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# EXTERNAL JENSEN-TYPE OPERATOR INEQUALITIES VIA SUPERQUADRATICITY

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In this paper we establish several Jensen-type operator inequalities for a class of superquadratic functions and self-adjoint operators. Our results are given in the so-called external form. As an application, we give improvements of the Hölder–McCarthy inequality and the classical discrete and integral Jensen inequality in the corresponding external forms. In addition, the established Jensen-type inequalities are compared with the previously known results and we show that our results provide more accurate estimates in some general settings.

## 1. INTRODUCTION AND PRELIMINARIES

The famous Jensen inequality in its basic form asserts that if $f : J \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function, then the inequality

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$$

holds for all $x, y \in J$ and $\lambda \in [0, 1]$. If $f$ is concave, then the inequality sign is reversed. Usually, the Jensen-type operator inequalities are accompanied by an operator convex function. Recall that a continuous function $f : J \subseteq \mathbb{R} \to \mathbb{R}$ is operator convex if

$$f(\lambda A + (1-\lambda)B) \le \lambda f(A) + (1-\lambda) f(B)$$

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holds for all $\lambda \in [0,1]$ and for every pair of self-adjoint operators $A$ and $B$ on a Hilbert space $\mathscr{H}$ whose spectra are contained in $J$.

However, some Jensen-type operator inequalities are accompanied by a usual convex function. The following operator extension of the Jensen inequality is also known (see, [**11**]): if $f : J \subseteq \mathbb{R} \to \mathbb{R}$ is a continuous convex function, then the inequality

$$f(\langle A\eta, \eta\rangle) \leq \langle f(A)\eta, \eta\rangle \tag{1}$$

holds for every self-adjoint operator $A$ with spectrum in $J$ and for every unit vector $\eta \in \mathscr{H}$. For a concave function $f$, the inequality sign in (1) is reversed. It should be noticed here that inequality (1) has been firstly established for traces on semi-finite von-Neumann algebras in [**4**] and then, in an inner product form, in [**11**]. This gives a tool for studying a class of continuous convex functions in the context of operators without any other assumption like operator convexity. The Jensen inequality (1) has been studied by numerous authors who applied and generalized it in various directions. For some extensions closely connected to (1), the reader is referred, for example, to [**5, 8**] and the references therein.

Fujii [**6**], noticed that concavity of a function can be characterized via externally dividing points. Namely, the function $f : J \to \mathbb{R}$ is concave if and only if

$$f((1+r)x - ry) \leq (1+r)f(x) - rf(y) \tag{2}$$

holds for all $r > 0$ and $x, y \in J$ such that $(1+r)x - ry \in J$. Motivated by (2), Fujii et al. [**7**], established a vector state generalization of (2).

**Theorem A.** [**7**, Theorem 2.1] A continuous function $f : J \to \mathbb{R}$ is concave if and only if the inequality

$$f(\langle A\eta, \eta\rangle - \langle B\zeta, \zeta\rangle) \leq \|\eta\|^2 f\left(\frac{1}{\|\eta\|^2}\langle A\eta, \eta\rangle\right) - \langle f(B)\zeta, \zeta\rangle \tag{3}$$

holds for all $\eta, \zeta \in \mathscr{H}$ such that $\|\eta\|^2 - \|\zeta\|^2 = 1$ and for all self-adjoint operators $A, B$ with spectra in $J$, provided that $\langle A\eta, \eta\rangle - \langle B\zeta, \zeta\rangle \in J$. If $f$ is a convex function, then the sign of inequality (3) is reversed.

In view of (2), inequality (3) can be regarded as an external form of (1). Therefore, for the reader's convenience, inequalities (2) and (3) will be referred to as the external Jensen-type inequalities.

It is interesting that, besides convexity and operator convexity, Jensen-type inequalities can also be studied for some other classes of functions. In 2004, Abramovich et al. [**2**], introduced the notion of a superquadratic function. A function $f : [0, \infty) \to \mathbb{R}$ is said to be superquadratic provided that for each $x \geq 0$ there exists a constant $C_x \in \mathbb{R}$ such that

$$f(y) - f(x) - f(|y-x|) \geq C_x(y-x) \tag{4}$$

for all $y \geq 0$. We say that $f$ is subquadratic if $-f$ is superquadratic.

Superquadratic functions are closely connected to convex functions. At the first sight, condition (4) appears to be stronger than convexity. However, if $f$ takes negative values, it may be considerably weaker. The negative superquadratic functions have been discussed in [**2**]. For example, it was shown that the function $f(x) = -\left(1 + x^{1/p}\right)^p$, $x \geq 0$, $p \geq 0$ is superquadratic. Note also that any function $f$ with values in the closed interval $[-2a, -a]$, $a > 0$, is superquadratic. This conclusion simply follows by observing that the left-hand side of (4) takes values in $[0, 3a]$, so in this case we can put $C_x = 0$ and (4) holds. On the other hand, non-negative superquadratic functions behave much better, in particular, they are convex.

The following properties of superquadratic functions have been proved in [**2**] (see also [**3**]):

(i) $f(0) \leq 0$;

(ii) if $f(0) = f'(0) = 0$ and $f$ is differentiable at $s$, then $C_s = f'(s)$;

(iii) if $f$ is non-negative, then $f$ is convex and $f(0) = f'(0) = 0$;

(iv) $f(x) = x^p$ is superquadratic for $p \geq 2$, while it is subquadratic for $0 < p \leq 2$.

