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THE MULTI-VARIABLE UNIFIED FAMILY OF GENERALIZED APOSTOL-TYPE POLYNOMIALS

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The aim of this paper is to investigate and give a new family of multi-variable Apostol-type polynomials. This family is related to Apostol-Euler, Apostol-Bernoulli, Apostol-Genocchi and Apostol-laguerre polynomials. Moreover, we derive some implicit summation formulae and general symmetry identities. The new family of polynomials introduced here, gives many interesting special cases.

1. INTRODUCTION

The 2-variable general polynomials (2VGP) $p_n(x,y)$ are defined by, [**10**]

$$e^{xt}\,\varphi(y,t)=\sum_{n=0}^{\infty}p_n(x,y)\frac{t^n}{n!},\ p_0(x,y)=1, \tag{1}$$

where $\varphi(y,t)$ has (at least the formal) series expansion

$$\varphi(y,t)=\sum_{n=0}^{\infty}\varphi_n(y)\frac{t^n}{n!},\ \varphi_0(y)\neq 0. \tag{2}$$

These polynomials $p_n(x,y)$ have many intersting special polynomials of two variables.
Generating functions for certain members belonging to the (2VGP) are given as

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follows.
The higher order Hermite polynomials, may be called the Kampé de Feriet polynomials of order m or the Gould– Hopper polynomials $H_n^{(m)}(x,y)$ are defined by, [7]

$$e^{xt+yt^m} = \sum_{n=0}^{\infty} H_n^{(m)}(x,y)\frac{t^n}{n!}. \tag{3}$$

The 2-variable Hermite Kampé de Feriet polynomials $H_n(x,y)$ are defined by, [1]

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!}. \tag{4}$$

The generalized Apostol-Bernoulli polynomials $B_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in \mathbb{C}$, the generalized Apostol-Euler polynomials $E_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in \mathbb{C}$ and the generalized Apostol-Genocchi polynomials $G_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in \mathbb{C}$ are defined, see ([13], [14], [15], [16], [17] and [23]) respectively, by

$$\left(\frac{t}{\lambda e^t - 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!},$$

$$(\mid t \mid < 2\pi, \text{ when } \lambda = 1; \mid t \mid < \mid \log\lambda \mid, \text{ when } \lambda \neq 1, \ 1^{\alpha} := 1). \tag{5}$$

$$\left(\frac{2}{\lambda e^t + 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!},$$

$$(\mid t \mid < \pi, \text{ when } \lambda = 1; \mid t \mid < \mid \log(-\lambda) \mid, \text{ when } \lambda \neq 1, \ 1^{\alpha} := 1). \tag{6}$$

and

$$\left(\frac{2t}{\lambda e^t + 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!},$$

$$(\mid t \mid < \pi, \text{ when } \lambda = 1; \mid t \mid < \mid \log(-\lambda) \mid, \text{ when } \lambda \neq 1, \ 1^{\alpha} := 1). \tag{7}$$

Khan at el. [11] defined the 2-variable Apostol type polynomials of order α, by

$$\sum_{n=0}^{\infty} {}_pF_n(x,y;\lambda;\mu,\nu)\frac{t^n}{n!} = \left(\frac{2^{\mu}t^{\nu}}{\lambda e^t + 1}\right)^{\alpha} e^{xt}\varphi(y,t), \quad |t| < |\log(-\lambda)|. \tag{8}$$

El-Desouky and Gomaa [5] introduced a unification of multiparameter Apostol-type polynomials by means of the following generating function

$$F_{\bar{\alpha}_r}^{[m-1,r]} = \frac{t^{rkm} \ 2^{rm(1-k)} \ c^{xt}}{\prod_{i=0}^{r-1}\left(\alpha_i b^t - a^t \sum_{\ell=0}^{m-1} \frac{t^{\ell}}{\ell!}\right)} = \sum_{n=0}^{\infty} M_n^{[m-1,r]}(x;k;a,b,c;\bar{\alpha}_r)\frac{t^n}{n!}$$

$$\left(|t\log\left(\frac{b}{a}\right)| < 2\pi \;\; when\; m=1\; and\; \alpha_i = 1\right);$$

$$\left(|t\log\left(\frac{b}{a}\right)| < |\log(\alpha_i)| \;\; when\; m=1\; and\; \alpha_i \neq 1; \forall i = 0,1,2,...,r-1\right), \tag{9}$$

where $k \in N_0$; $r \in \mathbb{C}$; $\bar{\alpha}_r = (\alpha_0, \alpha_1, ..., \alpha_{r-1})$ is a sequence of complex numbers. If $km = \gamma$ and $m(1-k) = \mu$ in (9), we obtain

$$F_{\bar{\alpha}_r}^{[r]} = \frac{t^{r\gamma}\;\; 2^{r\mu}\;\; c^{xt}}{\prod_{i=0}^{r-1}(\alpha_i b^t - a^t)} = \sum_{n=0}^{\infty} M_n^{[r]}(x;k;a,b,c;\bar{\alpha}_r)\frac{t^n}{n!}. \tag{10}$$

In this article, the multi-variable unified family of generalized Apostol-Euler, Bernoulli and Genocchi polynomials are introduced and investigate their properties. Also, some important summation formulas are given. Moreover, some general symmetric identities are derived .

