Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **14** (2020), 033–054.
https://doi.org/10.2298/AADM181018007G

ON CONTRACTIVE MAPPINGS AND DISCONTINUITY AT FIXED POINTS

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This paper deals with an interesting open problem of B.E. Rhoades (Contemporary Math. (Amer. Math. Soc.) 72(1988), 233–245) on the existence of general contractive conditions which have fixed points, but are not necessarily continuous at the fixed points. We propose some more solutions to this problem by introducing two new types of contractive mappings, that is, $\mathcal{A}$-contractive and $\mathcal{A}'$-contractive, which are, in some sense, more appropriate than those of the important previous attempts. We establish some new fixed point results involving these two contractive mappings in compact metric spaces and also in complete metric spaces and show that these contractive mappings are not necessarily continuous at their fixed points. Finally, we suggest an applicable area, where our main results may be employed.

1. INTRODUCTION

It is more than half of a century since the study of fixed points of contractive type mappings has been started. There are many contractive conditions to furnish existence (and uniqueness) of fixed points of different type of mappings in different structures. It is an interesting fact to observe that, among the several classes of contractive conditions, some conditions force the corresponding mapping to be continuous on the entire domain and some conditions to some particular points of the domain. However there are a number of contractive conditions which cannot guarantee the continuity of the mappings, although in most of the cases the continuity

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2010 Mathematics Subject Classification. 47H10, 54H25.
Keywords and Phrases. Fixed point, $\mathcal{A}$-contraction, $\mathcal{A}'$-contraction, orbitally continuous mapping, compact metric space.

of the mapping is assumed. One example of such type of contractive condition is the Kannan type contractive condition and others (see [9]).

In this direction, in the year 1977, Rhoades [14] did a comparitive study with 250 contractive definitions and he tried to find out the most general one. He observed that most of the contractive conditions do not force the mappings to be continuous on the entire domain, although in all cases the mappings are continuous at their respective fixed points. For this type of contractive conditions, (see [2–4,8,10]). In 1988, Rhoades [15] reexamined the continuity of a large number of contractive mappings in detail and showed that all of the contractive conditions assure that the mappings are continuous at the fixed points although continuity is not assumed in all the cases. After that, Rhoades [15] posed an exciting open problem and his question was, whether there exists any contractive condition which can generate fixed points of the underlying mappings, but which does not compel the mapping to be continuous at the fixed points.

The answer of this interesting open question was first achieved by Pant in [13]. In his paper, he proved the following theorem:

Theorem 1.1. [13, Theorem 1] *Let f be a self-mapping of a complete metric space (X, d) such that for any $x, y \in X$,*

(i) $d(fx, fy) \leq \phi(m(x, y))$;

(ii) for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\epsilon < m(x, y) < \epsilon + \delta \implies d(fx, fy) \leq \epsilon,$$

where $m(x, y) = \max\{d(x, fx), d(y, fy)\}$ and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a function such that $\phi(t) < t$ for all $t > 0$. Then f has a unique fixed point, say z. Moreover, f is continuous at z if and only if $\lim_{x \to z} m(x, z) = 0$.

After this, many researchers gave some interesting answers to the above mentioned open problem by taking different types of contractive definitions (see [1,12]).

In the present paper, our main objective is to provide some more general contractive definitions, so that the mappings, satisfying the contractive conditions, possess fixed points but are not necessarily continuous at their fixed points. In order to do this, we introduce the notions of two types of contractive mappings, which we shall call $\mathcal{A}$-contractive and $\mathcal{A}'$-contractive mappings. These two types of contractive mappings contain many well known contractive mappings, such as, Edelstein contractive, Kannan type contractive, Chatterjea type contractive, Hardy-Rogers type contractive and many more. We give another solutions to the open question of Rhoades, mentioned earlier, via these two contractive mappings by instigating some fixed point results concerning these two classes of contractive mappings. Our fixed point results improve and generalize some well known fixed point results in the literature related to contractive type mappings.

2. PRELIMINARIES

We first define the two new contractive mappings which we have mentioned in the previous section.

We denote by $\mathcal{A}$ the collection of all mappings $f : \mathbb{R}_+^3 \to \mathbb{R}_+$ which satisfy the following conditions:

($\mathcal{A}_1$) f is continuous;

($\mathcal{A}_2$) if $v > 0$ and $u < f(u, v, v)$ or $u < f(v, u, v)$ or $u < f(v, v, u)$, then $u < v$;

($\mathcal{A}_3$) $f(u, v, w) \leq u + v + w$ for all $u, v, w \in \mathbb{R}_+$.

Some examples of mappings belonging to the class $\mathcal{A}$ are given by the following:

(i) $f(u, v, w) = \frac{v+w}{2}$.

(ii) $f(u, v, w) = \frac{u+v}{2}$.

(iii) $f(u, v, w) = \frac{1}{2} \max\{u + v, v + w, w + u\}$.

(iv) $f(u, v, w) = \max\{u, v, w\}$.

(v) $f(u, v, w) = \max\{v, w\}$.

(vi) $f(u, v, w) = xu + yv + zw$ where x, y, z are positive real numbers such that $x + y + z = 1$.

(vii) $f(u, v, w) = \sqrt{vw}$.

(viii) $f(u, v, w) = u$.

(ix) $f(u, v, w) = (uvw)^{\frac{1}{3}}$.

Definition 2.2. Let (X, d) be a metric space and T be a self-mapping of X. Then the mapping T is said to be an $\mathcal{A}$-*contractive mapping* if there exists an $f \in \mathcal{A}$ such that

$$d(Tx, Ty) < f(d(x, y), d(x, Tx), d(y, Ty))$$

for all $x, y \in X$ with $x \neq y$.

We denote by $\mathcal{A}'$ the collection of all mappings $f : \mathbb{R}_+^3 \to \mathbb{R}_+$ which satisfy the following conditions:

($\mathcal{A}'_1$) f is continuous;

($\mathcal{A}'_2$) if $v > 0$ and $u < f(u, v, w)$ or $u < f(v, u, v)$ or $u < f(v, v, u)$, then $u < v$;

($\mathcal{A}'_3$) if $v > 0$ and $u < f(v, u + v, 0)$, then $u < v$;

($\mathcal{A}'_4$) if $v \le v_1$, then $f(u, v, w) \le f(u, v_1, w)$ for all $u, w \in \mathbb{R}_+$;

($\mathcal{A}'_5$) $f(u, u, u) \le u$ for all $u \in \mathbb{R}_+$;

($\mathcal{A}'_6$) $f(u, v, w) \le u + v + w$ for all $u, v, w \in \mathbb{R}_+$.