Moreover, the Jensen inequality for the superquadratic function $f$ in its basic discrete form (see [**3**]) asserts that

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) - \lambda f((1-\lambda)|y-x|) - (1-\lambda)f(\lambda|y-x|) \tag{5}$$

holds for every $\lambda \in [0, 1]$ and for all $x, y \geq 0$. For more properties and applications of superquadratic functions, the reader is referred to [**1, 2, 3**].

By virtue of (5), Kian [**9**], derived extension of (1) that corresponds to a class of superquadratic functions.

**Theorem B.** [**9**, Theorem 2.1] If $f : [0, \infty) \to \mathbb{R}$ is a continuous superquadratic function, then the inequality

$$f(\langle A\eta, \eta\rangle) \leq \langle f(A)\eta, \eta\rangle - \langle f(|A - \langle A\eta, \eta\rangle|)\eta, \eta\rangle \tag{6}$$

holds for every positive operator $A$ and every unit vector $\eta \in \mathscr{H}$.

If $f$ is a non-negative function, then inequality (6) represents refinement of (1). Yet another extension of (1), that corresponds to superquadratic functions, has been established in [**10**].

**Theorem C.** [**10**, Theorem 3.2] Let $g, \omega : [a, b] \to \mathbb{R}$ be non-negative continuous functions. If $f : [0, \infty) \to \mathbb{R}$ is a continuous superquadratic function, then the inequality

$$\begin{aligned} f\left(\frac{\langle g(A)\omega(A)\eta, \eta\rangle}{\langle \omega(A)\eta, \eta\rangle}\right) \leq{}& \frac{1}{\langle \omega(A)\eta, \eta\rangle}\langle (f \circ g)(A)\omega(A)\eta, \eta\rangle \\ &- \frac{1}{\langle \omega(A)\eta, \eta\rangle}\left\langle f\left(\left|g(A) - \frac{\langle g(A)\omega(A)\eta, \eta\rangle}{\langle \omega(A)\eta, \eta\rangle}\right|\right)\omega(A)\eta, \eta\right\rangle \end{aligned}$$

holds for every self-adjoint operator $A$ with spectrum in $[a, b]$ and for every $\eta \in \mathscr{H}$.

The main objective of the present article is to establish an analogue of Theorem A for a class of superquadratic functions. In other words, we derive external form of the Jensen operator inequality (1) that corresponds to superquadratic functions. The paper is divided into three sections as follows: after this introductory part, in Section 2 we establish the external Jensen-type operator inequality for a class of superquadratic functions. We show that our result provides more accurate estimates than Theorem A in some general settings. As an application, we give the external form of the Hölder-McCarthy inequality. Further, as a consequence, we also derive the classical discrete and integral Jensen inequalities for superquadratic functions in the external form. In Section 3, we present our results for non-negative continuous functions of operators.

In what follows, we assume that $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ is a complex Hilbert space. If $f : J \to \mathbb{R}$ is a continuous real function and $A$ is a self-adjoint operator on $\mathscr{H}$ whose spectrum is contained in $J$, then by $f(A)$ we mean a continuous functional calculus. More precisely, let the spectrum of $A$ be contained in the interval $[m, M]$ and let $\{E_x\}$ be its spectral family. Then, $f(A)$ can be expressed via the spectral representation as

$$f(A) = \int_{m-0}^{M} f(x) dE_x,$$

in which the integral is in terms of the Riemann–Stieltjes integral. In addition, if $\eta, \zeta \in \mathscr{H}$, then the inner product can be represented as

$$\langle f(A)\eta, \zeta\rangle = \int_{m-0}^{M} f(x) d\langle E_x \eta, \zeta\rangle.$$

For more details about the continuous functional calculus the reader is referred to [**12**].

## 2. MAIN RESULTS

The main objective of this section is to extend the external Jensen-type inequality (3) for a class of superquadratic functions. In such a way, we will obtain more accurate estimates in some general cases. Our first result is a variant of (3) for superquadratic functions.

**Theorem 1.** *Let $f : [0,\infty) \to \mathbb{R}$ be a continuous superquadratic function and let $\eta, \zeta \in \mathscr{H}$ be such that $\|\eta\|^2 - \|\zeta\|^2 = 1$. If $A$ and $B$ are positive operators, then holds the inequality*

$$\begin{aligned} f(\langle A\eta,\eta\rangle - \langle B\zeta,\zeta\rangle) \geq \|\eta\|^2 f\left(\frac{1}{\|\eta\|^2}\langle A\eta,\eta\rangle\right) - \langle f(B)\zeta,\zeta\rangle + \left\langle f\left(\left|B - \frac{1}{\|\zeta\|^2}\langle B\zeta,\zeta\rangle\right|\right)\zeta,\zeta\right\rangle \\ + f\left(\|\zeta\|^2\left|\frac{1}{\|\eta\|^2}\langle A\eta,\eta\rangle - \frac{1}{\|\zeta\|^2}\langle B\zeta,\zeta\rangle\right|\right) \\ + \|\zeta\|^2 f\left(\left|\frac{1}{\|\eta\|^2}\langle A\eta,\eta\rangle - \frac{1}{\|\zeta\|^2}\langle B\zeta,\zeta\rangle\right|\right), \end{aligned}$$

*provided that $\langle A\eta,\eta\rangle - \langle B\zeta,\zeta\rangle \geq 0$. If $f$ is subquadratic, then the sign of inequality is reversed.*

