Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **14** (2020), 198–220.
https://doi.org/10.2298/AADM190805017D

FULL HERMITE INTERPOLATION OF THE RELIABILITY OF A HAMMOCK NETWORK

Leonard Dăuş, Marilena Jianu*

Although the hammock networks were introduced more than sixty years ago, there is no general formula of the associated reliability polynomial. Using the full Hermite interpolation polynomial, we propose an approximation for the reliability polynomial of a hammock network of arbitrary size. In the second part of the paper, we provide combinatorial formulas for the first two non-zero coefficients of the reliability polynomial.

1. INTRODUCTION

In 1956, Von Neumann [**17**] and Moore and Shannon [**15**] introduced the concept of network reliability. Subsequently, in order to improve the reliability of a certain network, Moore and Shannon [**15**] wanted to replace an unreliable relay by a (redundant) network of such relays, called hammock network.

In general, finding the reliability polynomial of a network is one of the most important and also the challenging problems in reliability theory. The difficulty of calculating the reliability of a network is due to the fact that such a problem belongs to the class of $\#P$-complete problems, a class of computationally equivalent counting problems (introduced by Valiant in [**21**]) that are at least as difficult as the NP-complete problems ([**22**], [**2**]). Although the hammock networks were introduced more than sixty years ago, there is no general formula of the associated reliability polynomial. Recently, in [**3**], using an "in-house" algorithm and a powerful grid computing system, the reliability polynomials have been calculated exactly

*Corresponding author. Leonard Dăuş
2010 Mathematics Subject Classification. 05C31, 41A05, 68Rxx.
Keywords and Phrases. Reliability polynomial, full Hermite interpolation, hammock networks, lattice paths.

for a few particular cases of small size, more precisely for the 29 hammock networks presented by Moore and Shannon in their original paper. Another important step was achieved in [**5**] where the first and second non-zero coefficients of the reliability polynomial have been computed, for any hammock network. The methods used to prove the formulas for these leading coefficients involve the transition matrix of certain linear transformations, lattice paths and generating functions. In [**6**], a direct proof of duality properties for hammock networks is given, linking the reliability polynomial of the hammock network of dimensions (l, w) to the reliability polynomial of the hammock network of dimensions (w, l). Finally, we have to point out the paper [**7**], where compositions of series and parallel networks of two devices are compared to size-equivalent hammock networks, proving that compositions of series and parallel networks are not able to surpass hammock networks in terms of reliability.

This paper has two primary goals: the first one is to obtain an approximation for the reliability polynomial of a hammock network of arbitrary size, starting from the full Hermite interpolation polynomial. Introduced by Hermite in 1877 (see [**9**]), the full Hermite polynomial preserves not only the values of the function at certain nodes $x_i, i = 0, 1, ..., n,$ (as Lagrange polynomial does), but also the values of the derivatives of the function up to some arbitrary ranks $k_i, i = 0, 1, ..., n,$ (all $k_i = 1$ the classical Hermite polynomial is obtained). Last part of the paper is devoted to finding the first two non-zero coefficients b_l and b_{l+1} of the reliability polynomial, being the second purpose of this research article. As we already mention before, these two leading coefficients were also presented in [**5**], but in the form of matrix identities. In the current paper, Theorem 5 and Theorem 6 give explicit combinatorial expressions of b_l and b_{l+1}.

We refer the reader to [**2**] for elements of network reliability and to [**19**], [**10**] for definitions and results about lattice paths.

2. THE RELIABILITY OF A HAMMOCK NETWORK

A network is a probabilistic graph [**2**], $N = (V, E)$, where V is the set of nodes (vertices) and E is the set of (undirected) edges. “Probabilistic” means that all the edges can be represented as independent identically distributed random variables: each edge operates (is closed) with probability p and fails (is open) with probability $q = 1 - p$. We assume that nodes do not fail; the fail of the network is always a consequence of edge failures. Let S and T be two special nodes (terminals). The two-terminal reliability of the network, denoted by $h(p)$, is the probability that there exists a path (a sequence of adjacent edges) made of operational (closed) edges between S (source / input) and T (target / output). This probability can be expressed as a polynomial function in p.

A subset of E containing all the edges of a path between the nodes S and T is called *pathset*. A minimal pathset (*minpath*) is a pathset P such that, if any edge e of P is removed, then $P - \{e\}$ is no longer a pathset. Let $\mathcal{P}$ be the set of all the pathsets of the network. A *cutset* is subset of edges, $C \subset E$, with the property that the complementary set, $E - C$, is not a pathset. A minimal cutset (*mincut*) is a cutset C such that, if any edge e of C is removed, then $C - \{e\}$ is no longer a cutset. Let $\mathcal{C}$ be the set of all the cutsets of N.

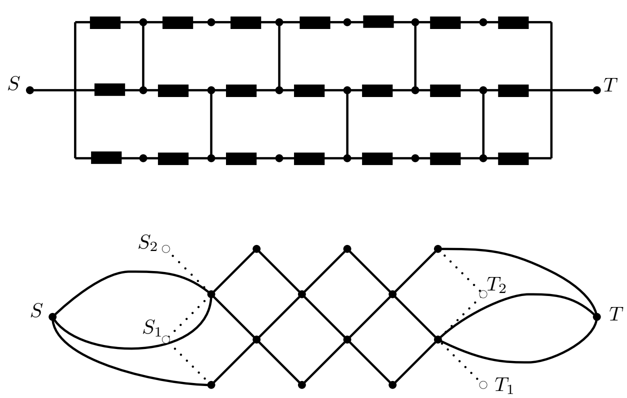
If $n = |E|$ is the size of the graph, N_i is the number of pathsets with exactly i edges and C_i, the number of cutsets with exactly i edges, then the reliability of the network can be expressed in the form (see [**2**]):

$$h(p) = \sum_{P \in \mathcal{P}} p^{|P|} q^{n-|P|} = \sum_{i=1}^{n} N_i p^i (1-p)^{n-i}, \tag{1}$$

or, in terms of cutsets, as

$$h(p) = 1 - \sum_{C \in \mathcal{C}} q^{|C|} p^{n-|C|} = 1 - \sum_{i=1}^{n} C_i (1-p)^i p^{n-i}. \tag{2}$$

A brick-wall network is formed by $w \times l$ identical devices disposed in w lines, each line consisting of l devices connected in series. Besides the horizontal connections, there exist also alternating vertical connections, creating the "brick-wall" pattern that can be seen in Figure **??**. Brick-wall networks were also named by Moore and Shannon *hammock networks* [**15**], because of their aspect when the terminal nodes S and T are pulled apart and every vertical connection becomes a node, as rectangular "bricks" turns into rhombs. As can be seen, the "hammock" representation fits the above definition of the probabilistic graph, unlike the "brick-wall" representation, where the vertical edges are assumed to be operational with probability 1.



In Fig.**??** a brick-wall network of dimensions $w = 3$, $l = 7$ and the corresponding hammock network are represented. Notice that, in order to preserve the regularity of the network, the terminal nodes S and T can be replaced by some "fictive" terminal nodes, $S_1, S_2, \ldots, S_k$, respectively, $T_1, T_2, \ldots, T_h$, where $k, h \in \left\{ \left[\frac{w}{2}\right], \left[\frac{w}{2}\right] + 1 \right\}$.

We introduced in [**6**] the concept of $\boldsymbol{X}$ – *path*: a lattice path that never passes twice through the same node and have steps in the set

$$S = \{(1,1), (-1,1), (1,-1), (-1,-1)\}.$$

Thus, from a lattice point (x, y) it is allowed to move in 4 directions and reach one of the 4 neighbor points: $(x+1, y+1)$, $(x-1, y+1)$, $(x+1, y-1)$, $(x-1, y-1)$. As one can notice, the sum of coordinates of any neighbor point has the same parity as $x + y$. We say that a lattice point (x, y) is even (odd) if $x + y$ is even (odd). An $\mathbf{X}$ – path is even (odd) if it contains even (respectively, odd) points.

Let $\mathcal{V}_{l,w} = \left\{A_{x,y} = (x, y) \in \mathbb{Z}^2 : 0 \le x \le l,\ 0 \le y \le w\right\}$ be the set of all lattice points in the rectangle $[0, l] \times [0, w]$ and $V_{l,w}^{[1]} = \{A_{x,y} \in \mathcal{V}_{l,w} : x + y = \text{even}\}$, $V_{l,w}^{[2]} = \{A_{x,y} \in \mathcal{V}_{l,w} : x + y = \text{odd}\}$ be the subsets of even (respectively, odd) points in $[0, l] \times [0, w]$. We denote by

$$\mathcal{E}_{l,w} = \{e = (A_{x,y}, A_{x',y'}) : A_{x,y}, A_{x',y'} \in \mathcal{V}_{l,w}, |x - x'| = |y - y'| = 1\}$$

the set of all the line segments of length $\sqrt{2}$ connecting points of $\mathcal{V}_{l,w}$. Let $E_{l,w}^{[1]} = \{(A_{x,y}, A_{x',y'}) \in \mathcal{E}_{l,w} : x + y = \text{even}\}$ be the subset of even edges and $E_{l,w}^{[2]} = \{(A_{x,y}, A_{x',y'}) \in \mathcal{E}_{l,w} : x + y = \text{odd}\}$, the subset of odd edges.