Some examples of mappings f belonging to $\mathcal{A}'$ are the following.

(i) $f(u, v, w) = \frac{1}{3}(u + v + w)$.

(ii) $f(u, v, w) = \frac{1}{2}\max\{u, v, w\}$.

(iii) $f(u, v, w) = \frac{1}{2}(v + w)$.

Definition 2.3. Let (X, d) be a metric space and $T : X \to X$ be a mapping such that there exists an $f \in \mathcal{A}'$ for which

$$d(Tx, Ty) < f(d(x, y), d(x, Ty), d(y, Tx))$$

holds for all $x, y \in X$ with $x \neq y$. Then T is said to be an $\mathcal{A}'$-*contractive mapping.*

In 1971, Ciric [3] introduced a weaker version of continuity for a mapping which is known as orbitally continuous. A mapping T on a metric space (X, d) is said to be *orbitally continuous* if, for any sequence (y_n) in $O_x(T)$, $y_n \to u$ implies $Ty_n \to Tu$ as $n \to \infty$, where $O_x(T) = \{T^n x : n \ge 0\}$ is the orbit of T at x. It is easy to observe that a continuous mapping is orbitally continuous, but not conversely.

In the next section, with the help of orbitally continuity, we derive some fixed point results concerning $\mathcal{A}$-contractive and $\mathcal{A}'$-contractive mappings.

3. MAIN RESULTS

At the beginning of the section, we prove a fixed point result for $\mathcal{A}$-contractive mapping in compact metric space.

Theorem 3.4. *Let (X, d) be a compact metric space and $T : X \to X$ be an $\mathcal{A}$-contractive mapping such that T is orbitally continuous. Then we have the following:*

(1) *T has a unique fixed point.*

(2) *Moreover, if the mapping $f \in \mathcal{A}$, arising in the $\mathcal{A}$-contractiveness of T, satisfies the condition that, if $c > f(u, 0, 0)$ for all $u > 0$, then the sequence $(T^n x)$ of iterates converges to that fixed point for each $x \in X$.*

(3) *Further, if $f(0, 0, u) = 0$ implies $u = 0$, then T is continuous at the fixed point z if and only if $\lim_{x \to z} m(z, x) = 0$, where*

$$m(z, x) = f\left(d(z, x), d(z, Tz), d(x, Tx)\right).$$

Proof. (1) Let $x_0 \in X$ be arbitrary and consider the sequence (x_n) where $x_n = T^n x_0$ for all natural numbers $n \geq 1$. Again, consider the sequence of real numbers (t_n), where $t_n = d(x_n, x_{n+1})$ for all natural numbers $n \geq 1$.

Now, we prove that (t_n) converges to 0. If $x_n = x_{n+1}$ for some natural number n, then $t_n = t_{n+1} = \ldots = 0$ and so (t_n) converges to 0. Without loss of generality, we may assume that no two consecutive terms of (t_n) are equal. Then we have

$$\begin{aligned} t_{n+1} = d(x_{n+1}, x_{n+2}) &< f(d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})) \\ &\implies t_{n+1} < f(t_n, t_n, t_{n+1}). \end{aligned}$$

Therefore, by $(\mathcal{A}_2)$, we have

$$t_{n+1} < t_n,$$

which shows that (t_n) is a decreasing sequence of non-negative real numbers and hence converges to some non-negative real number. Again, since X is compact, the sequence (x_n) has a convergent subsequence (x_{n_k}) and let $\lim_{k\to\infty} x_{n_k} = z$. Further, by using the orbitally continuity of T, we get

$$a = \lim_{k\to\infty} d(x_{n_k}, x_{n_k+1}) = d(z, Tz).$$

Again, we have

$$a = \lim_{k\to\infty} d(x_{n_k+1}, x_{n_k+2}) = d(Tz, T^2 z).$$

If $a > 0$, then $z \neq Tz$ and so, using $(\mathcal{A}_2)$, we have

$$\begin{aligned} d(Tz, T^2 z) &< f(d(z, Tz), d(z, Tz), d(Tz, T^2 z)) \\ &\implies d(Tz, T^2 z) < d(z, Tz) \\ &\implies a < a, \end{aligned}$$

which leads to a contradiction. So we must have $a = 0$, i.e., $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, and z is a fixed point of T.

Next, we prove the uniqueness of the fixed point. For this, let z_1 be another fixed point of T. Then we have

$$\begin{aligned} d(Tz, Tz_1) &< f(d(z, z_1), d(z, Tz), d(z_1, TZ_1)) \\ &\implies d(z, z_1) < f(d(z, z_1), 0, 0). \end{aligned} \tag{3.1}$$

Again, by $(\mathcal{A}_3)$, we have

$$d(z, z_1) \geq f(d(z, z_1), 0, 0),$$

which together with the equation (3.1) gives a contradiction.

(2) Next, we assume that $u > f(u, 0, 0)$ for all $c > 0$. We consider the sequence of real numbers (s_n) where $s_n = d(z, x_n)$. Next, we define a function $g : X \to \mathbb{R}$ by

$$g(x) = d(z, x)$$

for all $x \in X$. Then clearly g is continuous on X, and hence there is a positive real number M such that, for all $x \in X$ and $n \geq 1$,

$$|g(x)| \leq M \implies d(z,x) \leq M \implies d(z,x_n) \leq M \implies s_n \leq M.$$

Thus (s_n) is a bounded sequence of real numbers. Since the subsequence (x_{n_k}) of (x_n) converges to z, we get

$$\lim_{k\to\infty} d(z,x_{n_k}) = 0,$$

i.e., $\lim_{k\to\infty} s_{n_k} = 0$. Thus 0 is a cluster point of the sequence (s_n). Let c be any cluster point of (s_n). Then there exists a subsequence (s_{n_j}) of (s_n) which converges to c. So $d(z,x_{n_j}) \to c$ as $j \to \infty$. Therefore, we have

$$\begin{aligned} |s_{n_j+1} - s_{n_j}| &= |d(x_{n_j+1},z) - d(x_{n_j},z)| \\ &\leq d(x_{n_j+1},x_{n_j}) \ \to 0 \end{aligned}$$

as $j \to \infty$ and hence $\lim_{j\to\infty} s_{n_j+1} = \lim_{j\to\infty} s_{n_j} = c$.