*Proof.* Since $f$ is superquadratic, applying (5) with $\lambda = \frac{1}{\|\eta\|^2}$ implies that

$$\begin{aligned} f\left(\left\langle A\frac{\eta}{\|\eta\|}, \frac{\eta}{\|\eta\|}\right\rangle\right) &= f\left(\frac{1}{\|\eta\|^2}(\langle A\eta,\eta\rangle - \langle B\zeta,\zeta\rangle) + \frac{\|\zeta\|^2}{\|\eta\|^2}\left\langle B\frac{\zeta}{\|\zeta\|}, \frac{\zeta}{\|\zeta\|}\right\rangle\right) \\ &\le \frac{1}{\|\eta\|^2} f(\langle A\eta,\eta\rangle - \langle B\zeta,\zeta\rangle) + \frac{\|\zeta\|^2}{\|\eta\|^2} f\left(\left\langle B\frac{\zeta}{\|\zeta\|}, \frac{\zeta}{\|\zeta\|}\right\rangle\right) \\ &\quad - \frac{1}{\|\eta\|^2} f\left(\frac{\|\zeta\|^2}{\|\eta\|^2}\left|\langle A\eta,\eta\rangle - \langle B\zeta,\zeta\rangle - \frac{1}{\|\zeta\|^2}\langle B\zeta,\zeta\rangle\right|\right) \\ &\quad - \frac{\|\zeta\|^2}{\|\eta\|^2} f\left(\frac{1}{\|\eta\|^2}\left|\langle A\eta,\eta\rangle - \langle B\zeta,\zeta\rangle - \frac{1}{\|\zeta\|^2}\langle B\zeta,\zeta\rangle\right|\right) \\ &= \frac{1}{\|\eta\|^2} f(\langle A\eta,\eta\rangle - \langle B\zeta,\zeta\rangle) + \frac{\|\zeta\|^2}{\|\eta\|^2} f\left(\left\langle B\frac{\zeta}{\|\zeta\|}, \frac{\zeta}{\|\zeta\|}\right\rangle\right) \\ &\quad - \frac{1}{\|\eta\|^2} f\left(\|\zeta\|^2\left|\frac{1}{\|\eta\|^2}\langle A\eta,\eta\rangle - \frac{1}{\|\zeta\|^2}\langle B\zeta,\zeta\rangle\right|\right) \\ &\quad - \frac{\|\zeta\|^2}{\|\eta\|^2} f\left(\left|\frac{1}{\|\eta\|^2}\langle A\eta,\eta\rangle - \frac{1}{\|\zeta\|^2}\langle B\zeta,\zeta\rangle\right|\right), \end{aligned} \tag{7}$$

where we used the condition $\|\zeta\|^2 + 1 = \|\eta\|^2$ in the last step. In addition, applying inequality (6), we obtain

$$f\left(\left\langle B\frac{\zeta}{\|\zeta\|}, \frac{\zeta}{\|\zeta\|}\right\rangle\right) \le \frac{1}{\|\zeta\|^2}\langle f(B)\zeta,\zeta\rangle - \frac{1}{\|\zeta\|^2}\langle f\left(\left|B - \frac{1}{\|\zeta\|^2}\langle B\zeta,\zeta\rangle\right|\right)\zeta,\zeta\rangle. \tag{8}$$

Now, multiplying (7) by $\|\eta\|^2$ both-sidedly and utilizing (8), we get

$$\begin{aligned} \|\eta\|^2 f\left(\left\langle A\frac{\eta}{\|\eta\|}, \frac{\eta}{\|\eta\|}\right\rangle\right) &\le f(\langle A\eta,\eta\rangle - \langle B\zeta,\zeta\rangle) + \langle f(B)\zeta,\zeta\rangle \\ &\quad - \left\langle f\left(\left|B - \frac{1}{\|\zeta\|^2}\langle B\zeta,\zeta\rangle\right|\right)\zeta,\zeta\right\rangle \\ &\quad - f\left(\|\zeta\|^2\left|\frac{1}{\|\eta\|^2}\langle A\eta,\eta\rangle - \frac{1}{\|\zeta\|^2}\langle B\zeta,\zeta\rangle\right|\right) \\ &\quad - \|\zeta\|^2 f\left(\left|\frac{1}{\|\eta\|^2}\langle A\eta,\eta\rangle - \frac{1}{\|\zeta\|^2}\langle B\zeta,\zeta\rangle\right|\right), \end{aligned}$$

which proves our assertion for a superquadratic function.

On the other hand, if $f$ is subquadratic function, then $-f$ is superquadratic, so the inequality with a reversed sign holds due to the previous case. □

**Remark 2.** Obviously, our Theorem 1 can be regarded as an improvement of Theorem A in two general cases. In the first case, if $f$ is a non-negative superquadratic function, then $f$ is convex (for more details, see [**2**]), so in this setting Theorem 1 yields a sharper estimate than Theorem A. On the other hand, if $f$ is non-positive subquadratic function which is concave, we also obtain improved form of Theorem A. The basic examples of such particular superquadratic and subquadratic functions will be discussed below.

Our first application of Theorem 1 refers to the power function $f : [0, \infty) \to \mathbb{R}$, $f(x) = x^p$, $p > 0$. We have already discussed that if $p \geq 2$, then $f$ is superquadratic, while for $0 < p \leq 2$ $f$ is subquadratic (for more details, see [**2, 3**]). The following consequence is an extension of the Hölder-McCarthy inequality in a view of Theorem 1. Recall that the Hölder-McCarthy inequality asserts that

$$\langle A\zeta, \zeta\rangle^r \leq \langle A^r\zeta, \zeta\rangle \tag{9}$$

holds for all $r \geq 1$ and for every unit vector $\zeta \in \mathscr{H}$, while for $0 < r < 1$ the sign of inequality is reversed.