A *hammock network of the first kind* of dimensions (l, w) is the probabilistic graph $H_{l,w}^{[1]} = \left(V_{l,w}^{[1]}, E_{l,w}^{[1]}\right)$, while a *hammock network of the second kind* is $H_{l,w}^{[2]} = \left(V_{l,w}^{[2]}, E_{l,w}^{[2]}\right)$. We assume that each edge is closed with probability p and open with probability $1 - p$. The input (source) nodes are $S_j = A_{0,y}$ (with $y = \text{even}$ for the first kind and $y = \text{odd}$ for the second kind), and the output (target) nodes are $T_k = A_{l,z}$ (with $l + z =$ even, respectively, odd).

A subset of even (respectively, odd) edges $P \subset E_{l,w}^{[i]}$ is a *pathset* in $H_{l,w}^{[i]}$ if it contains an $\mathbf{X}$ – path connecting a source node S_j with a target node T_k. Let $\mathcal{P}_{l,w}^{[i]}$ be the set of all pathsets in $H_{l,w}^{[i]}$. A subset $C \subset E_{l,w}^{[i]}$ is a *cutset* in $H_{l,w}^{[i]}$ if $E_{l,w}^{[i]} - C$ contains no $\mathbf{X}$ – path connecting a source node S_j with a target node T_k. Let $\mathcal{C}_{l,w}^{[i]}$ be the set of all cutsets in $H_{l,w}^{[i]}$. By using these notations in formulas (1) and (2), the reliability polynomials of hammock networks of the first and of the second kind, $h_{l,w}^{[1]}(p)$ and $h_{l,w}^{[2]}(p)$, can be written:

$$(3) \qquad h_{l,w}^{[i]}(p) = \sum_{P \in \mathcal{P}_{l,w}^{[i]}} p^{|P|}(1-p)^{lw-|P|} = 1 - \sum_{C \in \mathcal{C}_{l,w}^{[i]}} (1-p)^{|C|} p^{lw-|C|},\ \ i = 1, 2$$

Remark 1. If $l = \text{odd}$ or $w = \text{odd}$, then the hammock networks $H_{l,w}^{[1]}$ and $H_{l,w}^{[2]}$ are isomorphic and the reliability polynomials are identical: $h_{l,w}^{[1]} = h_{l,w}^{[2]}$.

The dual network of a hammock network can be defined as follows (see [**6**]). For each edge $e \in \mathcal{E}_{l,w}$, $e = (A_{x,y}, A_{x+1,y\pm1})$, we denote by $\bar{e} = (A_{x+1,y}, A_{x,y\pm1})$ its complementary edge (the edge that *cuts* e). It can be seen that the complementary edge of an even edge is odd and the complementary edge of an odd edge is even. Thus, if $e \in E_{l,w}^{[i]}$, then $\bar{e} \in \overline{E_{l,w}^{[i]}} = \mathcal{E}_{l,w} - E_{l,w}^{[i]} = E_{l,w}^{[2/i]}$. By using the notation $\overline{V_{l,w}^{[i]}} = \mathcal{V}_{l,w} - V_{l,w}^{[i]} = V_{l,w}^{[2/i]}$, the dual network of $H_{l,w}^{[i]} = \left(V_{l,w}^{[i]}, E_{l,w}^{[i]}\right)$ is $\overline{H_{l,w}^{[i]}} = \left(\overline{V_{l,w}^{[i]}}, \overline{E_{l,w}^{[i]}}\right)$ with the source nodes $S_j' = A_{x,0} \in \overline{V_{l,w}^{[i]}}$ and the target nodes $T_k' = A_{z,w} \in \overline{V_{l,w}^{[i]}}$. The probability of an edge $\bar{e} \in \overline{E_{l,w}^{[i]}}$ to be closed is the probability of the edge $e \in E_{l,w}^{[i]}$ to be open ("cut"): $q = 1 - p$.

Remark 2. The networks $\overline{H_{l,w}^{[i]}}$ and $H_{w,l}^{[2/i]}$ are isomorphic (since they are symmetric with respect to the first bisectrix) and the reliability of the dual network can be written

$$\overline{h_{l,w}^{[i]}}(p) = h_{w,l}^{[2/i]}(1-p).$$

The next theorem and its corollary were proved in [**6**] and make the connection between the reliability polynomials of a hammock network $H_{l,w}^{[i]}$ and its dual network $\overline{H_{l,w}^{[i]}}$.

Theorem 1. *Let* $\Sigma = \{e_1, e_2, \ldots, e_n\} \subset E_{l,w}^{[i]}$ *be a subset of edges of the network* $H_{l,w}^{[i]}$ *and let* $\overline{\Sigma} = \{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n\} \subset \overline{E_{l,w}^{[i]}}$ *be the set of complementary edges. The following statements hold:*

i) If Σ *is a mincut in* $H_{l,w}^{[i]}$ *then* $\overline{\Sigma}$ *is a minpath in* $\overline{H_{l,w}^{[i]}}$.

ii) If Σ *is a minpath in* $H_{l,w}^{[i]}$ *then* $\overline{\Sigma}$ *is a mincut in* $\overline{H_{l,w}^{[i]}}$.

By Theorem 1 it follows that a set of edges $\Sigma = \{e_1, e_2, \ldots, e_n\} \subset E_{l,w}^{[i]}$ is a minpath in the network $H_{l,w}^{[i]}$ if and only if the set of complementary edges, $\overline{\Sigma} = \{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n\} \subset \overline{E_{l,w}^{[i]}}$ is a mincut in the dual network, $\overline{H_{l,w}^{[i]}}$. As a consequence, by using the equation (3) and Remark 2, the next corollary follows.

Corollary 1. *For any* $l, w \geq 1$ *and* $i = 1, 2$ *the following relation is true for all* $p \in [0, 1]$*:*

$$h_{l,w}^{[i]}(p) = 1 - h_{w,l}^{[2/i]}(1-p) \tag{4}$$

If at least one of l *and* w *is odd then* $h_{l,w}^{[1]} = h_{l,w}^{[2]} =: h_{l,w}$ *and we have:*

$$h_{l,w}(p) = 1 - h_{w,l}(1-p). \tag{5}$$

Before ending this section, we should add some remarks about the values of the reliability polynomial and its derivatives at the endpoints $p = 0$ and $p = 1$.

First of all, it is easy to see that any pathset of a hammock network has at least l edges, therefore, by equation (1), we have:

$$h(p) = \sum_{i=l}^{wl} N_i p^i (1-p)^{wl-i}, \tag{6}$$

so the reliability polynomial can be written as

$$h(p) = \sum_{i=l}^{wl} b_i p^i \tag{7}$$

for some integers b_i, $i = l, \ldots, wl$ and the next theorem follows.

Theorem 2. *Let $h(p) = h_{l,w}^{[i]}(p)$ be the reliability polynomial of a hammock network of dimensions (l, w), either of kind 1 or 2. Then the following relations hold:*

$$h(0) = 0, h(1) = 1 \tag{8}$$

$$h^{(k)}(0) = 0, \text{ for all } k = 1, 2, \ldots, l-1 \tag{9}$$

$$h^{(k)}(1) = 0, \text{ for all } k = 1, 2, \ldots, w-1. \tag{10}$$

Proof. The relations (8) and (9) follows immediately by (6) and (7) .

In order to prove (10), we denote by $\bar{h}(p) = h_{w,l}^{[2/i]}(p)$ the reliability polynomial of the dual network. Since w is the length of the dual network, it follows by (9) that $\bar{h}^{(k)}(0) = 0$, for all $k = 1, 2, \ldots, w-1$. By Corollary 1 we have that $h(p) = 1 - \bar{h}(1-p)$, and it follows that $h^{(k)}(p) = (-1)^{k+1}\bar{h}^{(k)}(1-p)$. For $p = 1$ we obtain $h^{(k)}(1) = (-1)^{k+1}\bar{h}^{(k)}(0) = 0$, for all $k = 1, 2, \ldots, w-1$. □

3. FULL HERMITE INTERPOLATION

Let $x_0 < x_1 < \ldots < x_n$ be $n+1$ distinct points in the interval $[a, b]$, let $k_0, k_1, \ldots, k_n > 0$ be $n+1$ positive integers and let $f : [a, b] \to \mathbb{R}$ be a function of class $C^k[a, b]$, where $k = \max_{j=\overline{0,n}} k_j$. The *full Hermite interpolation problem* ([**9**], [**4**]) is the problem of finding a polynomial $H(x)$ of degree at most $N = \sum_{j=0}^{n} k_j - 1$ such that

$$H^{(i)}(x_j) = f^{(i)}(x_j)\,, \text{ for all } i = 0, 1, \ldots, k_j - 1,\ j = 0, 1, \ldots, n.$$

The next theorem (firstly proved by Hermite in [**9**]) states the existence and uniqueness the full Hermite interpolation polynomial. The expression of this polynomial (equations (11), (12)) is given in [**18**].