We now show that $c = 0$. If $c > 0$, then $\lim_{j\to\infty} d(x_{n_j},z) > 0$ and so we may assume that $x_{n_j} \neq z$ for all $j \geq 1$. Then we have

$$\begin{aligned} s_{n_j+1} = d(Tx_{n_j},Tz) &< f(d(x_{n_j},z), d(x_{n_j},x_{n_j+1}), d(z,Tz)) \\ &= f(s_{n_j}, t_{n_j}, 0). \end{aligned}$$

Taking the limit as $j \to \infty$ in both sides of above equation, we get

$$c \leq f(c,0,0).$$

Since $c > 0$, we have $c > f(c,0,0)$ and this gives $c < c$, which is a contradiction. So $c = 0$. Therefore 0 is the only cluster point of the bounded sequence (s_n) and so this sequence must converge to 0. Hence (x_n) converges to z. Since $x_0 \in X$ was arbitrary, it follows that $(T^n x)$ converges to the fixed point z for each $x \in X$.

(3) Next, we assume that $f(0,0,u) = 0$ implies $u = 0$. Let T be continuous at the fixed point z. For showing $\lim_{x\to z} m(z,x) = 0$, let (y_n) be a sequence in X converging to z. Then

$$\begin{aligned} \lim_{n\to\infty} m(z,y_n) &= \lim_{n\to\infty} f(d(z,y_n), d(z,Tz), d(y_n,Ty_n)) \\ &= f(0,0,d(z,Tz)) \leq 0. \end{aligned}$$

Therefore, $\lim_{n\to\infty} m(z,y_n) = 0$ and hence $\lim_{x\to z} m(z,x) = 0$.

Conversely, let $\lim_{x\to z} m(z,x) = 0$. To show that T is continuous at the fixed point z, let (y_n) be a sequence in X converging to z. Therefore, we have

$$\begin{aligned} &\lim_{n\to\infty} m(z,y_n) = 0 \\ &\implies \lim_{n\to\infty} f(d(z,y_n), d(z,Tz), d(y_n,Ty_n)) = 0 \\ &\implies f(0,0,\lim_{n\to\infty} d(y_n,Ty_n)) = 0 \\ &\implies \lim_{n\to\infty} d(y_n,Ty_n) = 0 \\ &\implies \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} y_n = z = Tz. \end{aligned}$$

So T is continuous at the fixed point z. This completes the proof. □

The following two examples support the above theorem:

Example 3.5. Let $X = [0, 1]$ and consider the usual metric d on X. Then clearly (X, d) is a compact metric space. Define a function $T : X \to X$ by

$$Tx = \begin{cases} 1 - x, & \text{if } x \leq \frac{1}{2}, \\ \frac{1}{2}, & \text{if } x > \frac{1}{2}. \end{cases}$$

We also consider $f \in \mathcal{A}$, where $f(u, v, w) = \max\{v, w\}$ for all $u, v, w \in \mathbb{R}$. Let $x, y \in X$ be arbitrary with $x \neq y$. Then the following cases arise:

Case I: Let $x, y \leq \frac{1}{2}$. Then $Tx = 1 - x$ and $Ty = 1 - y$. Therefore,

$$|Tx - Ty| = |x - y| < \begin{cases} |x - Tx|, & \text{if } x < y, \\ |y - Ty|, & \text{if } x > y. \end{cases}$$

Thus $d(Tx, Ty) < \max\{d(x, Tx), d(y, Ty)\}$, i.e., we have

$$d(Tx, Ty) < f(d(x, y), d(x, Tx), d(y, Ty)).$$

Case II: Let $x, y > \frac{1}{2}$. Then it is easy to check that

$$d(Tx, Ty) < f(d(x, y), d(x, Tx), d(y, Ty)).$$

Case III: Let $x \leq \frac{1}{2}$, $y > \frac{1}{2}$. Then $Tx = 1 - x$ and $Ty = \frac{1}{2}$. Therefore, we have

$$d(Tx, Ty) = \frac{1 - 2x}{2}.$$

Also, $d(x, Tx) = 1 - 2x$ and $d(y, Ty) = y - \frac{1}{2} > 0$. If $x = \frac{1}{2}$, then $d(Tx, Ty) = 0$, but $d(y, Ty) > 0$. If $x < \frac{1}{2}$, then $\frac{1-2x}{2} < 1 - 2x$, i.e., $d(Tx, Ty) < d(x, Tx)$. So we have

$$d(Tx, Ty) < f(d(x, y), d(x, Tx), d(y, Ty)).$$

Thus T is an $\mathcal{A}$-contractive mapping. Also it is easy to verify that T is orbitally continuous. By Theorem 3.4, T has a unique fixed point. Indeed $\frac{1}{2}$ is the unique fixed point of T. Again, $m(\frac{1}{2}, x) = \max\{|\frac{1}{2} - x|, 0, |x - Tx|\}$ and $\lim_{x \to \frac{1}{2}} m(\frac{1}{2}, x) = 0$. Thus, by Theorem 3.4, T is continuous at the fixed point $\frac{1}{2}$ and one can easily verify that T is indeed continuous at $\frac{1}{2}$.

Example 3.6. Let $X = [2, 9]$ and take the usual metric d on X. Then (X, d) is a compact metric space. Also, let $f \in \mathcal{A}$ be a function defined by $f(u, v, w) = \max\{u, v, w\}$. Define a function $T : X \to X$ by

$$Tx = \begin{cases} \frac{x+3}{2}, & \text{if } x \leq 3, \\ 2, & \text{if } x > 3. \end{cases}$$

Let $x, y \in X$ be arbitrary with $x \neq y$. If $x, y > 3$, then $d(Tx, Ty) = 0$ and, if $x, y \leq 3$, then $d(Tx, Ty) = \frac{1}{2}|x - y|$ and so

$$d(Tx, Ty) < f(d(x, y), d(x, Tx), d(y, dy)).$$

Finally, let $x \leq 3$ and $y > 3$. Then we have

$$d(Tx, Ty) = \left|\frac{x+3}{2} - 2\right| = \frac{1}{2}|x - 1| \leq 1,$$

whereas

$$d(y, Ty) = |y - 2| > 1.$$

Thus we have

$$d(Tx, Ty) < f(d(x, y), d(x, Tx), d(y, dy)).$$

Therefore, T is an $\mathcal{A}$-contractive mapping and also T is orbitally continuous. Note that 3 is the only fixed point of T. Also, we have

$$m(3, x) = f(d(3, x), d(3, T3), d(x, Tx))$$

and one can easily verify that $\lim_{x \to 3} m(3, x) \neq 0$. Thus Theorem 3.4 implies that T is not continuous at the fixed point 3.