2. THE MULTI-VARIABLE UNIFIED FAMILY OF GENERALIZED APOSTOL-EULER, BERNOULLI AND GENOCCHI POLYNOMIALS

The multi-variable unified family of generalized Apostol-Euler, Bernoulli and Genocchi polynomials of order r, denoted by ${}_p\mathbb{M}_n^{(r)}(x, y_1, y_2, ..., y_m, k, a, b; \bar{\alpha}_r)$ will be defined as the discrete Apostol type convolution of the multi-variable general polynomials $p_n(x, y_1, y_2, ..., y_m)$.

Definition 1. The multi-variable general polynomials are defined by

$$c^{xt}\phi(y_1, y_2, ..., y_m, t) = \sum_{n=0}^{\infty} p_n(x, y_1, y_2, ..., y_m)\frac{t^n}{n!}, \tag{11}$$

where

$$\phi(y_1, y_2, ..., y_m, t) = \sum_{n=0}^{\infty} \phi_n(y_1, y_2, ..., y_m)\frac{t^n}{n!}. \tag{12}$$

Definition 2. Let $a,\ b,\ c \in \mathbf{R}^+$ $(a \neq b)$, $n \in N_0$ and $m \in N$, the multi-variable unified family of generalized Apostol-type polynomials is defined in terms of a unification of multi-parameter Apostol-type polynomial and multi-variables general polynomials as follows

$${}_p\mathbb{M}_n^{(r)}(x, y_1, y_2, ..., y_m, k, a, b; \bar{\alpha}_r) = \sum_{m=0}^{n}\binom{n}{m} M_{n-m}^{[r]}(x; a, b; \bar{\alpha}_r)\phi_m(x, y_1, y_2, ..., y_m). \tag{13}$$

Theorem 1. Let $a, b, c \in \mathbb{R}^+$ $(a \neq b)$, $n \in N_0$ and $m \in N$, the generating function of ${}_p\mathbb{M}_n^{(r)}(x, y_1, y_2, ..., y_m; k, a, b, c; \bar{\alpha}_r)$ is given by

$$\sum_{n=0}^{\infty} {}_p\mathbb{M}_n^{(r)}(x, y_1, y_2, ..., y_m, k, a, b; \bar{\alpha}_r)\frac{t^n}{n!} = \frac{t^{r\gamma}\ 2^{r\mu}}{\prod\limits_{i=0}^{r-1}(\alpha_i b^t - a^t)} c^{xt}\ \phi(y_1, y_2, ..., y_m, t) \tag{14}$$

$$\left(|t \log\left(\frac{b}{a}\right)| < 2\pi \ \ \textit{when } m = 1 \textit{ and } \alpha_i = 1\right),$$

$$\left(|t \log\left(\frac{b}{a}\right)| < |\log(\alpha_i)| \ \ \textit{when } m = 1 \textit{ and } \alpha_i \neq 1; \forall i = 0, 1, 2, ..., r-1\right),$$

where $k \in N_0$; $r \in \mathbb{C}$; $\bar{\alpha}_r = (\alpha_0, \alpha_1, ..., \alpha_{r-1})$ is a sequence of complex numbers.

Proof. From Eq.(13), we have

$$\begin{aligned}\sum_{n=0}^{\infty} {}_p\mathbb{M}_n^{(r)}(x, y_1, y_2, ..., y_m, k, a, b; \bar{\alpha}_r)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m} M_{n-m}^{[r]}(x; a, b; \bar{\alpha}_r) \\ &\quad p(x, y_1, y_2, ..., y_m)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} M_n^{[r]}(x; a, b; \bar{\alpha}_r)\frac{t^n}{n!} \\ &\quad \sum_{m=0}^{\infty} \phi_m(x, y_1, y_2, ..., y_m)\frac{t^m}{m!}.\end{aligned}$$

From Eq. (10) and Eq. (12), we get (14). □

Remark 1. Setting $x = 0$ and $y_j = 0$, $j = 1, 2, ..., m$ in (14), then we obtain the new multi-variable Apostol-type numbers, as

$${}_p\mathbb{M}_n^{(r)}(0, 0, k, a, b; \bar{\alpha}_r) = {}_p\mathbb{M}_n^{(r)}(k, a, b; \bar{\alpha}_r). \tag{15}$$