Theorem 3. *Given the distinct points $x_0 < x_1 < \ldots < x_n$, the positive integers $k_0, k_1, \ldots, k_n > 0$ and the arbitrary real numbers $f_j^{(i)}$, $i = 0, 1, \ldots, k_j - 1$, $j = 0, 1, \ldots, n$, there exists a unique polynomial $H(x)$ of degree at most $N = \sum_{j=0}^{n} k_j - 1$ such that*

$$H^{(i)}(x_j) = f_j^{(i)}\,, \text{ for all } i = 0, 1, \ldots, k_j - 1,\ j = 0, 1, \ldots, n.$$

The expression of this polynomial (called full Hermite interpolation polynomial) is

$$H(x) = \sum_{j=0}^{n} \sum_{i=0}^{k_j-1} f_j^{(i)} l_{i,j}(x)\,, \tag{11}$$

where, for every $i = 0, 1, \ldots, k_j - 1$ and $j = 0, 1, \ldots, n$,

$$l_{i,j}(x) = u_j(x) \frac{(x - x_j)^i}{i!} \sum_{r=0}^{k_j-i-1} \frac{1}{r!} v_j^{(r)}(x_j)(x - x_j)^r, \tag{12}$$

$$u_j(x) = \prod_{\substack{t=0\\ t\neq j}}^{n} (x - x_t)^{k_t},$$

$$v_j(x) = \frac{1}{u_j(x)}.$$

The full Hermite interpolation polynomial generalizes both the Taylor polynomial (obtained for $n = 0$) and the Lagrange polynomial (when $k_j = 1$, for all $j = 0, 1, \ldots, n$). We can also notice that for $k_j = 2$, $j = 0, 1, \ldots, n$, the classical Hermite polynomial is obtained. This type of interpolation is also known as *osculating* interpolation. The Latin word *osculum*, literally *kiss*, when applied to a curve indicates that it just touches and has the same shape. Hermite interpolation polynomial has this osculating property: it matches a given curve and its derivative forces the interpolating curve to "kiss" the given curve ([**1**], p.136).

We apply Theorem 3 for the case $n = 1$. Suppose we have two distinct points, which may be the limits of the interval: $x_0 = a$, $x_1 = b$. We use the notations $k_0 = k$, $k_1 = m$ and we denote by $H_{k,m}(x)$ the full Hermite interpolation polynomial of order (k, m) for the function $f \in C^{\max(k,m)}[a, b]$. Thus, $H_{k,m}(x)$ is the (unique) polynomial of degree at most $k + m - 1$ such that

$$f^{(i)}(a) = H_{k,m}^{(i)}(a)\,,\ i = 0, 1, \ldots, k - 1,$$

and

$$f^{(i)}(b) = H_{k,m}^{(i)}(b)\,,\ i = 0, 1, \ldots, m - 1.$$

The next theorem gives the expression of this polynomial.

Theorem 4. *The full Hermite polynomial of order* (k, m) *interpolating* f *at the points* $x_0 = a,\ x_1 = b$ *can be written in the following form:*

$$H_{k,m}(x) = \sum_{i=0}^{k-1} f^{(i)}(a)A_i(x) + \sum_{i=0}^{m-1} f^{(i)}(b)B_i(x)\,,$$

where

$$A_i(x) = \left(\frac{b-x}{b-a}\right)^m \frac{(x-a)^i}{i!} \sum_{r=0}^{k-i-1} \binom{m+r-1}{m-1} \left(\frac{x-a}{b-a}\right)^r,$$

for $i = 0, 1, \ldots, k-1$, *and*

$$B_i(x) = \left(\frac{x-a}{b-a}\right)^k \frac{(x-b)^i}{i!} \sum_{r=0}^{m-i-1} \binom{k+r-1}{k-1} \left(\frac{b-x}{b-a}\right)^r,$$

for $i = 0, 1, \ldots, m-1$.

Taking $a = 0$ and $b = 1$, we apply Theorem 4 to the reliability polynomial of a hammock network of dimensions (l, w), $h_{l,w}(p)$. Using the formulas (8) - (10), we obtain the following result:

Corollary 2. *The full Hermite interpolation polynomial of order* (l, w) *corresponding to the reliability polynomial of a hammock network of dimensions* (l, w), $h_{l,w}(p)$, *at* $p_0 = 0,\ p_1 = 1$ *can be written in the following form:*

$$H_{l,w}(p) = p^l \cdot \sum_{r=0}^{w-1} \binom{l+r-1}{l-1} (1-p)^r. \tag{13}$$

Example 1. We compare the reliability polynomial of a hammock network of dimensions $l = 5, w = 4$ with the full Hermite interpolation polynomial of order $(5, 4)$, calculated by the formula (13). The expression of the reliability polynomial is

$$h_{5,4}(p) = 36p^5 - 30p^6 - 26p^7 - 92p^8 + 122p^9 + 62p^{10} + 328p^{11} - 760p^{12} - 306p^{13} \\ +1708p^{14} - 1234p^{15} - 312p^{16} + 932p^{17} - 566p^{18} + 156p^{19} - 17p^{20},$$

while the full Hermite interpolation polynomial of order $(5, 4)$ is given by the formula:

$$H_{5,4}(p) = 56p^5 - 140p^6 + 120p^7 - 35p^8.$$

In Figure 1 the graphs of the two polynomials are represented. It is worth emphasizing how accurate is the approximation of the polynomial of degree $w \cdot l$ by a polynomial of degree $w + l - 1$.

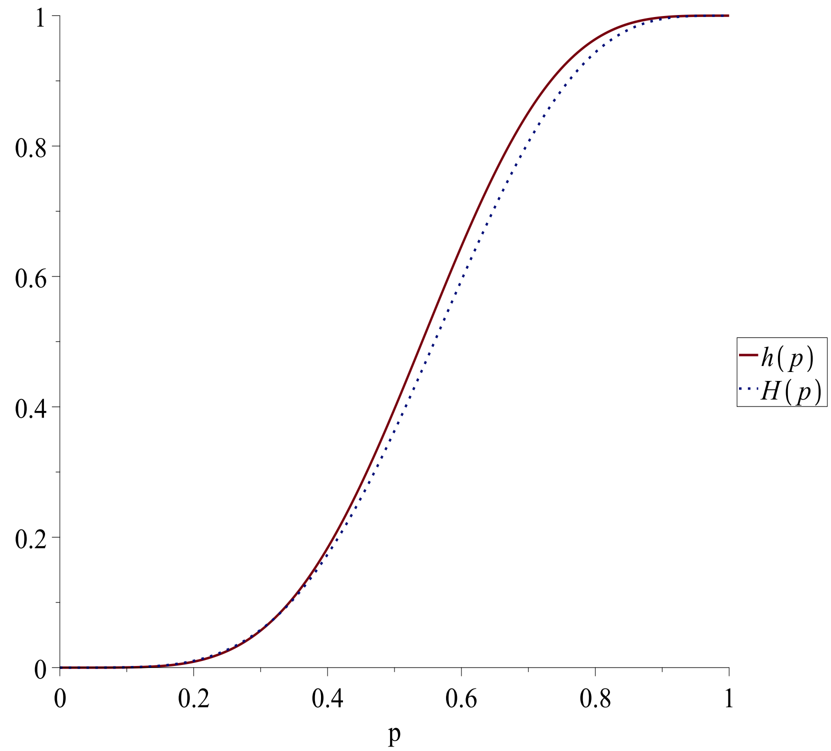


Figure 1: The reliability polynomial of a network of dimensions $l = 5, w = 4$, $h(p) = h_{5,4}(p)$, and the full Hermite interpolation polynomial, $H(p) = H_{5,4}(p)$.

When w and l are both even numbers, by Corollary 2 it follows that the full Hermite interpolation polynomial of order (l, w), $H_{l,w}$, is the same for both kinds of networks $h^{[1]}_{l,w}$ and $h^{[2]}_{l,w}$. The next example deals with networks of even dimensions.