Next, we give another example in which we replace the compactness of the underlying space by the completeness.

Example 3.7. Let $X = \{\frac{1}{n} : n \in \mathbb{N}\}$ and define a function $d : X \times X \to \mathbb{R}$ by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1 + \left|\frac{1}{x} - \frac{1}{y}\right|, & \text{if } x \neq y. \end{cases}$$

Then it is easy to check that (X, d) is a complete but non-compact metric space. We define a function $T : X \to X$ by

$$T\left(\frac{1}{n}\right) = \frac{1}{4n}$$

for all $n \in \mathbb{N}$. Also, we take $f \in \mathcal{A}$, where $f(u, v, w) = \frac{u+v+w}{3}$. Let $x, y \in X$ be arbitrary with $x \neq y$ and take $x = \frac{1}{n}$, $y = \frac{1}{m}$. First, we assume that $n > m$. Then we have

$$d(Tx, Ty) = 1 + \frac{1}{4m} - \frac{1}{4n}, \quad d(x, y) = 1 + \frac{1}{m} - \frac{1}{n},$$

$$d(x, Tx) = 1 + \frac{3}{4n}, \quad d(y, Ty) = 1 + \frac{3}{4m}.$$

Therefore, we have

$$
\begin{aligned}
&d(Tx,Ty) - f(d(x,y), d(x,Tx), d(y,Ty))\\
&= 1 + \frac{1}{4m} - \frac{1}{4n} - \frac{1}{3}\Big(1 + \frac{1}{m} - \frac{1}{n} + 1 + \frac{3}{4n} + 1 + \frac{3}{4m}\Big)\\
&= -\frac{4}{12m} - \frac{2}{12n} < 0\\
&\implies d(Tx,Ty) < f(d(x,y), d(x,Tx), d(y,Ty)).
\end{aligned}
$$

In a similar manner, we can show that

$$d(Tx,Ty) < f(d(x,y), d(x,Tx), d(y,Ty))$$

if $m > n$. Thus T is an $\mathcal{A}$-contractive mapping. Also, it is easy to check that T is orbitally continuous. Further, T has no fixed point.

From above example, we see that the conclusions of Theorem 3.4 may not hold if compactness is replaced by completeness. Thus, if we take the underlying space as complete, then we need additional condition on $f \in \mathcal{A}$ and/or T to make true the conclusions of that theorem. We add such an additional condition in the next Theorem.

Theorem 3.8. *Let (X,d) be a complete metric space and $T : X \to X$ be an $\mathcal{A}$-contractive mapping such that either T is orbitally continuous or T^k is continuous for some $k \in \mathbb{N}$. Also, assume that the mapping $f \in \mathcal{A}$ arising in the $\mathcal{A}$-contractiveness of T satisfies the following conditions:*

(i) for any $\epsilon > 0$, there is a $\delta > 0$ such that

$$f(d(x,y), d(x,Tx), d(y,Ty)) < \epsilon + \delta \implies d(T^2x, T^2y) \leq \frac{\epsilon}{4}$$

for all $x, y \in X$;

(ii) $f(0,0,u) = 0$ implies $u = 0$.

Then we have the following:

(1) *T has a unique fixed point z.*

(2) *The sequence $(T^n x)$ of iterates converges to z for each $x \in X$.*

(3) *Moreover, T is continuous at z if and only if $\lim_{x\to z} m(z,x) = 0$, where*

$$m(z,x) = f(d(z,x), d(z,Tz), d(x,Tx)).$$

Proof. (1) Let $x_0 \in X$ be arbitrary but fixed, and consider the sequence (x_n), where $x_n = T^n x_0$ for all natural numbers $n \geq 1$. Consider the sequence of real numbers (s_n), where $s_n = d(x_n, x_{n+1})$ for all $n \geq 1$.

We now show that (s_n) converges to 0. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then clearly (s_n) converges to 0 and so we shall assume that $x_n \neq x_{n+1}$ for each $n \geq 1$. Then we have

$$\begin{aligned} s_{n+1} &= d(Tx_n, Tx_{n+1}) \\ &< f(d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})) \\ &= f(s_n, s_n, s_{n+1}). \end{aligned}$$

Using $(\mathcal{A}_2)$, we get $s_{n+1} < s_n$. Therefore, (s_n) is a decreasing sequence of non-negative real numbers and hence it converges to some $a \geq 0$.

Our claim is that $a = 0$. If not, then, by the condition (i), there exists a $\delta > 0$ such that

$$f(d(x, y), d(x, Tx), d(y, Ty)) < 4a + \delta \implies d(T^2x, T^2y) \leq a.$$

Since (s_n) converges to 0, for the above $\delta > 0$, there exists an $n \in \mathbb{N}$ such that

$$s_n < a + \frac{\delta}{4},$$

i.e.,

$$d(x_n, x_{n+1}) < a + \frac{\delta}{4}.$$

So we have

$$\begin{aligned} f(d(x_{n+1}, x_n), d(x_{n+1}, x_{n+2}), d(x_n, x_{n+1})) \\ &\leq d(x_{n+1}, x_n) + d(x_{n+1}, x_{n+2}) + d(x_n, x_{n+1}) \\ &= s_n + s_{n+1} + s_n \\ &< 3s_n \\ &< 4a + \delta. \end{aligned}$$

Therefore, $d(x_{n+3}, x_{n+2}) \leq a$, i.e., $s_{n+2} \leq a$. But this is a contradiction to the fact that (s_n) converges to a and so we must have $a = 0$, i.e., $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

Next, we show that (x_n) is a Cauchy sequence. Let $\epsilon > 0$ be arbitrary. Then, by the condition (i), there exists a $\delta > 0$ such that

$$f(d(x, y), d(x, Tx), d(y, Ty)) < \epsilon + \delta \implies d(T^2x, T^2y) \leq \frac{\epsilon}{4}$$

for all $x, y \in X$. Without loss of generality, we may assume that $\delta < \epsilon$. Since $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, there exists an $N \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) < \frac{\delta}{10} < \frac{\epsilon}{10} < \epsilon$$

for all $n \geq N$. By induction on p, we show that

$$d(x_N, x_{N+p}) < \epsilon \tag{3.2}$$

for all $p \in \mathbb{N}$. Clearly, the equation (3.2) is true for $p = 1$. Let the equation (3.2) be true for p, i.e., $d(x_N, x_{N+p}) < \epsilon$. Then we have

$$\begin{aligned} f(d(x_N, x_{N+p}), &d(x_N, x_{N+1}), d(x_{N+p}, x_{N+p+1})) \\ &\leq d(x_N, x_{N+p}) + d(x_N, x_{N+1}) + d(x_{N+p}, x_{N+p+1}) \\ &< \epsilon + \frac{\delta}{10} + \frac{\delta}{10} \\ &< \epsilon + \delta. \end{aligned}$$