**Corollary 3.** *Let $p \geq 2$ and let $\eta, \zeta \in \mathscr{H}$ be such that $\|\eta\|^2 - \|\zeta\|^2 = 1$. If $A$ and $B$ are positive operators, then holds the inequality*

$$\begin{aligned}(\langle A\eta, \eta\rangle - \langle B\zeta, \zeta\rangle)^p \geq \|\eta\|^{2(1-p)} \langle A\eta, \eta\rangle^p - \langle B^p\zeta, \zeta\rangle + \left\langle \left| B - \frac{1}{\|\zeta\|^2}\langle B\zeta, \zeta\rangle\right|^p \zeta, \zeta\right\rangle \\ + \|\zeta\|^{2p} \left| \frac{1}{\|\eta\|^2}\langle A\eta, \eta\rangle - \frac{1}{\|\zeta\|^2}\langle B\zeta, \zeta\rangle\right|^p \\ + \|\zeta\|^{2} \left| \frac{1}{\|\eta\|^2}\langle A\eta, \eta\rangle - \frac{1}{\|\zeta\|^2}\langle B\zeta, \zeta\rangle\right|^p,\end{aligned} \tag{10}$$

*provided that $\langle A\eta, \eta\rangle - \langle B\zeta, \zeta\rangle \geq 0$. If $0 < p \leq 2$, then the sign of inequality* (10) *is reversed.*

**Remark 4.** It should be noticed here that if $p = 2$, then holds the equality in (10).

**Remark 5.** If the superquadratic function $f$ is non-positive and concave, then our Theorem 1 also provides a reverse of Theorem A. Indeed, in this case we have

$$\begin{aligned}0 &\geq f(\langle A\eta, \eta\rangle - \langle B\zeta, \zeta\rangle) - \|\eta\|^2 f\left(\frac{1}{\|\eta\|^2}\langle A\eta, \eta\rangle\right) + \langle f(B)\zeta, \zeta\rangle \\ &\geq f\left(\|\zeta\|^2 \left|\frac{1}{\|\eta\|^2}\langle A\eta, \eta\rangle - \frac{1}{\|\zeta\|^2}\langle B\zeta, \zeta\rangle\right|\right) + \|\zeta\|^2 f\left(\left|\frac{1}{\|\eta\|^2}\langle A\eta, \eta\rangle - \frac{1}{\|\zeta\|^2}\langle B\zeta, \zeta\rangle\right|\right) \\ &\quad + \left\langle f\left(\left|B - \frac{1}{\|\zeta\|^2}\langle B\zeta, \zeta\rangle\right|\right)\zeta, \zeta\right\rangle,\end{aligned}$$

where the second inequality sign provides the corresponding reverse. For example, if $1 \leq p \leq 2$, then $f(x) = -x^p$ is superquadratic and concave function. It should be noticed here that this case has been covered by Corollary 3. More precisely, it corresponds to the reverse relation in (10).

**Remark 6.** We have previously discussed the case of a non-negative superquadratic function. Such function is convex, so in this case inequality (6) provides a sharper estimate for the Jensen operator inequality (1). In this case, Theorem 1 can be regarded as an extension of inequality (6). Indeed, if $\|\zeta\| = 1$ and $\eta = \sqrt{2}\zeta$, then $\|\eta\|^2 - \|\zeta\|^2 = 1$, so by putting $B := A$ Theorem 1 reduces to inequality (6). In particular, in this setting relation (10) reduces to

$$\langle A\zeta, \zeta\rangle^p \leq \langle A^p\zeta, \zeta\rangle - \langle |A - \langle A\zeta, \zeta\rangle|^p\zeta, \zeta\rangle,$$

which holds for $p \geq 2$, while for $0 < p \leq 2$, the sign of inequality is reversed (see also [**9**]). It should be noticed here that this inequality represents a refinement of the Hölder-McCarthy inequality (9).

In order to conclude this section, we give a real discrete form of Theorem 1. For that sake, assume that $a_1, a_2, \dots, a_k$ and $b_1, b_2 \dots, b_m$ are non-negative real numbers. Assume in addition that $\alpha_1, \alpha_2, \dots, \alpha_k$ and $\beta_1, \beta_2, \dots, \beta_m$ are also non-negative real numbers such that $\sum_{i=1}^k \alpha_i - \sum_{j=1}^m \beta_j = 1$. Now, define diagonal matrices $A = \operatorname{diag}(a_1, a_2, \dots, a_k)$, $B = \operatorname{diag}(b_1, b_2, \dots, b_m)$ and vectors $\eta = [\sqrt{\alpha_1}, \sqrt{\alpha_2}, \dots, \sqrt{\alpha_k}]^T \in \mathbb{C}^k$, $\zeta = [\sqrt{\beta_1}, \sqrt{\beta_2}, \dots, \sqrt{\beta_m}]^T \in \mathbb{C}^m$. Obviously, $A, B$ are positive semi-definite matrices and $\|\eta\|^2 - \|\zeta\|^2 = 1$. Therefore, in this setting we have:

$$\langle A\eta, \eta\rangle = \sum_{i=1}^k \alpha_i a_i, \qquad \langle B\zeta, \zeta\rangle = \sum_{j=1}^m \beta_j b_j,$$

$$\left\langle f\left(\left|B - \frac{1}{\|\zeta\|^2}\langle B\zeta, \zeta\rangle\right|\right)\zeta, \zeta\right\rangle = \sum_{j=1}^m \beta_j f\left(\left|b_j - \frac{1}{\sum_{\ell=1}^m \beta_\ell}\sum_{j=1}^m \beta_j b_j\right|\right),$$

$$\frac{1}{\|\eta\|^2}\langle A\eta, \eta\rangle - \frac{1}{\|\zeta\|^2}\langle B\zeta, \zeta\rangle = \frac{1}{\sum_{i=1}^k \alpha_i}\sum_{i=1}^k \alpha_i a_i - \frac{1}{\sum_{j=1}^m \beta_j}\sum_{j=1}^m \beta_j b_j.$$

As a consequence, considering Theorem 1 in the above setting, we obtain the following external Jensen-type inequality in a real discrete form.