Example 2. The reliability polynomials for the two hammock networks of dimensions $l = w = 4$ ($h^{[1]}_{4,4}(p)$ and $h^{[2]}_{4,4}(p)$) are given by the formulas:

$$h^{[1]}_{4,4}(p) = 24p^4 - 24p^5 - 18p^6 - 40p^7 + 98p^8 + 40p^9 - 6p^{10} - 472p^{11} + 852p^{12} - 696p^{13} + 308p^{14} - 72p^{15} + 7p^{16},$$

$$h^{[2]}_{4,4}(p) = 18p^4 - 12p^5 - 14p^6 - 32p^7 + 10p^8 + 156p^9 - 126p^{10} - 128p^{11} + 188p^{12} - 24p^{13} - 68p^{14} + 40p^{15} - 7p^{16},$$

while the full Hermite interpolation polynomial of order $(4, 4)$ is written:

$$H_{4,4}(p) = 35p^4 - 84p^5 + 70p^6 - 20p^7.$$

The graphs of these polynomials are represented in Figure 2.

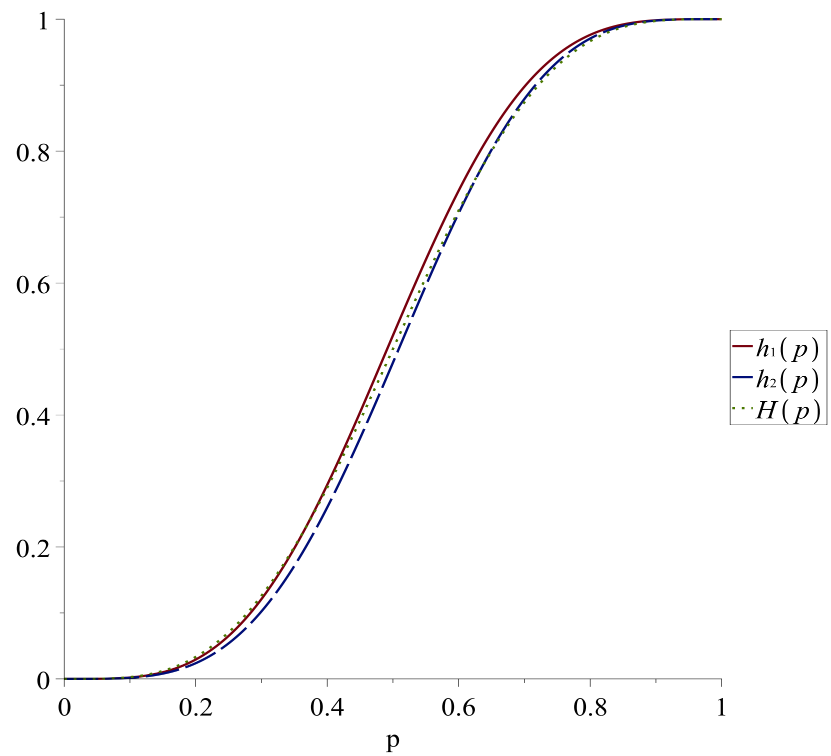


Figure 2: The reliability polynomials of the networks of dimensions $l = w = 4$ ($h_1 = h^{[1]}_{4,4}, h_2 = h^{[2]}_{4,4}$) and the full Hermite interpolation polynomial ($H = H_{4,4}$).

Remark 3. If k, m are two positive integers, $f : [a, b] \to \mathbb{R}$ is a function of class $C^r[a, b]$ ($r = \max\{k-1, m-1\}$), and $g : [a, b] \to \mathbb{R}$ is the function defined by

$$g(x) = f(a) + f(b) - f(a + b - x),$$

for all $x \in [a, b]$, then:

$$g(a) = f(a),\ g(b) = f(b),$$

and, for any $i = 1, 2, \ldots, r$,

$$g^{(i)}(a) = (-1)^{i+1} f^{(i)}(b),\ g^{(i)}(b) = (-1)^{i+1} f^{(i)}(a).$$

It follows that, if $H_{k,m}(x)$ is the full Hermite interpolation polynomial of order (k, m) interpolating f at the points $x_0 = a$, $x_1 = b$, then the full Hermite interpolation polynomial of order (m, k) interpolating the function g at the points $x_0 = a$, $x_1 = b$ is $\bar{H}_{m,k}(x) = f(a) + f(b) - H_{k,m}(a + b - x)$.

Now, let $h^{[i]}_{l,w}(p)$ be the reliability polynomial of a hammock network of kind i, ($i = 1$ or $i = 2$) and $h^{[2/i]}_{w,l}(p)$ be the reliability polynomial of the dual network.

By Corollary 1 we have $h_{w,l}^{[2/i]}(p) = 1 - h_{l,w}^{[i]}(1-p)$. Using Remark 3 it follows that, if $H_{k,m}(p)$ is the full Hermite polynomial of order (k,m) interpolating $h_{l,w}^{[i]}(p)$, then the polynomial

$$\bar{H}_{m,k}(p) = 1 - H_{k,m}(1-p)$$

is the full Hermite polynomial of order (m,k) for $h_{w,l}^{[2/i]}(p)$. As a consequence, it suffices to find the full Hermite polynomial for $h_{l,w}^{[i]}(p)$ with $l \le w$.

Corollary 2 establishes the formula of the full Hermite interpolation polynomial of order (l,w) for the reliability polynomial $h_{l,w}(p)$, using the fact that all the derivatives $h_{l,w}^{(i)}$ at 0 and 1 are equal to 0 for $i \le l-1$ and, respectively, $i \le w-1$. We could construct a better interpolation polynomial (of higher order) if we knew, for instance, $h_{l,w}^{(l)}(0)$, $h_{l,w}^{(l+1)}(0)$, and so on. By formula (7) it is easy to see that

$$h_{l,w}^{(l)}(0) = l!\, b_l \quad \text{and} \quad h_{l,w}^{(l+1)}(0) = (l+1)!\, b_{l+1}, \tag{14}$$

so, for a better approximation of the reliability, we should know the coefficients b_l and b_{l+1}. That is why we dedicate the next section to the calculation of these two coefficients.

4. THE FIRST AND THE SECOND NON-ZERO COEFFICIENTS OF THE RELIABILITY POLYNOMIAL OF A HAMMOCK NETWORK

By the formulas (6) and (7), we have that $b_l = N_l$, so the first non-zero coefficient is equal to the number of pathsets with exactly l edges, i.e. the number of $\mathbf{X}$ – paths with exactly l edges from an input node $(0,i)$ to an output node (l,j), and containing only points in the rectangle $[0,l] \times [0,w]$. So, in order to find b_l, we need to solve a problem of counting lattice paths.

Theorem 5. *If* $h(p) = \sum_{j=l}^{wl} b_j p^j$ *is the reliability polynomial of a hammock network of dimensions* (l,w) *with* $l \le w+1$*, then the coefficient* b_l *is given by one of the following formulas:*

$$b_l = b_l^{(l,w)} = (w+2) \cdot 2^{l-1} - (l+1) \cdot \binom{l}{\left[\frac{l}{2}\right]}, \tag{15}$$

if l *is odd,*

$$b_l = b_l^{(l,w)} = (w+2) \cdot 2^{l-1} - \left(l + \tfrac{1}{2}\right) \cdot \binom{l}{\frac{l}{2}}, \tag{16}$$

if l *is even and* w *is odd, and*

$$b_l = b_l^{(l,w,i)} = (w+2) \cdot 2^{l-1} - \left(l + \left[\tfrac{i}{2}\right]\right) \cdot \binom{l}{\frac{l}{2}}, \tag{17}$$

if l and w are both even and the network is of the kind i, where $i = 1, 2$.

Proof. The main idea of this proof is modeled on that of Myers [**16**], which counts the number of zig-zag paths on an $n \times n$ chessboard. However, in our case the domain is a rectangle, not a square. For any integers i and j, we denote by $N_{i,j}^{(l)}$ the number of $\mathbf{X}$ – paths with l edges connecting the points $(0, i)$ and (l, j) (all of them, not only those lying into the rectangle $[0, l] \times [0, w]$). We have:

$$(18) \qquad N_{i,j}^{(l)} = \begin{cases} 0, & \text{if } |i-j| > l \ \text{ or } \ i-j \equiv l+1 \pmod 2 \\ \binom{l}{\frac{l+i-j}{2}}, & \text{if } |i-j| \le l \ \text{ and } \ i-j \equiv l \pmod 2. \end{cases}$$

For any $i, j \in \{0, 1, \ldots, w\}$, we denote by $n_{i,j}^{(l,w)}$ the number of $\mathbf{X}$ – paths with l edges connecting the points $(0, i)$ and (l, j) lying into the rectangle $[0, l] \times [0, w]$. Obviously, $n_{i,j}^{(l,w)} = 0$ if $i - j \equiv l + 1 \pmod 2$ or $|i - j| > l$. If $i - j \equiv l \pmod 2$ and $|i-j| \le l$, then $n_{i,j}^{(l,w)}$ is equal to the difference between the total number of $\mathbf{X}$ – paths, $N_{i,j}^{(l)}$ and the number of $\mathbf{X}$ – paths running bellow Ox axis or above the straight line $y = w$. An $\mathbf{X}$ – path with l edges from $(0, i)$ to (l, j) can run out the rectangle $[0, l] \times [0, w]$ only if $i + j < l$ or $i + j > 2w - l$. By the condition $l \le w + 1$, these cases cannot occur simultaneously.