Therefore, $d(x_{N+2}, x_{N+p+2}) \leq \frac{\epsilon}{4}$. So we have

$$\begin{aligned} &d(x_N, x_{N+p+1}) \\ &\leq d(x_N, x_{N+1}) + d(x_{N+1}, x_{N+2}) + d(x_{N+2}, x_{N+p+2}) + d(x_{N+p+2}, x_{N+p+1}) \\ &< \frac{\delta}{10} + \frac{\delta}{10} + \frac{\epsilon}{4} + \frac{\delta}{10} \\ &< \epsilon. \end{aligned}$$

Thus the equation (3.2) is true for $p + 1$. Hence the equation (3.2) holds for all $p \in \mathbb{N}$. In a similar manner, we can show that

$$d(x_n, x_{n+p}) < \epsilon$$

for all $n \geq N$ and $p \geq 1$. This shows that (x_n) is a Cauchy sequence in the complete metric space (X, d) and hence convergent to some $z \in X$.

If T is orbitally continuous, then (x_n) converges to Tz also, and so $Tz = z$. If T^k is continuous for some $k \in \mathbb{N}$, then (x_n) converges to $T^k z$ also, and so $T^k z = z$. We claim that $Tz = z$. If $Tz \neq z$, then $T^{k-1} z \neq z$ also. So we have

$$\begin{aligned} d(Tz, z) &= d(Tz, T^k z) \\ &= d(Tz, T(T^{k-1} z)) \\ &< f(d(z, T^{k-1} z), d(z, Tz), d(T^{k-1} z, T^k z)) \\ &= f(d(T^k z, T^{k-1} z), d(z, Tz), d(T^{k-1} z, T^k z)). \end{aligned}$$

Using $(\mathcal{A}_2)$, we get $d(Tz, z) < d(T^{k-1} z, T^k z)$. But we also have

$$d(T^{k-1} z, T^k z) \leq d(T^{k-2} z, T^{k-1} z) \leq \cdots \leq d(Tz, z)$$

and this leads to a contradiction. So we must have $Tz = z$, i.e., z is a fixed point of T.

(2) and (3) The remaining parts of the proof of this theorem is similar to that of Theorem 3.4 and so is omitted. This completes the proof. □

Next, we present the following example to validate the above theorem:

Example 3.9. Let $X = [0, \infty)$ be equipped with the usual metric d. Then clearly (X, d) is a complete metric space. Define $T : X \to X$ by

$$Tx = \begin{cases} 4, & \text{if } x \leq 4, \\ 1, & \text{if } x > 4. \end{cases}$$

Also, we take $f \in \mathcal{A}$ defined by $f(u, v, w) = \max\{u, v, w\}$. Let $x, y \in X$ be arbitrary with $x \neq y$. Then, if $x, y \leq 4$ or if $x, y > 4$, it is obvious that

$$d(Tx, Ty) < f(d(x, y), d(x, Tx), d(y, dy)).$$

Let $x \leq 4$ and $y > 4$. Then $d(Tx, Ty) = 3$, but $d(y, Ty) = |y - 1| > 3$ and so

$$d(Tx, Ty) < f(d(x, y), d(x, Tx), d(y, dy))$$

and hence T is an $\mathcal{A}$-contractive mapping. Also, T^2 is continuous. Let $\epsilon > 0$ be arbitrary. Then, if we choose $\delta = \epsilon$, then it follows that

$$f(d(x, y), d(x, Tx), d(y, dy)) < \epsilon + \delta \implies d(T^2x, T^2y) \leq \epsilon.$$

Note that 4 is the unique fixed point of T. Now

$$m(4, x) = \max\{|4 - x|, 0, |x - Tx|\}.$$

If we define a sequence (x_n) in X by setting $x_n = 4 + \frac{1}{n}$, then we have

$$m(4, x_n) = \max\left\{\frac{1}{n}, 0, 3 + \frac{1}{n}\right\} \to 3$$

as $n \to \infty$. Thus $\lim_{x \to 4} m(4, x) \neq 0$. So, by Theorem 3.8, it follows that T is not continuous at the fixed point 4.

We now obtain a fixed point result concerning $\mathcal{A}'$-contractive mapping in compact metric spaces.

Theorem 3.10. *Let (X, d) be a compact metric space and $T : X \to X$ be an orbitally continuous $\mathcal{A}'$-contractive mapping. Then we have the following:*

(1) *T has a unique fixed point z.*

(2) *For any $x \in X$, the sequence $(T^n x)$ converges to the fixed point z.*

(3) *Moreover, if $f \in \mathcal{A}'$, arising in the $\mathcal{A}'$-contractiveness of T, satisfies the condition that $f(0, u, 0) = 0$ implies $u = 0$, then T is continuous at the fixed point z if and only if $\lim_{x \to z} m(z, x) = 0$, where*

$$m(z, x) = f(d(z, x), d(z, Tx), d(x, Tz)).$$

Proof. (1) Let $x_0 \in X$ be arbitrary but fixed and consider the sequence (x_n), where $x_n = T^n x_0$ for all natural numbers $n \geq 1$. Since X is compact, the sequence

(x_n) has a convergent subsequence, say (x_{n_k}), and let $\lim_{k\to\infty} x_{n_k} = z$. Also, we consider the sequences (s_n) and (t_n) of real numbers, where $s_n = d(x_n, z)$ and $t_n = d(x_n, x_{n+1})$ for all natural numbers $n \geq 1$.