1.2. SPECIAL CASES

1. Setting $\alpha_i = -\lambda,\ b = e,\ c = e,\ m = 1 \textit{ and } a = 1 \textit{ in }$ (14), we have

$$\begin{aligned}\sum_{n=0}^{\infty} {}_p\mathbb{M}_n^{(r)}(x, y; \gamma, \mu, 1; -\lambda)\frac{t^n}{n!} &= \frac{2^{r\mu}t^{r\gamma}}{(-\lambda e^t - 1)^r}e^{xt}\ \ \varphi(y, t) \\ &= (-1)^r\left(\frac{2^{\mu}t^k}{\lambda e^t + 1}\right)^r e^{xt}\ \ \varphi(y, t) \\ &= \sum_{n=0}^{\infty}(-1)^r {}_pF_n^{(r)}(x, y; \lambda, \mu, \gamma)\frac{t^n}{n!}.\end{aligned}$$

Thus, equating the coefficients of t^n on both sides, gives
${}_p\mathbb{M}_n^{(r)}(x,y;k,1;-\lambda)=(-1)^r{}_pF_n^{(r)}(x,y;\lambda,\mu,\gamma)$.
(The 2-variable Apostol type polynomials of order α, see [**11**]).

2. Setting $\alpha_i=\lambda,\ b=e,\ a=1,\ c=e,\ m=1,\ \mu=0$ *and* $\gamma=1$ *in* (14), we have

$$\begin{aligned}\sum_{n=0}^{\infty}{}_p\mathbb{M}_n^{(r)}(x,y;1,1,e;\lambda)\frac{t^n}{n!} &= \frac{t^r}{(\lambda e^t-1)^r}e^{xt}\ \varphi(y,t)\\ &= \left(\frac{t}{\lambda e^t-1}\right)^r e^{xt}\ \varphi(y,t)\\ &= \sum_{n=0}^{\infty}{}_pB_n^{(r)}(x,y;\lambda)\frac{t^n}{n!}.\end{aligned}$$

Thus, equating the coefficients of t^n on both sides, we obtain
${}_p\mathbb{M}_n^{(r)}(x,y;1,1,e;\lambda)={}_pB_n^{(r)}(x,y;\lambda)$.
(The 2-variable Apostol – Bernoulli polynomials of order r, see [**11**]).

3. Setting $\alpha_i=-\lambda,\ b=e,\ a=1,\ c=e,\ m=1,\ \mu=1$ *and* $\gamma=0$ *in* (14), we have

$$\begin{aligned}\sum_{n=0}^{\infty}{}_p\mathbb{M}_n^{(r)}(x,y;0,1,e;-\lambda)\frac{t^n}{n!} &= \frac{2^r}{(-\lambda e^t-1)^r}e^{xt}\ \varphi(y,t)\\ &= (-1)^r\left(\frac{2}{\lambda e^t+1}\right)^r e^{xt}\ \varphi(y,t)\\ &= \sum_{n=0}^{\infty}(-1)^r{}_pE_n^{(r)}(x,y;\lambda)\frac{t^n}{n!}.\end{aligned}$$

Thus, equating the coefficients of t^n on both sides, we get
${}_p\mathbb{M}_n^{(r)}(x,y;0,1,e;-\lambda)=(-1)^{-r}{}_pE_n^{(r)}(x,y;\lambda)$.
(The 2-variable Apostol - Euler polynomials of order r, see [**11**]).

4. Setting $\alpha_i=-\lambda,\ b=e,\ a=1,\ c=e,\ m=1,\ \mu=1$ *and* $\gamma=1$ *in* (14), we have

$$\begin{aligned}\sum_{n=0}^{\infty}{}_p\mathbb{M}_n^{(r)}(x,y;1,1,e;-\lambda)\frac{t^n}{n!} &= \frac{(2t)^r}{(-\lambda e^t-1)^r}e^{xt}\ \varphi(y,t)\\ &= \left(\frac{2t}{\lambda e^t+1}\right)^r e^{xt}\ \varphi(y,t)\\ &= \sum_{n=0}^{\infty}{}_pG_n^{(r)}(x,y;\lambda)\frac{t^n}{n!}.\end{aligned}$$

Thus, equating the coefficients of t^n on both sides, we get
${}_p\mathbb{M}_n^{(r)}(x,y;1,1,e;-\lambda)={}_pG_n^{(r)}(x,y;\lambda)$.
(The 2-variable Apostol - Genocchi polynomials of order r, see [**11**]).