Suppose that $i + j < l$. Let η be an $\mathbf{X}$ – path from $(0, i)$ to (l, j) that runs bellow Ox axis and let $(k, -1)$ be the rightmost point of intersection between η and the straight line $y = -1$. Then, by reflecting w.r.t the line $y = -1$ the part of η from $(k, -1)$ to (l, j), we obtain an $\mathbf{X}$ – path from $(0, i)$ to $(l, -j - 2)$. By the reflection principle (see [**8**]), the number of $\mathbf{X}$ – paths from $(0, i)$ to (l, j) running bellow Ox axis is equal to the number of $\mathbf{X}$ – paths from $(0, i)$ to $(l, -j - 2)$, so $n_{i,j}^{(l,w)} = N_{i,j}^{(l)} - N_{i,-j-2}^{(l)}$.

In the same way, if $i + j > 2w - l$, the number of $\mathbf{X}$ – paths from $(0, i)$ to (l, j) running above the straight line $y = w$ is equal to the number of $\mathbf{X}$ – paths from $(0, i)$ to $(l, 2(w + 1) - j)$, so $n_{i,j}^{(l,w)} = N_{i,j}^{(l)} - N_{i,2(w+1)-j}^{(l)}$. Thus, for any $i, j \in \{0, 1, \ldots, w\}$ such that $i - j \equiv l \pmod 2$, we have

$$(19) \qquad n_{i,j}^{(l,w)} = \begin{cases} 0, & \text{if } |i-j| > l \\ \binom{l}{\frac{l+i-j}{2}}, & \text{if } \ |l-i| \le j \le \min\{l+i, 2w-l-i\} \\ \binom{l}{\frac{l+i-j}{2}} - \binom{l}{\frac{l+i+j+2}{2}}, & \text{if } \ j < l - i \\ \binom{l}{\frac{l+i-j}{2}} - \binom{l}{\frac{l+i+j-2(w+1)}{2}}, & \text{if } \ j > 2w - l - i. \end{cases}$$

Let $n_i^{(l,w)}$ be the number of $\mathbf{X}$ – paths from the input node $(0, i)$ to a target node, lying in the rectangle $[0, l] \times [0, w]$, i.e. $n_i^{(l,w)} \stackrel{\text{def}}{=} \sum_{j=0}^{w} n_{i,j}^{(l,w)}$. As we shall see,

$n_i^{(l,w)}$ can be written as a sum of binomial coefficients:

$$n_i^{(l,w)} = \sum_{s=0}^{l} \delta_{i,s} \binom{l}{s}, \tag{20}$$

where the coefficients $\delta_{i,s}$ can take the values $0, 1, -1$, depending on the parity of w and l and on the kind of the hammock network.

We study now the hammock networks of of the first kind (with even nodes). By Remark 1, if one of the numbers l and w is odd, it is sufficient to study the first kind network.

Case I. If l and w are both odd: $l = 2k+1$, $w = 2m+1$, $k \le m$, then $b_l = \sum_{r=0}^{m} n_{2r}^{(l,w)}$, where, for all $r = 0, 1, \ldots, m$, the numbers $n_{2r}^{(l,w)}$ are computed using (19) as follows:

$$n_0^{(l,w)} = n_{0,l}^{(l,w)} + n_{0,l-2}^{(l,w)} + \ldots + n_{0,3}^{(l,w)} + n_{0,1}^{(l,w)}$$

$$= \binom{l}{0} + \binom{l}{1} - \binom{l}{l} + \binom{l}{2} - \binom{l}{l-1} + \ldots + \binom{l}{k-1} - \binom{l}{k+3} + \binom{l}{k} - \binom{l}{k+2},$$

so we have:

$$n_0^{(l,w)} = \binom{l}{0} + \binom{l}{1} + \ldots + \binom{l}{k-1} + \binom{l}{k} + 0 \cdot \binom{l}{k+1} - \binom{l}{k+2} - \ldots - \binom{l}{l}.$$

In the same manner we can calculate:

$$n_2^{(l,w)} = \binom{l}{0} + \binom{l}{1} + \ldots + \binom{l}{k} + \binom{l}{k+1} + 0 \cdot \binom{l}{k+2} - \binom{l}{k+3} - \ldots - \binom{l}{l},$$

....

$$n_{2(m-1)}^{(l,w)} = -\binom{l}{0} - \binom{l}{1} - \ldots - \binom{l}{k-2} + \binom{l}{k-1} + \binom{l}{k} + \ldots + \binom{l}{l},$$

$$n_{2m}^{(l,w)} = -\binom{l}{0} - \binom{l}{1} - \ldots - \binom{l}{k-2} - \binom{l}{k-1} + \binom{l}{k} + \ldots + \binom{l}{l}.$$

Thus, we obtained that

$$n_{2r}^{(l,w)} = \sum_{s=0}^{l} \varepsilon(r,s) \binom{l}{s}, \tag{21}$$

where the coefficients $\varepsilon(r,s)$ are given by the formula:

$$\varepsilon(r,s) = \begin{cases} 1\,, & \text{if} \quad k-m+r \le s \le k+r \\ 0\,, & \text{if} \quad s = k+r+1 \\ -1\,, & \text{if} \;\; s < k-m+r \;\; \text{or} \;\; s > k+r+1 \end{cases} \tag{22}$$

It follows that

$$b_l = \sum_{r=0}^{m}\sum_{s=0}^{l}\varepsilon(r,s)\binom{l}{s} = \sum_{s=0}^{l}\binom{l}{s}\sum_{r=0}^{m}\varepsilon(r,s)$$

$$= \sum_{s=0}^{k}\binom{l}{s}(m+1-2k+2s) + \sum_{s=k+1}^{2k+1}\binom{l}{s}(m+2k+2-2s)$$

$$= \sum_{s=0}^{k}\binom{l}{s}(m+1-2k+2s) + \sum_{s=0}^{k}\binom{l}{l-s}(m-2k+2s) = \sum_{s=0}^{k}\binom{l}{s}(2m+1-4k+4s)$$

$$= (2m+1-4k)\sum_{s=0}^{k}\binom{2k+1}{s} + 4\sum_{s=0}^{k}s\binom{2k+1}{s} = (2m+1-4k)\cdot 2^{2k} + 4(2k+1)\sum_{s=1}^{k}\binom{2k}{s-1}$$

$$= (2m+1-4k)\cdot 2^{2k} + 2(2k+1)\left(2^{2k} - \binom{2k}{k}\right) = (2m+3)\cdot 2^{2k} - 2(2k+1)\binom{2k}{k}$$

$$= (2m+3)\cdot 2^{2k} - (2k+2)\binom{2k+1}{k} = (w+2)\cdot 2^{l-1} - (l+1)\binom{l}{\left[\frac{l}{2}\right]},$$

so the equation (15) is verified in this case.

Case II. If l is odd and w is even : $l = 2k+1$, $w = 2m$, $k \le m$ then it can be proved similarly that the coefficients $\varepsilon(r,s)$ in the equation (21) are given by the formula:

$$\varepsilon(r,s) = \begin{cases} 1\,, & \text{if} \quad k-m+r+1 \le s \le k+r \\ 0\,, & \text{if} \quad s = k-m+r \ \text{ or } \ s = k+r+1 \\ -1\,, & \text{if} \quad s < k-m+r \ \text{ or } \ s > k+r+1 \end{cases} \tag{23}$$

It follows that

$$b_l = \sum_{r=0}^{m}\sum_{s=0}^{l}\varepsilon(r,s)\binom{l}{s} = \sum_{s=0}^{l}\binom{l}{s}\sum_{r=0}^{m}\varepsilon(r,s)$$

$$= \sum_{s=0}^{k}\binom{l}{s}(m-2k+2s) + \sum_{s=k+1}^{2k+1}\binom{l}{s}(m+2k+2-2s)$$

$$= \sum_{s=0}^{k}\binom{l}{s}(2m-4k+4s) = (2m-4k)\cdot 2^{2k} + 4(2k+1)\sum_{s=1}^{k}\binom{2k}{s-1}$$

$$= (2m+2)\cdot 2^{2k} - 2(2k+1)\binom{2k}{k} = (2m+2)\cdot 2^{2k} - (2k+2)\binom{2k+1}{k}$$

and, since $l = 2k+1$ and $w = 2m$, the formula (15) follows.