Now, we show that (t_n) converges to 0. If $x_n = x_{n+1}$ for some natural number n, then clearly (t_n) converges to 0. So we assume that $x_n \neq x_{n+1}$ for all natural numbers $n \geq 1$. Therefore, we have

$$\begin{aligned} t_{n+1} &= d(Tx_n, Tx_{n+1}) \\ &< f(d(x_n, x_{n+1}), d(x_n, x_{n+2}), d(x_{n+1}, x_{n+1})) \\ &\leq f(d(x_n, x_{n+1}), d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}), 0) \\ &= f(t_n, t_n + t_{n+1}, 0). \end{aligned}$$

Using $(\mathcal{A}_3')$ we get $t_{n+1} < t_n$. This is true for all natural numbers $n \geq 1$. Therefore, (t_n) is a decreasing sequence of non-negative real numbers and hence convergent to some $a \geq 0$. Then, by using the orbital continuity of T, we have

$$a = \lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{k\to\infty} d(x_{n_k}, x_{n_{k+1}}) = d(z, Tz).$$

Again, we have

$$a = \lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{k\to\infty} d(x_{n_{k+1}}, x_{n_{k+2}}) = d(Tz, T^2z).$$

If $a > 0$, then $z \neq Tz$ and so

$$\begin{aligned} d(Tz, T^2z) &< f(d(z, Tz), d(z, T^2z), d(Tz, Tz)) \\ &\leq f(d(z, Tz), d(z, Tz) + d(Tz, T^2z), 0) \\ &\Longrightarrow d(Tz, T^2z) < d(z, Tz) \\ &\Longrightarrow a < a, \end{aligned}$$

which is a contradiction. So we must have $a = 0$ and hence $z = Tz$; i.e., z is a fixed point of T.

Next, we prove the uniqueness of the fixed point. To prove this, let z_1 be another fixed point of T. Then we have

$$\begin{aligned} d(z, z_1) = d(Tz, Tz_1) &< f(d(z, z_1), d(z, Tz_1), d(z_1, Tz)) \\ &= f(d(z, z_1), d(z, z_1), d(z, z_1)) \\ &\leq d(z, z_1), \end{aligned}$$

which is a contradiction. So z is the unique fixed point of T.

(2) Now, we show that (s_n) converges to 0. If $x_n = z$ for some natural number n, then clearly (s_n) converges to 0. So we assume that $x_n \neq z$ for all natural numbers n. Then we have

$$\begin{aligned} s_{n+1} &= d(Tx_n, Tz) \\ &< f(d(x_n, z), d(x_n, z), d(z, x_{n+1})) \\ &= f(s_n, s_n, s_{n+1}). \end{aligned}$$

Therefore, we have $s_{n+1} < s_n$, and this is true for all natural numbers n. Then (s_n) is a decreasing sequence of real numbers. Also, since (x_{n_k}) converges to z, is follows that (s_{n_k}) converges to 0. Hence (s_n) must converge to 0, i.e., (x_n) converges to the fixed point z.

(3) Now, let $f \in \mathcal{A}'$ satisfy the condition that $f(0,u,0) = 0$ implies $u = 0$. Let T be continuous at the fixed point z. To show that $\lim_{x\to z} m(z,x) = 0$, let (y_n) be any sequence in X converging to z. Then (Ty_n) converges to z also. Therefore,

$$\begin{aligned}\lim_{n\to\infty} m(z,y_n) = \lim_{n\to\infty} f(d(z,y_n), d(z,Ty_n), d(y_n,z)) &= f(0,0,0) \le 0\\ \implies \lim_{n\to\infty} m(z,y_n) &= 0\\ \implies \lim_{x\to z} m(z,x) &= 0.\end{aligned}$$

Conversely, assume that $\lim_{x\to z} m(z,x) = 0$. To show that T is continuous at the fixed point z, let (y_n) be any sequence in X converging to z. Then since $\lim_{x\to z} m(z,x) = 0$, we have

$$\begin{aligned}&\lim_{n\to\infty} m(z,y_n) = 0\\ &\implies \lim_{n\to\infty} f(d(z,y_n), d(z,Ty_n), d(y_n,z)) = 0\\ &\implies f(0, \lim_{n\to\infty} d(z,Ty_n), 0) = 0\\ &\implies \lim_{n\to\infty} d(z,Tx_n).\end{aligned}$$

This shows that (Ty_n) converges to $z = Tz$. Hence T is continuous at z. This completes the proof. □

The following example illustrates the above theorem:

Example 3.11. Let $X = \{0, \frac{1}{n} : n \in \mathbb{N}\}$ and define a function $d : X \times X \to \mathbb{R}$ by

$$d(x,y) = \begin{cases} 0, & \text{if } x = y,\\ x + y, & \text{if } x \neq y.\end{cases}$$

Then one can easily verify that (X,d) is a compact metric space. Next, we define a function $T : X \times X$ by

$$Tx = \begin{cases} 0, & \text{if } x = 0,\\ \frac{1}{n+1}, & \text{if } x = \frac{1}{n}.\end{cases}$$

Let $f \in \mathcal{A}'$ be a function defined by $f(u,v,w) = \frac{v+w}{2}$. Let $x, y \in X$ be arbitrary with $x \neq y$. Then the following cases occur.

Case I: Let $x = \frac{1}{n}$ and $y = \frac{1}{m}$ with $n \neq m$. So, we have

$$Tx = \frac{1}{n+1},\ Ty = \frac{1}{m+1}.$$

Therefore, we have

$$\begin{aligned}&d(Tx,Ty)-f(d(x,y),d(x,Ty),d(y,Tx))\\&=\frac{1}{n+1}+\frac{1}{m+1}-\frac{1}{2}\left(\frac{1}{n}+\frac{1}{m+1}+\frac{1}{m}+\frac{1}{n+1}\right)<0\\&\Longrightarrow d(Tx,Ty)<f(d(x,y),d(x,TY),d(y,Tx)).\end{aligned}$$

Case II: Let $x=\frac{1}{n}$ and $y=0$. So, we have

$$Tx=\frac{1}{n+1},\quad Ty=0.$$

Therefore, we have

$$\begin{aligned}&d(Tx,Ty)-f(d(x,y),d(x,Ty),d(y,Tx))\\&=\frac{1}{n+1}+0-\frac{1}{2}\left(\frac{1}{n}+\frac{1}{n+1}\right)<0\\&\Longrightarrow d(Tx,Ty)<f(d(x,y),d(x,Ty),d(y,Tx)).\end{aligned}$$

Also, T is orbitally continuous and 0 is the unique fixed point of T. Next, if $x=\frac{1}{n}$, then we have

$$\begin{aligned}m(0,x)&=f(d(0,x),d(0,Tx),d(x,T0))\\&=\frac{1}{2}\left(\frac{1}{n+1}+\frac{1}{n}\right)\\&\to 0\end{aligned}$$

as $n\to\infty$, that is, as $x\to 0$. Therefore, $\lim_{x\to 0}m(0,x)=0$. So, by Theorem 3.10, it follows that T is continuous at 0 and also the continuity of T at 0 is clearly seen from the formulation of T.