**Corollary 7.** *Let $a_i, \alpha_i$, $i = 1, 2, \dots, k$, and $b_j, \beta_j$, $j = 1, 2, \dots, m$, be non-negative real numbers such that $\sum_{i=1}^k \alpha_i - \sum_{j=1}^m \beta_j = 1$. If $f : [0, \infty) \to \mathbb{R}$ is a continuous superquadratic function, then holds the inequality*

$$\begin{aligned} & f\left(\sum_{i=1}^k \alpha_i a_i - \sum_{j=1}^m \beta_j b_j\right) \\ & \geq \left(\sum_{i=1}^k \alpha_i\right) f\left(\frac{1}{\sum_{i=1}^k \alpha_i}\sum_{i=1}^k \alpha_i a_i\right) - \sum_{j=1}^m \beta_j f(b_j) + \sum_{j=1}^m \beta_j f\left(\left|b_j - \frac{1}{\sum_{\ell=1}^m \beta_\ell}\sum_{j=1}^m \beta_j b_j\right|\right) \\ & \quad + f\left(\sum_{\ell=1}^m \beta_\ell \left|\frac{1}{\sum_{i=1}^k \alpha_i}\sum_{i=1}^k \alpha_i a_i - \frac{1}{\sum_{j=1}^m \beta_j}\sum_{j=1}^m \beta_j b_j\right|\right) \\ & \quad + \left(\sum_{\ell=1}^m \beta_\ell\right) f\left(\left|\frac{1}{\sum_{i=1}^k \alpha_i}\sum_{i=1}^k \alpha_i a_i - \frac{1}{\sum_{j=1}^m \beta_j}\sum_{j=1}^m \beta_j b_j\right|\right), \end{aligned}$$

*provided that $\sum_{i=1}^k \alpha_i a_i - \sum_{j=1}^m \beta_j b_j \geq 0$. If $f$ is subquadratic, then the sign of inequality is reversed.*

**Remark 8.** In particular, if $k = m = 1$, $a_1 = x$, $b_1 = y$, $\alpha_1 = 1 + r$, $\beta_1 = r$, $r \geq 0$, the previous corollary yields the relation

$$f((1+r)x - ry) \geq (1+r)f(x) - rf(y) + rf(0) + f(r|x-y|) + rf(|x-y|), \tag{11}$$

provided that $(1+r)x - ry \geq 0$. Clearly, if $f : [0,\infty) \to \mathbb{R}$ is a subquadratic function, then the sign of inequality (11) is reversed. Therefore, if $f$ is non-positive concave subquadratic function (for example, $f(x) = -x^p$, $p \geq 2$), then the reverse inequality in (11) provides a refinement of the basic external Jensen-type inequality (2).

On the other hand, rewriting the Jensen inequality (5) with $(1+r)x - ry \geq 0$ instead of $x$, and with $\lambda = \frac{1}{1+r}$, we arrive at the inequality

$$f((1+r)x - ry) \geq (1+r)f(x) - rf(y) + f(r|x-y|) + rf(|x-y|), \tag{12}$$

which holds for superquadratic functions. Obviously, inequality (12) is more accurate than (11) since $f(0) \leq 0$. The same conclusion can be drawn for the reversed inequality and subquadratic functions. Of course, this is in accordance with the proof of Theorem 1 since we utilize some further estimates there.

## 3. A CONTINUOUS EXTENSION

Motivated by Theorem C, in this section we give a continuous extension of Theorem 1. Roughly speaking, we will establish a variant of Theorem 1 in which the operators $A$ and $B$ are replaced by non-negative continuous functions of these operators.

The crucial step in this direction is to derive the integral version of Corollary 7. Clearly, this can be done by applying a continuity argument to the corresponding discrete form. However, the corresponding result will be derived directly, by virtue of relations (4) and (5), that are characteristic for a class of superquadratic functions.

**Theorem 9.** *Let $\mu$ and $\nu$ be positive measures on $[a,b]$ and $[c,d]$, respectively, such that $\mu([a,b]) - \nu([c,d]) = 1$. Further, assume that $\phi$ and $\psi$ are non-negative integrable functions with respect to $\mu$ and $\nu$. If $f : [0,\infty) \to \mathbb{R}$ is a continuous superquadratic function, then holds the inequality*

$$\begin{aligned} f\left(\int_{[a,b]} \phi d\mu - \int_{[c,d]} \psi d\nu\right) &\geq \mu([a,b]) f\left(\frac{1}{\mu([a,b])}\int_{[a,b]} \phi d\mu\right) - \int_{[c,d]} (f\circ\psi) d\nu \\ &\quad + \int_{[c,d]} f\left(\left|\psi - \frac{1}{\nu([c,d])}\int_{[c,d]}\psi d\nu\right|\right) d\nu \\ &\quad + f\left(\nu([c,d])\left|\frac{1}{\mu([a,b])}\int_{[a,b]}\phi d\mu - \frac{1}{\nu([c,d])}\int_{[c,d]}\psi d\nu\right|\right) \\ &\quad + \nu([c,d]) f\left(\left|\frac{1}{\mu([a,b])}\int_{[a,b]}\phi d\mu - \frac{1}{\nu([c,d])}\int_{[c,d]}\psi d\nu\right|\right), \end{aligned} \tag{13}$$