Case III. If l is even and w is odd : $l = 2k$, $w = 2m+1$, $k \le m+1$ then

$$\varepsilon(r,s) = \begin{cases} 1, & \text{if} \quad k-m+r \le s \le k+r \\ 0, & \text{if} \quad s = k-m+r-1 \\ -1, & \text{if} \quad s < k-m+r-1 \text{ or } s > k+r \end{cases} \tag{24}$$

Hence,

$$b_l = \sum_{s=0}^{l} \binom{l}{s} \sum_{r=0}^{m} \varepsilon(r,s)$$

$$= \sum_{s=0}^{k-1} \binom{l}{s}(m-2k+2+2s) + (m+1)\binom{l}{k} + \sum_{s=k+1}^{2k} \binom{l}{s}(m+2k+1-2s)$$

$$= \sum_{s=0}^{k-1} \binom{l}{s}(m-2k+2+2s) + (m+1)\binom{l}{k} + \sum_{j=0}^{k-1} \binom{l}{l-s}(m-2k+1+2s)$$

$$= \sum_{s=0}^{k-1} \binom{l}{s}(2m-4k+3+4s) + (m+1)\binom{l}{k} =$$

$$= (2m-4k+3)\cdot\left(2^{2k-1} - \frac{1}{2}\binom{l}{k}\right) + (m+1)\binom{l}{k} + 4\sum_{s=1}^{k-1} s\binom{l}{s}$$

$$= (2m-4k+3)\cdot 2^{2k-1} + \frac{4k-1}{2}\binom{2k}{k} + 8k\sum_{s=1}^{k-1}\binom{2k-1}{s-1}$$

$$= (2m-4k+3)\cdot 2^{2k-1} + \frac{4k-1}{2}\binom{2k}{k} + 4k\sum_{s=0}^{2k-1}\binom{2k-1}{s} - 8k\binom{2k-1}{k}$$

$$= (2m+3)\cdot 2^{2k-1} + \frac{4k-1}{2}\binom{2k}{k} - 4k\binom{2k}{k} = (2m+3)\cdot 2^{2k-1} - \frac{4k+1}{2}\binom{2k}{k}$$

and, since $l = 2k$ and $w = 2m+1$, the formula (16) follows.

Case IV. If l and w are both even $l = 2k$, $w = 2m$, $k \le m$, then we can have two different networks: of the first kind and, respectively, of the second kind. To shorten the proof of the formula (17) we notice that, for the first kind network, the number of $\mathbf{X}$ – paths from a source node $(0, 2i)$, $i = 0, 1, \ldots, m$ to a target node, $(l, 2j)$, $j = 0, 1, \ldots, m$ is twice the number of $\mathbf{X}$ – paths in the network of dimensions $(l-1, w)$ from a source node $(0, 2i)$, $i = 0, 1, \ldots, m$ to a target node, $(l-1, 2j+1)$, $j = 0, 1, \ldots, m-1$ (for any $\mathbf{X}$ – path in the network of length $l-1$ there are exactly two final edges to obtain an $\mathbf{X}$ – path in the network of length l). We can also see that, in the network of dimensions $(l+1, w)$, the number of $\mathbf{X}$ – paths from a source node $(0, 2i)$, $i = 0, 1, \ldots, m$ to a target node, $(l+1, 2j+1)$, $j = 0, 1, \ldots, m-1$ is twice the number of $\mathbf{X}$ – paths in the second kind network of dimensions (l,w), from a source node $(0,2i+1)$, $i=0,1,\ldots,m-1$ to a target node, $(l,2j+1)$, $j=0,1,\ldots,m-1$. Thus, we can write:

$$b_l^{(l,w,1)} = 2b_{l-1}^{(l-1,w)} \tag{25}$$

and

$$b_l^{(l,w,2)} = \frac{1}{2}b_{l+1}^{(l+1,w)}. \tag{26}$$

Since $l-1$ and $l+1$ are odd numbers, we can use the equation (15) and the formula (17) follows. □

Dăuş et al. investigated in [**5**] the sequences of general term $a_l = b_l^{(l,w)}$, where w is constant and $l=1,2,\ldots$. For odd values of w they found the same sequences as the ones obtained by Malešević in [**12**], [**13**], expressing the number of meaningful differential operations of order l on the space $\mathbb{R}^w$. The same sequences give the number of compositions of differential operations and Gateaux directional derivative on the space $\mathbb{R}^{w-1}$ (see [**14**]). In the particular case $w=3$, the Fibonacci sequence is obtained (which is also related to the number of higher order non-trivial compositions of the differential operations and the directional derivative on the space $\mathbb{R}^n$ [**11**]). For even values of w, some other significant sequences are obtained (see [**16**], [**20**]).

In this paper, by Theorem 5, we introduce a different approach: by keeping l constant and varying w such that $w \geq l-1$, the sequence of general term $c_w = b_l^{(l,w)}$, $w=l-1,l,\ldots$ looks like an *arithmetic progression*. Indeed, for odd values of l, by (15), the sequence $\{c_w\}_{w\geq l-1}$ is an arithmetic progression with common difference 2^{l-1}. This is also true for even values of l, if we use the notations $c_{2m}^{[1]} = b_l^{(l,2m,1)}$, $c_{2m}^{[2]} = b_l^{(l,2m,2)}$ and $c_{2m} = \frac{1}{2}\left(c_{2m}^{[1]} + c_{2m}^{[2]}\right)$. We also note that in the case of even l, all the three sequences $c_{2m}^{[1]} = b_l^{(l,2m,1)}$, $c_{2m}^{[2]} = b_l^{(l,2m,2)}$ and $c_{2m+1} = b_l^{(l,2m+1)}$ are arithmetic progressions with common difference 2^l.

Remark 4. If l and w are both even then, by (17), we have:

$$b_l^{(l,w,1)} = b_l^{(l,w,2)} + \binom{l}{\frac{l}{2}}. \tag{27}$$

In a hammock network of dimensions (l,w), *an extreme node* is a node of coordinates $(0,0)$, $(0,w)$, $(l,0)$ or (l,w). We notice that any extreme node is either a source, or a target node. A network may have 4 extreme nodes (if w and l are both even and the network is of the first kind), 2 extreme nodes (if at least one of w and l is odd), or 0 (if w and l are both even and the network is of the second kind). We can see that, for a given network, the number of $\mathbf{X}$ – paths lying in the rectangle $[0,l]\times[0,w]$ formed by l edges connecting an extreme node of the form $(0,0)$, $(0,w)$ to a target node, or an extreme node of the form $(l,0)$, (l,w) to a source node is the same, $n_0 = n_0^{(l,w)}$.

Lemma 1. *In a hammock network of dimensions (l,w) with $l \leq w+1$ having at least one extreme node, n_0 is given by the formula:*

$$n_0 = \binom{l}{[\frac{l}{2}]}. \tag{28}$$

Proof. By using equation (21) from the proof of Theorem 5, for $r=0$, we have:

$$n_0 = n_0^{(l,w)} = \sum_{s=0}^{l} \varepsilon(0,s)\binom{l}{s}.$$

Since $l \leq w+1$, n_0 depends on l only. If $l = 2k+1$, by applying (22) we obtain:

$$n_0 = \binom{l}{0} + \ldots + \binom{l}{k-1} + \binom{l}{k} + 0 \cdot \binom{l}{k+1} - \binom{l}{k+2} - \ldots - \binom{l}{l} = \binom{l}{k}.$$

If $l = 2k$, we apply the equation (24) and we obtain:

$$n_0 = \binom{l}{0} + \ldots + \binom{l}{k-1} + \binom{l}{k} - \binom{l}{k+1} - \ldots - \binom{l}{l} = \binom{l}{k},$$

so the formula (28) is verified in both cases. □

Theorem 6. *If $h(p) = \sum_{j=l}^{wl} b_j p^j$ is the reliability polynomial of a hammock network of dimensions (l,w) with $l \leq w+1$, then the coefficient b_{l+1} is given by the formulas:*

$$b_{l+1} = b_{l+1}^{(l,w)} = \binom{l-1}{\left[\frac{l-1}{2}\right]} - b_l^{(l,w)}, \tag{29}$$

whether l or w is odd,

$$b_{l+1} = b_{l+1}^{(l,w,1)} = -b_l^{(l,w,1)}, \tag{30}$$

if l and w are both even and the network is of the first kind and

$$b_{l+1} = b_{l+1}^{(l,w,2)} = \binom{l}{\frac{l}{2}} - b_l^{(l,w,2)}, \tag{31}$$

if l and w are both even and the network is of the second kind.

Proof. By the formulas (6) and (19) we can see that the coefficient of p^{l+1} is

$$b_{l+1} = N_{l+1} - (wl - l)N_l.$$

Since the coefficient $N_l = b_l$ was calculated above, we have to find N_{l+1}, the number of pathsets with exactly $l+1$ edges containing an $\mathbf{X}$ − path from a source node to a target node. Obviously,

$$N_{l+1} = N_l(wl - l) - \bar{N} \Rightarrow b_{l+1} = -\bar{N},$$

where $\bar{N}$ is the number of such pathsets counted twice (because they contain two $\mathbf{X}$ – paths from a source node to a target node). We calculate this number for each case.