In the next example, we show that the compactness of X in Theorem 3.10 cannot be replaced by the completeness:

Example 3.12. Let $X=\{\frac{1}{2^n}:n\in\mathbb{N}\}$ and define a function $d:X\times X\to\mathbb{R}$ by

$$d(x,y)=\begin{cases}0, & \text{if } x=y,\\ 1+x+y, & \text{if } x\neq y.\end{cases}$$

Then clearly (X,d) is a complete but non-compact metric space. Next, we define a function $T:X\to X$ by

$$T\left(\frac{1}{2^n}\right)=\frac{1}{2^{n+1}}$$

for all $n \geq 1$. Also, we take $f \in \mathcal{A}'$ defined by $f((u,v,w)) = \frac{1}{3}(u+v+w)$. Let $x, y \in X$ be arbitrary with $x \neq y$ and take $x = \frac{1}{2^n}$, $y = \frac{1}{2^m}$ with $n \neq m$. Therefore, we have

$$d(Tx,Ty) = 1 + \frac{1}{2}\left(\frac{1}{2^n} + \frac{1}{2^m}\right)$$

and

$$f(d(x,y), d(x,Ty), d(y,Tx)) = 1 + \frac{2}{3}\left(\frac{1}{2^n} + \frac{1}{2^m}\right) + \frac{1}{6}\left(\frac{1}{2^n} + \frac{1}{2^m}\right).$$

Therefore, it is easy to observe that

$$d(Tx,Ty) < f(d(x,y), d(x,Ty), d(y,Tx)).$$

Hence T is an $\mathcal{A}'$-contractive mapping and also T is orbitally continuous, but T is fixed point free.

Then we need some additional condition on $f \in \mathcal{A}'$ and/or T to obtain a fixed point of T, which is given in the next theorem:

Theorem 3.13. *Let (X,d) be a complete metric space and $T : X \to X$ be an $\mathcal{A}'$-contractive mapping such that either T is orbitally continuous or T^k is continuous for some $k \in \mathbb{N}$. Also, assume that the mapping $f \in \mathcal{A}'$, which arises in the $\mathcal{A}$-contractiveness of T, satisfies the following conditions:*

(i) for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$f(d(x,y), d(x,Tx), d(y,Ty)) < \epsilon + \delta \implies d(T^2x, T^2y) \leq \frac{\epsilon}{4}$$

for all $x, y \in X$;

(ii) $f(0,u,0) = 0$ implies $u = 0$.

Then we have the following:

(1) *T has a unique fixed point $z \in X$.*

(2) *The sequence $(T^n x)$ of iterates converges to z for each $x \in X$.*

(3) *Moreover, T is continuous at z if and only if $\lim_{x\to z} m(z,x) = 0$, where*

$$m(z,x) = f(d(z,x), d(z,Tx), d(x,Tz)).$$

Proof. (1) Let $x_0 \in X$ be an arbitrary but fixed element and consider the sequence (x_n), where $x_n = T^n x_0$ for all natural numbers $n \geq 1$. Also, we consider the sequence (t_n), where $t_n = d(x_n, x_{n+1})$ for all natural numbers $n \geq 1$.

Next, we show that (t_n) converges to 0. If $x_n = x_{n+1}$ for some natural number n, then it is easy to note that (t_n) converges to 0. So, we assume that no two consecutive terms of (x_n) are equal. Then, proceeding in a manner similar to that of Theorem 3.10, we can show that (t_n) is a decreasing sequence of non-negative real numbers and hence it is convergent to some $a \geq 0$. If $a > 0$, then, by the condition (i), there exists a $\delta > 0$ such that

$$f(d(x,y), d(x,Ty), d(y,Tx)) < 4a + \delta \implies d(T^2x, T^2y) \leq a.$$

Since (t_n) converges to a, there exists a natural number n such that $t_n < a + \frac{\delta}{5}$, i.e., $d(x_n, x_{n+1}) < a + \frac{\delta}{5}$. Then we have

$$\begin{aligned} f(d(x_n, x_{n+1}), &d(x_n, x_{n+2}), d(x_{n+1}, x_{n+1})) \\ &\leq d(x_n, x_{n+1}) + d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1}) \\ &\leq d(x_n, x_{n+1}) + d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) \\ &< 3\left(a + \frac{\delta}{5}\right) < 4a + \delta. \end{aligned}$$

This implies that $d(T^2x_n, T^2x_{n+1}) \leq a$, i.e., $t_{n+2} \leq a$, which is a contradiction to the fact that (t_n) converges to a and so we must have $a = 0$. Thus $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

Now, we show that (x_n) is a Cauchy sequence. Let $\epsilon > 0$ be arbitrary. From the condition (i), there exists a $\delta > 0$ such that

$$f(d(x,y), d(x,Ty), d(y,Tx)) < 3\epsilon + \delta \implies d(T^2x, T^2y) \leq \frac{3\epsilon}{4}.$$

Without loss of generality, we may assume that $\delta < \epsilon$. Since $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, for the above $\delta > 0$, there exists a natural number N such that

$$d(x_n, x_{n+1}) < \frac{\delta}{12} < \frac{\epsilon}{12} < \epsilon$$

for all $n \geq N$. Using induction on p, we show that

$$d(x_N, x_{N+p}) < \epsilon \tag{3.3}$$

for all $p \in \mathbb{N}$. Clearly, the equation (3.3) is true for $p = 1$. Let the equation (3.3) be true for some $p \in \mathbb{N}$, i.e., $d(x_N, x_{N+p}) < \epsilon$. Then we have

$$\begin{aligned} f(d(x_N, x_{N+p}), &d(x_N, x_{N+p+1}), d(x_{N+p}, x_{N+1})) \\ &\leq d(x_N, x_{N+p}) + d(x_N, x_{N+p+1}) + d(x_{N+p}, x_{N+1}) \\ &\leq d(x_N, x_{N+p}) + d(x_N, x_{N+p}) \\ &\quad + d(x_{N+p}, x_{N+p+1}) + d(x_{N+p}, x_N) + d(x_N, x_{N+1}) \\ &< 3\epsilon + \frac{2\delta}{12} < 3\epsilon + \delta. \end{aligned}$$

This implies that $d(x_{N+2}, x_{N+p+2}) \leq \frac{3\epsilon}{4}$. Then we have

$$\begin{aligned} d(x_N, x_{N+p+1}) &\leq d(x_N, x_{N+1}) + d(x_{N+1}, x_{N+2}) + d(x_{N+2}, x_{N+p+2}) \\ &\quad + d(x_{N+p+2}, x_{N+p+1}) \\ &< \frac{3\delta}{12} + \frac{3\epsilon}{4} < \epsilon. \end{aligned}$$

Thus the equation (3.3) is true for $p+1$. Hence we have

$$d(x_N, x_{N+p}) < \epsilon$$

for all $p \geq 1$. In a similar manner, we can show that

$$d(x_n, x_{n+p}) < \epsilon$$

for all $n \geq N$ and $p \geq 1$. Therefore, (x_n) is a Cauchy sequence in the complete metric space (X, d) and hence convergent to some $z \in X$.