*provided that $\int_{[a,b]} \phi d\mu - \int_{[c,d]} \psi d\nu \geq 0$. If $f$ is subquadratic, then the inequality sign is reversed.*

*Proof.* We start by finding an external form of relation (5). Namely, with $x = (1+r)v - rw$, $y = w$, and $\lambda = \frac{1}{1+r}$, inequality (5) can be rewritten as

$$f((1+r)v - rw) \geq (1+r)f(v) - rf(w) + f(r|v-w|) + rf(|v-w|),$$

provided that $(1+r)v - rw \geq 0$. Now, considering this relation with $r = \nu([c,d])$, $v = \frac{1}{\mu([a,b])}\int_{[a,b]}\phi d\mu$, $w = \frac{1}{\nu([c,d])}\int_{[c,d]}\psi d\nu$, it follows that

$$\begin{aligned} f\left(\int_{[a,b]}\phi d\mu - \int_{[c,d]}\psi d\nu\right) \geq\ & \mu([a,b])f\left(\frac{1}{\mu([a,b])}\int_{[a,b]}\phi d\mu\right) \\ & - \nu([c,d])f\left(\frac{1}{\nu([c,d])}\int_{[c,d]}\psi d\nu\right) \\ & + f\left(\nu([c,d])\left|\frac{1}{\mu([a,b])}\int_{[a,b]}\phi d\mu - \frac{1}{\nu([c,d])}\int_{[c,d]}\psi d\nu\right|\right) \\ & + \nu([c,d])f\left(\left|\frac{1}{\mu([a,b])}\int_{[a,b]}\phi d\mu - \frac{1}{\nu([c,d])}\int_{[c,d]}\psi d\nu\right|\right). \end{aligned} \tag{14}$$

On the other hand, utilizing superquadraticity definition (4) with $x$ and $y$ respectively replaced by $\frac{1}{\nu([c,d])}\int_{[c,d]}\psi d\nu$ and $\psi(y)$, we have

$$\begin{aligned} & (f\circ\psi)(y) - f\left(\frac{1}{\nu([c,d])}\int_{[c,d]}\psi d\nu\right) - f\left(\left|\psi(y) - \frac{1}{\nu([c,d])}\int_{[c,d]}\psi d\nu\right|\right) \\ & \geq C\left(\psi(y) - \frac{1}{\nu([c,d])}\int_{[c,d]}\psi d\nu\right), \end{aligned}$$

where $C$ is the constant that corresponds to the value $x = \frac{1}{\nu([c,d])}\int_{[c,d]}\psi d\nu$. Now, integrating the last inequality over $[c,d]$ with respect to measure $\nu$ and rearranging, we obtain the relation

$$\begin{aligned} -\nu([c,d])f\left(\frac{1}{\nu([c,d])}\int_{[c,d]}\psi d\nu\right) \geq\ & -\int_{[c,d]}(f\circ\psi)d\nu \\ & + \int_{[c,d]} f\left(\left|\psi - \frac{1}{\nu([c,d])}\int_{[c,d]}\psi d\nu\right|\right)d\nu. \end{aligned} \tag{15}$$

Finally, combining (14) and (15), we obtain (13), as claimed. If $f$ is subquadratic function, then $-f$ is superquadratic function, so the inequality with a reversed sign holds due to the previous case. □

Now, we are ready to state and prove a continuous extension of Theorem 1. In order to derive the proof, besides integral relation (13), we will utilize the spectral representation for a continuous function of a self-adjoint operator.

**Theorem 10.** *Let $\eta,\zeta\in\mathscr{H}$ and let $g,\phi:[a,b]\to\mathbb{R}$, $h,\psi:[c,d]\to\mathbb{R}$ be non-negative continuous functions. Further, assume that $A$ and $B$ are self-adjoint operators with spectra contained in $[a,b]$ and $[c,d]$, respectively. If $f:[0,\infty)\to\mathbb{R}$ is a continuous superquadratic function, then holds the inequality*

$$\begin{aligned}
&(\langle g(A)\eta,\eta\rangle-\langle h(B)\zeta,\zeta\rangle)\, f\left(\frac{\langle\phi(A)g(A)\eta,\eta\rangle-\langle\psi(B)h(B)\zeta,\zeta\rangle}{\langle g(A)\eta,\eta\rangle-\langle h(B)\zeta,\zeta\rangle}\right)\\
&\geq \langle g(A)\eta,\eta\rangle f\left(\frac{\langle\phi(A)g(A)\eta,\eta\rangle}{\langle g(A)\eta,\eta\rangle}\right)-\langle f\circ\psi(B)h(B)\zeta,\zeta\rangle\\
&\quad+\left\langle f\left(\left|\psi(B)-\frac{\langle\psi(B)h(B)\zeta,\zeta\rangle}{\langle h(B)\zeta,\zeta\rangle}\right|\right)h(B)\zeta,\zeta\right\rangle\\
&\quad+\langle h(B)\zeta,\zeta\rangle f\left(\left|\frac{\langle\phi(A)g(A)\eta,\eta\rangle}{\langle g(A)\eta,\eta\rangle}-\frac{\langle\psi(B)h(B)\zeta,\zeta\rangle}{\langle h(B)\zeta,\zeta\rangle}\right|\right)\\
&\quad+(\langle g(A)\eta,\eta\rangle-\langle h(B)\zeta,\zeta\rangle)\\
&\qquad\times f\left(\frac{\langle h(B)\zeta,\zeta\rangle}{\langle g(A)\eta,\eta\rangle-\langle h(B)\zeta,\zeta\rangle}\left|\frac{\langle\phi(A)g(A)\eta,\eta\rangle}{\langle g(A)\eta,\eta\rangle}-\frac{\langle\psi(B)h(B)\zeta,\zeta\rangle}{\langle h(B)\zeta,\zeta\rangle}\right|\right),
\end{aligned}$$