Case I. Consider, firstly, the case when w and l are both even numbers: $l = 2k$, $w = 2m$ and the network is of the first kind. It is easy to see that any $(l+1)$-pathset counted twice either is formed by an $(l-1)$-$\mathbf{X}$ – path from a node $A_{1,2i+1}$ to a target node $A_{2k,2j}$ and the edges $A_{0,2i}A_{1,2i+1}$, $A_{0,2i+2}A_{1,2i+1}$, or it contains an $(l-1)$-$\mathbf{X}$ – path from a source node $A_{0,2i}$ to a node $A_{2k-1,2j+1}$ and the edges $A_{2k-1,2j+1}A_{2k,2j}$, $A_{2k-1,2j+1}A_{2k,2j+2}$ (where $i, j \in \{0, 1, \ldots, m-1\}$). It follows that $\bar{N} = 2b_{l-1}^{(l-1,w)} = b_l^{(l,w,1)}$ (by equation (25)). Since $b_{l+1} = -\bar{N}$, the equation (30) follows.

Case II. If w and l are both even numbers: $l = 2k$, $w = 2m$ and the network is of the second kind, we apply the same method to calculate $\bar{N}$, but in this case there are some $(l-1)$-$\mathbf{X}$ – paths which can be connected to a source node (or, respectively, to a target node) by a single edge. It is the case of $(l-1)$-$\mathbf{X}$ – paths connecting $A_{1,0}$ or $A_{1,2m}$ to a target node, or the $(l-1)$-$\mathbf{X}$ – paths connecting $A_{2k-1,0}$ or $A_{2k-1,2m}$ to a source node. These nodes are extreme nodes in the corresponding networks of dimensions $(l-1, w)$, so, by Lemma 1, we have:

$$n_0 = n_0^{(l-1,w)} = \binom{2k-1}{k-1} = \frac{1}{2}\binom{2k}{k}.$$

Using (25) it follows that:

$$\bar{N} = 2b_{l-1}^{(l-1,w)} - 4n_0 = b_l^{(l,w,1)} - 2\binom{2k}{k},$$

and, by Remark 4, the equation (31) is obtained.

If at least one of the numbers l and w is odd, we can consider only the networks of the first kind.

Case III. If w is odd then, by the same reasoning as above, one can write:

$$\bar{N} = 2b_{l-1}^{(l-1,w)} - 2n_0^{(l-1,w)} = 2b_{l-1}^{(l-1,w)} - 2\binom{l-1}{\left[\frac{l-1}{2}\right]},$$

by Lemma 1. By the formulas (15) and (16) we can see that

$$2b_{l-1}^{(l-1,w)} = b_l^{(l,w)} + \binom{l-1}{\left[\frac{l-1}{2}\right]} \tag{32}$$

so it follows that:

$$\bar{N} = b_l^{(l,w)} - \binom{l-1}{\left[\frac{l-1}{2}\right]},$$

and, since $b_{l+1} = -\bar{N}$, the equation (29) is obtained.

Case IV. If l is odd and w is even: $l = 2k+1$, $w = 2m$, then one can write:

$$\bar{N} = b_{l-1}^{(l-1,w,2)} + b_{l-1}^{(l-1,w,1)} - 2n_0^{(l-1,w)} = b_{l-1}^{(l-1,w,2)} + b_{l-1}^{(l-1,w,1)} - 2\binom{2k}{k}.$$

By the relation (27) and equation (26) we obtain that

$$\bar{N} = 2b_{l-1}^{(l-1,w,2)} - \binom{2k}{k} = b_l^{(l,w)} - \binom{2k}{k},$$

and the equation (29) is obtained. □

Let $h(p) = \sum_{j=l}^{wl} b_j p^j$ be the reliability polynomial of a hammock network of dimensions (l,w) with $l \leq w+1$. Now, since we know the coefficients b_l and b_{l+1}, we can apply the formulas (14), (8)-(10) and Theorem 4 to calculate the full Hermite polynomial of order $(l+2,w)$ interpolating $h(p)$ and the next corollary follows instantly.

Corollary 3. *If $l \leq w+1$, then the full Hermite polynomial of order $(l+2,w)$, interpolating the reliability polynomial of a hammock network of dimensions (l,w) at $p_0 = 0,\ p_1 = 1$ can be written:*

$$H_{l+2,w}(p) = b_l(1-p)^w p^l(1 + w\cdot p) + b_{l+1}(1-p)^w p^{l+1}$$
$$+ p^{l+2}\sum_{r=0}^{w-1}\binom{l+r-1}{l-1}(1-p)^r.$$

If the dimensions (l,w) of the hammock network are such that $|w-l| \leq 1$, then we can apply Theorems 5 and 6 for the dual network as well. Let $h(p) = \sum_{j=l}^{wl} b_j p^j$ be the reliability polynomial of a hammock network of dimensions (l,w), and $\bar{h}(p) = \sum_{j=w}^{wl} \bar{b}_j p^j$ be the reliability polynomial of the dual network. By Corollary 1 , since $\bar{h}(p) = 1 - h(1-p)$, it follows that

$$\bar{h}^{(j)}(0) = j!\,\bar{b}_j = (-1)^{j-1}h^{(j)}(1) \text{ for } j = 1,2,\ldots,$$

so we have:

$$h^{(w)}(1) = (-1)^{w-1}w!\,\bar{b}_w \ \text{ and } \ h^{(w+1)}(1) = (-1)^w(w+1)!\,\bar{b}_{w+1}.$$

Consequently, since $h^{(w)}(1)$ and $h^{(w+1)}(1)$ are known, we can apply Theorem 4 to calculate the full Hermite polynomial of order $(l+2,w+2)$.

Corollary 4. *If $|w-l| \leq 1$, then the full Hermite polynomial of order $(l+2,w+2)$, interpolating the reliability polynomial $h_{l,w}(p)$ at $p_0 = 0,\ p_1 = 1$ can be written in the following form:*

$$H_{l+2,w+2}(p) = p^{l+2}\sum_{r=0}^{w+1}\binom{l+1+r}{l+1}(1-p)^r + p^l(1-p)^{w+2}\left[b_l(1+(w+2)p) + b_{l+1}p\right]$$
$$-\,p^{l+2}(1-p)^w\left[\bar{b}_w(1+(l+2)(1-p)) + \bar{b}_{w+1}(1-p)\right].$$

Consider, firstly, the case of a square network ($l = w$).

Corollary 5. *The full Hermite polynomial of order* $(l+2, l+2)$, *interpolating the reliability polynomial* $h_{l,l}(p)$ *at* $p_0 = 0,\ p_1 = 1$ *can be written in the following form:*

$$H_{l+2,l+2}(p) = p^{l+2}\sum_{r=0}^{l+1}\binom{l+1+r}{l+1}(1-p)^r \tag{33}$$

$$+\, b_l p^l (1-p)^l (1-2p)[1+(l+1)p(1-p)] + \pi(p).$$

where

(34)

$$\pi(p) = \begin{cases} \binom{l-1}{\frac{l-1}{2}} p^{l+1}(1-p)^{l+1}(1-2p)\ , & \text{if} \quad l = 2k+1 \\ \binom{l}{\frac{l}{2}} p^{l+2}(1-p)^l(1+l(1-p))\ , & \text{if} \quad l = 2k, \text{ and } i = 1 \\ \binom{l}{\frac{l}{2}} p^{l+1}(1-p)^l(1-lp(1-p)+2(1-p)^2)\ , & \text{if} \quad l = 2k, \text{ and } i = 2 \end{cases}$$

and

$$b_l = (l+2)\cdot 2^{l-1} - 2\left(l + \left[\tfrac{i}{2}\right]\right)\binom{l-1}{\left[\frac{l-1}{2}\right]}, \tag{35}$$

where i *is the kind of the network if* l *is even and* $i = 0$ *if* l *is odd.*

Proof. First of all, the formula (35) expressing b_l when $w = l$ and l is either odd, or even and the network is of the first type, easily follows by unifying the formulas (15) and (17) (for $i = 1$) in Theorem 5.

Now, we write the Corollary 4 for $w = l$. If $l = w = 2k+1$, then $h_{l,l}(p) \equiv \bar{h}_{l,l}(p)$, so $b_l = \bar{b}_l$ and $b_{l+1} = \bar{b}_{l+1}$. On the other hand, we can use the formula (29) for b_{l+1}, so we obtain (33) with $\pi(p) = \binom{l-1}{\frac{l-1}{2}} p^{l+1}(1-p)^{l+1}(1-2p)$.

