)]^H \right) \\ &\geq \frac{1}{h_r^{\alpha}} \mu(K_{\theta}^r(A, \varphi), \varepsilon, i) \min \left( [f(\varepsilon)]^{\inf p_n}, [f(\varepsilon)]^H \right), \end{aligned}$$

where

$$I_r^1 = \left\{ n \in I_r : \sum_{k=1}^{\infty} a_{nk}(i) \varphi(|x_k|) \geq \varepsilon \right\}.$$

Finally, if  $x \in N_{\theta}^0(A, \varphi, f, p)$  then  $x \in S_{\theta}^0(A, \varphi, f)$ .

(b) Let us suppose that  $x \in S_{\theta}^{\alpha}(A, \varphi)_0$ . If the modulus function  $f$  is a bounded function, then there exists an integer  $K$  such that  $f(x) < K$  for  $x \geq 0$ . Let us take

$$I_r^2 = \left\{ n \in I_r : \sum_{k=1}^{\infty} a_{nk}(i) \varphi(|x_k|) < \varepsilon \right\}.$$

Thus we have, for all  $i$

$$\begin{aligned}
& \frac{1}{h_r^\alpha} \sum_{n \in I_r} f \left( \left| \sum_{k=1}^{\infty} a_{nk}(i) \varphi(|x_k|) \right| \right)^{p_n} \\
& \leq \frac{1}{h_r^\alpha} \sum_{n \in I_r^1} f \left( \left| \sum_{k=1}^{\infty} a_{nk}(i) \varphi(|x_k|) \right| \right)^{p_n} \\
& \quad + \frac{1}{h_r^\alpha} \sum_{n \in I_r^2} f \left( \left| \sum_{k=1}^{\infty} a_{nk}(i) \varphi(|x_k|) \right| \right)^{p_n} \\
& \leq \frac{1}{h_r^\alpha} \sum_{n \in I_r^1} \max(K^h, K^H) + \frac{1}{h_r} \sum_{n \in I_r^2} [f(\varepsilon)]^{p_n} \\
& \leq \max(K^h, K^H) \frac{1}{h_r^\alpha} \mu(K_\theta^r((A, \varphi), \varepsilon, i)) + \max([f(\varepsilon)]^h, [f(\varepsilon)]^H).
\end{aligned}$$

Taking the limit as  $\varepsilon \rightarrow 0$ , we observe that  $x \in N_\theta^\alpha(A, \varphi, f, p)_0$ .  
This completes the proof.  $\square$

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(Received 18.12.2018)

(Revised 15.08.2019)