*provided that $\langle\phi(A)g(A)\eta,\eta\rangle-\langle\psi(B)h(B)\zeta,\zeta\rangle\geq 0$ and $\langle g(A)\eta,\eta\rangle-\langle h(B)\zeta,\zeta\rangle>0$. If $f$ is subquadratic, then the sign of inequality is reversed.*

*Proof.* Let $\lambda:[a,b]\to\mathbb{R}$ and $\theta:[c,d]\to\mathbb{R}$ be non-decreasing continuous functions. From the hypotheses, $g$ and $h$ are non-negative continuous functions on intervals $[a,b]$ and $[c,d]$, respectively. Now, for every $t\in[a,b]$ and $s\in[c,d]$, we define

$$\mu(t)=\frac{1}{\int_a^b g(r)d\lambda(r)-\int_c^d h(r)d\theta(r)}\int_a^t g(r)d\lambda(r)$$

and

$$\nu(s)=\frac{1}{\int_a^b g(r)d\lambda(r)-\int_c^d h(r)d\theta(r)}\int_c^s h(r)d\theta(r).$$

Obviously, $\mu$ and $\nu$ are positive measures on $[a,b]$ and $[c,d]$ satisfying condition $\int_a^b d\mu(t)-\int_c^d d\nu(s)=1$. Since measures $\mu$ and $\nu$ satisfy conditions as in Theorem 9, utilizing (13), we obtain the inequality

$$\begin{aligned}
&f\left(\frac{1}{\Gamma}\left(\int_a^b\phi(t)g(t)d\lambda(t)-\int_c^d\psi(s)h(s)d\theta(s)\right)\right)\\
&\geq\left(\frac{1}{\Gamma}\int_a^b g(r)d\lambda(r)\right)f\left(\frac{1}{\int_a^b g(r)d\lambda(r)}\int_a^b\phi(t)g(t)d\lambda(t)\right)-\frac{1}{\Gamma}\int_c^d(fo\psi)(s)h(s)d\theta(s)\\
&\quad+\frac{1}{\Gamma}\int_c^d f\left(\left|\psi(s)-\frac{1}{\int_c^d h(r)d\theta(r)}\int_c^d\psi(r)h(r)d\theta(r)\right|\right)h(s)d\theta(s)\\
&\quad+f\left(\frac{1}{\Gamma}\int_c^d h(s)d\theta(s)\left|\frac{1}{\int_a^b g(r)d\lambda(r)}\int_a^b\phi(t)g(t)d\lambda(t)-\frac{1}{\int_c^d h(r)d\theta(r)}\int_c^d\psi(s)h(s)d\theta(s)\right|\right)\\
&\quad+\frac{1}{\Gamma}\int_c^d h(s)d\theta(s)f\left(\left|\frac{1}{\int_a^b g(r)d\lambda(r)}\int_a^b\phi(t)g(t)d\lambda(t)-\frac{1}{\int_c^d h(r)d\theta(r)}\int_c^d\psi(s)h(s)d\theta(s)\right|\right),
\end{aligned}$$

where $\Gamma = \int_a^b g(r)d\lambda(r) - \int_c^d h(r)d\theta(r)$.

Our next step is to rewrite the previous inequality in terms of the spectral representation. For that sake, fix vectors $\eta, \zeta \in \mathscr{H}$ and the real number $\epsilon > 0$. In addition, assume that $A$ and $B$ are self-adjoint operators with spectral families $\{E_t\}$ and $\{F_s\}$, respectively. Now, let the functions $\lambda : [m_A - \epsilon, M_A] \to \mathbb{R}$ and $\theta : [m_B - \epsilon, M_B] \to \mathbb{R}$ be defined by

$$\lambda(t) = \langle E_t\eta, \eta\rangle \quad \text{and} \quad \theta(s) = \langle F_s\zeta, \zeta\rangle, \qquad t \in [m_A - \epsilon, M_A], \quad s \in [m_B - \epsilon, M_B],$$

where $m_A = \min \operatorname{sp}(A)$, $M_A = \max \operatorname{sp}(A)$, $m_B = \min \operatorname{sp}(B)$ and $M_B = \max \operatorname{sp}(B)$. In this setting, the previous inequality can be rewritten as