If $l = w = 2k$ and the network is of the first kind, then $b_l = b_l^{(l,l,1)}$, $\bar{b}_l = b_l^{(l,l,2)} = b_l - \binom{l}{\frac{l}{2}}$. By Theorem 6 we know that $b_{l+1} = b_{l+1}^{(l,l,1)} = -b_l$ and $\bar{b}_{l+1} = b_{l+1}^{(l,l,2)} = \binom{l}{\frac{l}{2}} - \bar{b}_l$, so we obtain (33) with $\pi(p) = \binom{l}{\frac{l}{2}} p^{l+2}(1-p)^l(1+l(1-p))$. For a network of the second kind, the result follows similarly. □

Example 3. For a hammock network of dimensions $w = l = 9$, the reliability polynomial ($h(p) = h_{9,9}(p)$) and the full Hermite interpolation polynomials of order $(9,9)$ and, respectively, $(11,11)$ ($H(p) = H_{9,9}(p)$ and $H_1(p) = H_{11,11}(p)$) are represented in Figure 3. As we were expecting, the interpolation polynomial $H_{11,11}(p)$ is closer to the exact reliability polynomial $h = h_{9,9}(p)$ than $H_{9,9}(p)$.

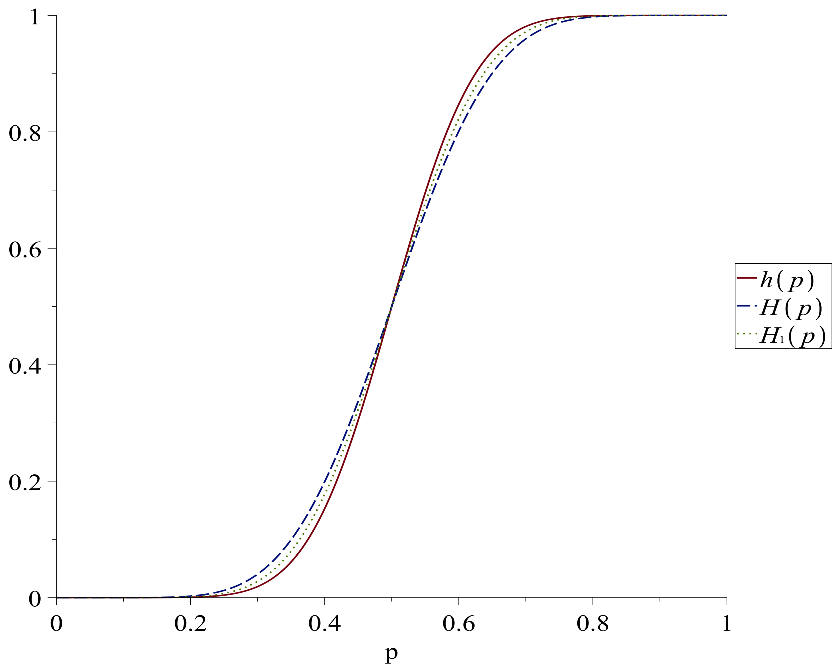


Figure 3: The reliability ($h = h_{9,9}(p)$) and the full Hermite interpolation polynomials ($H = H_{9,9}(p)$ and $H_1 = H_{11,11}(p)$) for a network of dimensions $w = l = 9$.

Finally, we study the case of a hammock network of dimensions $(l, l+1)$. Let $b_j,\ j = l, l+1, \ldots, wl$, be the coefficients of the reliability polynomial of this hammock network, and $\bar{b}_j,\ j = l+1, l+2, \ldots, l(l+1)$, be the coefficients of the reliability polynomial of the dual network (the hammock network of dimensions $(l+1, l)$). Since one of the dimensions l and $l+1$ is odd, by Theorem 6 it follows that

$$b_{l+1} = \binom{l-1}{[\frac{l-1}{2}]} - b_l \quad \text{and} \quad \bar{b}_{l+2} = \binom{l}{[\frac{l}{2}]} - \bar{b}_{l+1}$$

and, from Corollary 4, the following form of the full Hermite interpolation polynomial is obtained.

Corollary 6. *The full Hermite polynomial of order $(l+2, l+3)$, interpolating the reliability polynomial $h_{l,l+1}(p)$ at $p_0 = 0,\ p_1 = 1$ can be written:*

$$H_{l+2,l+3}(p) = p^{l+2} \sum_{r=0}^{l+2} \binom{l+1+r}{l+1} (1-p)^r$$

$$+ b_l p^l (1-p)^{l+3}[1+(l+2)p] + \binom{l-1}{[\frac{l-1}{2}]} p^{l+1}(1-p)^{l+3}$$
$$- \bar{b}_{l+1} p^{l+2}(1-p)^{l+1}[1+(l+3)(1-p)] - \binom{l}{[\frac{l}{2}]} p^{l+2}(1-p)^{l+2}.$$

Acknowledgments. This research was partially funded by the European Union through the European Regional Development Fund under the Competitiveness Operational Program (BioCell-NanoART = Novel Bio-inspired Cellular Nano-architectures, POC-A1-A1.1.4-E nr. 30/2016).

The authors are grateful to the referee for the valuable comments and suggestions on improving the paper.

REFERENCES

1. R. Burden, D. Faires: *Numerical Analysis* (Chapter 3: Interpolation and Polynomial Approximation), Brooks/Cole, Cengage Learning, 2011.
2. C.J. Colbourn: *The Combinatorics of Network Reliability*, Oxford University Press, Oxford, 1987.
3. S.R. Cowell, V. Beiu, L. Dăuş, P. Poulin: *On the exact reliability enhancements of small hammock networks*, IEEE Access **6** (2018), 25411–25426.
4. P. Davis: *Interpolation and Approximation*, Dover Publications, Inc., New York, 1975.
5. L. Dăuş, V. Beiu, S.R. Cowell, P. Poulin: *Brick-wall lattice paths and applications*, Tech. Rep. arXiv, 17 Apr. 2018, http://arxiv.org/abs/222.7960.
6. L. Dăuş, M. Jianu: *The shape of the reliability polynomial of a hammock network*, arXiv 1901.0436 [math.CO], 2019.
7. V. Drăgoi, S.R. Cowell, V. Beiu, S. Hoară, P. Gaşpar: *How reliable are compositions of series and parallel networks compared with hammocks?*, International Journal of Computers, Communications and Control, **13 (5)** (2018) 772–791.
8. W. Feller: *An Introduction to Probability Theory and its Applications, vol. 1, third ed., John Wiley & Sons, Inc., 1968.*
9. Ch. Hermite: *Sur la formule d'interpolation de Lagrange*, J. Reine Angew. Math., **84** (1878), 70–79.
10. K. Humphreys: *A history and a survey of lattice path enumeration*, Journal of Statistical Planning and Inference **140** (2010), 2237–2254.
11. I. Jovović, B. Malešević: *Some enumerations of non-trivial composition of the differential operations and the directional derivative*, Notes on Number Theory and Discrete Mathematics, **23 (1)** (2017) 28—38.
12. B. Malešević: *Some combinatorial aspects of differential operation composition on the space R^n*, Univ. Beograd, Publ. Elektrotehn. Fak., Ser. Mat. **9** (1998), 29–33.
13. B. Malešević: *Some combinatorial aspects of composition of a set of functions*, Novi Sad Journ. Of Math., **36 (1)** (2006) 3–9.

14. B. Malešević, I. Jovović: *The compositions of differential operations and the Gateaux directional derivative*, Journal of Integer Sequences, **10(8)** (2007), 1–11.

15. E.F. Moore, C.E. Shannon: *Reliable circuits using less reliable relays*, Part I, J. Frankl. Inst. **262 (3)** (1956), 191–208.

16. J. Myers: British Mathematical Olympiad 2008 - 2009, Round 1, Problem 1, https://www.polyomino.org.uk/publications/2008/bmo1-2009-q1.pdf (2008).

17. J. von Neumann: *Probabilistic logics and the synthesis of reliable organisms from unreliable components*, Automata studies, Princeton University Press, 1956, 43–98.

18. A. Spitzbart: *A generalization of Hermite's interpolation formula, Amer. Math. Monthly*, 67 (1960), pp. 42-46.

19. R.P. Stanley: *Enumerative Combinatorics:* Volume I, 2nd Edition, Cambridge University Press, 2012.

20. N.J.A. Sloane: The On-Line Encyclopedia of Integer Sequences, https://oeis.org.

21. L. Valiant: *The Complexity of Computing the Permanent:* Theoretical Computer Science **8 (2)** (1979) 189–201.

22. L. Valiant: *The Complexity of Enumeration and Reliability Problems*, SIAM J. Comput. Vol. 8, No. 3, (1979) 410–421.

(Received 05.08.2019)
(Revised 31.10.2019)

Leonard Dăuş
Department of Mathematics
and Computer Science,
Technical University of Civil Engineering,
Bucharest.
E-mail: *daus@utcb.ro*

Marilena Jianu
Department of Mathematics
and Computer Science,
Technical University of Civil Engineering,
Bucharest.
E-mail: *marilena_jianu@yahoo.com*