Next, we show that z is a fixed point of T. If T is orbitally continuous, then (x_n) converges to Tz also and so we have $Tz = z$. Now, assume that T^k is continuous for some natural number k. Then also (x_n) converges to $T^k z$ and thus we have $T^k z = z$.

Our claim is that $Tz = z$. If not, then $T^{k-1}z \neq z$ and so we get

$$\begin{aligned} d(T^k z, Tz) &< f(d(T^{k-1}z, z), d(T^{k-1}z, Tz), d(z, T^k z)) \\ &\leq f(d(T^{k-1}z, z), d(T^{k-1}z, z) + d(z, Tz), 0) \\ &= f(d(T^{k-1}z, z), d(T^{k-1}z, z) + d(T^k z, Tz), 0). \end{aligned}$$

Using $(\mathcal{A}_3')$, we get

$$d(T^k z, Tz) < d(T^{k-1}z, z),$$

i.e.,

$$d(z, Tz) < d(T^{k-1}z, T^k z).$$

But we have

$$d(T^{k-1}z, T^k z) \leq d(T^{k-2}z, T^{k-1}z) \leq \cdots \leq d(z, Tz),$$

which is a contradiction. So we must have $Tz = z$, i.e., z is a fixed point of T.

(2) and (3) The rests of the proof is similar to the proof of Theorem 3.10 and so is omitted. This completes the proof. □

Finally, we cite the following example to illustrate Theorem 3.13:

Example 3.14. Let $X = [0, \infty)$ and take the usual metric d on X. Then (X, d) is a complete metric space. We define a function $T : X \to X$ by

$$Tx = \begin{cases} \frac{3}{2}, & \text{if } x < 1, \\ 2, & \text{if } x \geq 1. \end{cases}$$

Take $f \in \mathcal{A}$ defined by $f(u, v, w) = \frac{v+w}{2}$. Let $x, y \in X$ be arbitrary with $x \neq y$. Then, if $x, y < 1$ or $x, y \geq 1$, then it is easy to check that

$$d(Tx, Ty) < f(d(x, y), d(x, Ty), d(y, Tx)).$$

Now, let $x < 1$ and $y \geq 1$. Then $d(Tx, Ty) = \frac{1}{2}$ and

$$\frac{1}{2}\Big(d(x, Ty) + d(y, Tx)\Big) = \frac{1}{2}\Big(|2 - x| + |y - \frac{3}{2}|\Big) > \frac{1}{2}.$$

Thus we have

$$d(Tx, Ty) < f(d(x, y), d(x, Ty), d(y, Tx)).$$

Therefore, T is an $\mathcal{A}$-contractive mapping. Also, for any $\epsilon > 0$, if we choose $\delta = \epsilon$, then we get

$$f(d(x, y), d(x, Ty), d(y, Tx)) < \epsilon + \delta \implies d(T^2x, T^2y) \leq \frac{\epsilon}{4}$$

for all $x, y \in X$. Thus T satisfies all of the conditions of Theorem 3.13 and so, by Theorem 3.13, T has a unique fixed point. Note that 2 is the unique fixed point of T. Also, we have

$$m(2, x) = \frac{1}{2}\left(d(2, Tx), d(x, T2)\right) \to 0$$

as $x \to 2$. So, by Theorem 3.13, T is continuous at the fixed point 2.

4. CONSEQUENCES

Using the fixed point results in Section 3, we obtain the following corollaries:

Corollary 4.15. *Let (X, d) be a compact metric space and T be a self-mapping on X such that*

$$d(Tx, Ty) < d(x, y)$$

for all $x, y \in X$ with $x \neq y$. Then T has a unique fixed point.

Proof. The proof of the corollary easily follows from Theorem 3.4 by taking $f(u, v, w) = u$. □

Corollary 4.16. *Let (X, d) be a compact metric space and T be a continuous self-mapping on X such that*

$$d(Tx, Ty) < \frac{1}{2}(d(x, Tx) + d(y, Ty))$$

for all $x, y \in X$ with $x \neq y$. Then T has a unique fixed point.

Proof. The proof of this corollary follows from Theorem 3.4 by taking $f(u, v, w) = \frac{1}{2}(v + w)$. □

Corollary 4.17. *Let (X,d) be a compact metric space and T be a continuous self-mapping on X such that*

$$d(Tx,Ty) < (d(x,Tx)d(y,Ty))^{\frac{1}{2}}$$

for all $x,y \in X$ with $x \neq y$. Then T has a unique fixed point.

Proof. The proof follows from Theorem 3.4 by taking $f(u,v,w) = (v.w)^{\frac{1}{2}}$. □

Corollary 4.18. *Let (X,d) be a compact metric space and T be a continuous self-mapping on X such that*

$$d(Tx,Ty) < \frac{1}{2}(d(x,Ty) + d(y,Tx))$$

for all $x,y \in X$ with $x \neq y$. Then T has a unique fixed point.

Proof. The proof of this corollary follows easily from Theorem 3.10 by taking $f(u,v,w) = \frac{1}{2}(v+w)$. □

Remark 4.19. It is worth mentioning that the above corollaries actually give many important results in the literature. To be precise, Corollary 4.15 and Corollary 4.16 follow Remark 3.1 in [5] and Theorem 2.2 in [9], respectively. Further, Corollary 4.17 and Corollary 4.18 give one of the main results in [11] and [6], respectively.

Remark 4.20. In this remark we specify a suitable area where our main results may be useful. Neural networks are frequently used in character recognition, image compression, stock market prediction, travelling salesman's problem. In [7] the authors introduced discontinuous type neural networks which can be applied to solve non-negative sparse approximation problems. This type of neural networks can also be used as a model of the mammalian olfactory system. Thus our main results can be applied to neural networks by using suitable conditions.

Acknowledgements. The authors express their deep gratitudes to Prof. B. E. Rhoades for his valuable suggestions and corrections for improvement of this paper. The first named author is heartily thankful to CSIR, New Delhi, India for their financial supports during this research period under the CSIR-JRF fellowship scheme (Award Number: 09/973(0018)/2017-EMR-I).

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(Received 18.10.2018)
(Revised 10.02.2020)

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