$$\begin{aligned}
& f\left(\frac{1}{\widetilde{\Gamma}}\left(\int_{m_A-\epsilon}^{M_A} \phi(t)g(t)d\langle E_t\eta, \eta\rangle - \int_{m_B-\epsilon}^{M_B} \psi(s)h(s)d\langle F_s\zeta, \zeta\rangle\right)\right) \\
& \geq \left(\frac{1}{\widetilde{\Gamma}}\int_{m_A-\epsilon}^{M_A} g(r)d\langle E_r\eta, \eta\rangle\right) f\left(\frac{1}{\int_{m_A-\epsilon}^{M_A} g(r)d\langle E_r\eta, \eta\rangle}\int_{m_A-\epsilon}^{M_A} \phi(t)g(t)d\langle E_t\eta, \eta\rangle\right) \\
& \quad - \frac{1}{\widetilde{\Gamma}}\int_{m_B-\epsilon}^{M_B} (f \circ \psi)(s)h(s)d\langle F_s\zeta, \zeta\rangle \\
& \quad + \frac{1}{\widetilde{\Gamma}}\int_{m_B-\epsilon}^{M_B} f\left(\left|\psi(s) - \frac{1}{\int_{m_B-\epsilon}^{M_B} h(r)d\langle F_r\zeta, \zeta\rangle}\int_{m_B-\epsilon}^{M_B} \psi(r)h(r)d\langle F_r\zeta, \zeta\rangle\right|\right) h(s)d\langle F_s\zeta, \zeta\rangle \\
& \quad + f\Big(\frac{1}{\widetilde{\Gamma}}\int_{m_B-\epsilon}^{M_B} h(s)d\langle F_s\zeta, \zeta\rangle\Big| \frac{1}{\int_{m_A-\epsilon}^{M_A} g(r)d\langle E_t\eta, \eta\rangle}\int_{m_A-\epsilon}^{M_A} \phi(t)g(t)d\langle E_r\eta, \eta\rangle \\
& \qquad\qquad - \frac{1}{\int_{m_B-\epsilon}^{M_B} h(r)d\langle F_r\zeta, \zeta\rangle}\int_{m_B-\epsilon}^{M_B} \psi(s)h(s)d\langle F_s\zeta, \zeta\rangle\Big|\Big) \\
& \quad + \frac{1}{\widetilde{\Gamma}}\int_{m_B-\epsilon}^{M_B} h(s)d\langle F_s\zeta, \zeta\rangle f\Big(\Big|\frac{1}{\int_{m_A-\epsilon}^{M_A} g(r)d\langle E_r\eta, \eta\rangle}\int_{m_A-\epsilon}^{M_A} \phi(t)g(t)d\langle E_t\eta, \eta\rangle \\
& \qquad\qquad - \frac{1}{\int_{m_B-\epsilon}^{M_B} h(r)d\langle F_r\zeta, \zeta\rangle}\int_{m_B-\epsilon}^{M_B} \psi(s)h(s)d\langle F_s\zeta, \zeta\rangle\Big|\Big),
\end{aligned}$$

where $\widetilde{\Gamma} = \int_{m_A-\epsilon}^{M_A} g(r)d\langle E_r\eta, \eta\rangle - \int_{m_B-\epsilon}^{M_B} h(r)d\langle F_r\zeta, \zeta\rangle$.

Finally, letting $\epsilon \to 0^+$ and utilizing the spectral representation for the function of an operator, the last inequality yields

$$\begin{aligned}
& f\left(\frac{\langle\phi(A)g(A)\eta, \eta\rangle - \langle\psi(B)h(B)\zeta, \zeta\rangle}{\langle g(A)\eta, \eta\rangle - \langle h(B)\zeta, \zeta\rangle}\right) \geq \left(\frac{\langle g(A)\eta, \eta\rangle}{\langle g(A)\eta, \eta\rangle - \langle h(B)\zeta, \zeta\rangle}\right) f\left(\frac{\langle\phi(A)g(A)\eta, \eta\rangle}{\langle g(A)\eta, \eta\rangle}\right) \\
& \quad - \frac{\langle fo\psi(B)h(B)\zeta, \zeta\rangle}{\langle g(A)\eta, \eta\rangle - \langle h(B)\zeta, \zeta\rangle} + \frac{\left\langle f\left(\left|\psi(B) - \frac{\langle\psi(B)h(B)\zeta, \zeta\rangle}{\langle h(B)\zeta, \zeta\rangle}\right|\right) h(B)\zeta, \zeta\right\rangle}{\langle g(A)\eta, \eta\rangle - \langle h(B)\zeta, \zeta\rangle} \\
& \quad + f\left(\frac{\langle h(B)\zeta, \zeta\rangle}{\langle g(A)\eta, \eta\rangle - \langle h(B)\zeta, \zeta\rangle}\left|\frac{\langle\phi(A)g(A)\eta, \eta\rangle}{\langle g(A)\eta, \eta\rangle} - \frac{\langle\psi(B)h(B)\zeta, \zeta\rangle}{\langle h(B)\zeta, \zeta\rangle}\right|\right) \\
& \quad + \frac{\langle h(B)\zeta, \zeta\rangle}{\langle g(A)\eta, \eta\rangle - \langle h(B)\zeta, \zeta\rangle} f\left(\left|\frac{\langle\phi(A)g(A)\eta, \eta\rangle}{\langle g(A)\eta, \eta\rangle} - \frac{\langle\psi(B)h(B)\zeta, \zeta\rangle}{\langle h(B)\zeta, \zeta\rangle}\right|\right),
\end{aligned}$$

which proves our assertion for the case of a superquadratic function. If $f$ is subquadratic function, then $-f$ is superquadratic function, so the inequality with a reversed sign holds due to the previous case. □

**Remark 11.** Obviously, Theorem 10 is a continuous extension of Theorem 1. To see this, put $g(t) = 1$, $\phi(t) = t$, $t \in [a, b]$, and $h(s) = 1$, $\psi(s) = s$, $s \in [c, d]$. In this setting, if $\eta, \zeta \in \mathscr{H}$ are such that $\|\eta\|^2 - \|\zeta\|^2 = 1$, Theorem 10 reduces to Theorem 1. In addition, according to Remark 2, our Theorem 10 can also be regarded as an improvement of Theorem A for the case of non-negative superquadratic function and for the case of non-positive subquadratic function which is concave.

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