5. Setting $\alpha_i = \lambda,\ b = e,\ a = 1,\ c = e,\ \mu = 1,\ \gamma = 1$ *and* $\phi(y_1, y_2, ..., y_m) = e^{y_1 t^2 + y_2 t^3}$ *in* (14), we have

$$\begin{aligned}\sum_{n=0}^{\infty} {}_p\mathbb{M}_n^{(r)}(x, y_1, y_2; 1, 1, e; \lambda)\frac{t^n}{n!} &= \frac{(2t)^r}{(-\lambda e^t - 1)^r} e^{xt} \ e^{y_1 t^2 + y_2 t^3} \\ &= \left(\frac{2t}{\lambda e^t - 1}\right)^r e^{xt + y_1 t^2 + y_2 t^3} \\ &= \sum_{n=0}^{\infty} {}_H B_n^{(r)}(x, y_1, y_2; \lambda)\frac{t^n}{n!}.\end{aligned}$$

Thus, equating the coefficients of t^n on both sides, we get
${}_p\mathbb{M}_n^{(r)}(x, y; 1, 1, e; \lambda) = {}_p B_n^{(r)}(x, y_1, y_2; \lambda)$.
(The Hermite-based generalized Apostol- Bernoulli polynomials, see [**18**]).

6. Setting $\alpha_i = -\lambda,\ b = e,\ a = 1,\ c = e,\ \mu = 1,\ \gamma = 0$ *and* $\phi(y_1, y_2, ..., y_m) = e^{y_1 t^2 + y_2 t^3}$ *in* (14), we have

$$\begin{aligned}\sum_{n=0}^{\infty} {}_p\mathbb{M}_n^{(r)}(x, y_1, y_2; 0, 1, e; -\lambda)\frac{t^n}{n!} &= \frac{2^r}{(-\lambda e^t - 1)^r} e^{xt} \ e^{y_1 t^2 + y_2 t^3} \\ &= (-1)^{-r}\left(\frac{2}{\lambda e^t + 1}\right)^r e^{xt + y_1 t^2 + y_2 t^3} \\ &= \sum_{n=0}^{\infty} {}_H E_n^{(r)}(x, y_1, y_2; \lambda)\frac{t^n}{n!}.\end{aligned}$$

Thus, equating the coefficients of t^n on both sides, we get
${}_p\mathbb{M}_n^{(r)}(x, y; 0, 1, e; -\lambda) = (-1)^{-r}{}_H E_n^{(r)}(x, y_1, y_2; \lambda)$.
(The Hermite-based generalized Apostol - Euler polynomials, see [**18**]).

7. Setting $\alpha_i = -\lambda,\ b = e,\ a = 1,\ c = e,\ \mu = 1,\ \gamma = 1$ *and* $\phi(y_1, y_2, ..., y_m) = e^{y_1 t^2 + y_2 t^3}$ *in* (14), we have

$$\begin{aligned}\sum_{n=0}^{\infty} {}_p\mathbb{M}_n^{(r)}(x, y_1, y_2; 1, 1, e; -\lambda)\frac{t^n}{n!} &= \frac{(2t)^r}{(-\lambda e^t - 1)^r} e^{xt} \ e^{y_1 t^2 + y_2 t^3} \\ &= \left(\frac{2t}{\lambda e^t + 1}\right)^r e^{xt + y_1 t^2 + y_2 t^3} \\ &= \sum_{n=0}^{\infty} {}_H G_n^{(r)}(x, y_1, y_2; \lambda)\frac{t^n}{n!}.\end{aligned}$$

Thus, equating the coefficients of t^n on both sides, we get
${}_p\mathbb{M}_n^{(r)}(x, y_1, y_2; 1, 1, e; -\lambda) = {}_H G_n^{(r)}(x, y_1, y_2; \lambda)$.
(The Hermite-based generalized Apostol- Genocchi polynomials, see [**18**]).

8. Setting $\alpha_i = \lambda,\ \mu = 0,\ \gamma = 1$ *and* $y_j = 0$ *in* (14), we have

$$\begin{aligned}\sum_{n=0}^{\infty} {}_p\mathbb{M}_n^{(r)}(x,0;1,a,b;\lambda)\frac{t^n}{n!} &= \frac{t^r}{(\lambda b^t - a^t)^r} c^{xt} \\ &= \left(\frac{t}{\lambda b^t - a^t}\right)^r c^{xt} \\ &= \sum_{n=0}^{\infty} \mathfrak{B}_n^{(r)}(x;a,b,c;\lambda)\frac{t^n}{n!}.\end{aligned}$$

Thus, equating the coefficients of t^n on both sides, we get
${}_p\mathbb{M}_n^{(r)}(x,0;1,a,b;\lambda) = \mathfrak{G}_n^{(r)}(x;a,b,c;\lambda)$.
(The generalized Apostol- Bernoulli polynomials, see [**21**]).

9. Setting $\alpha_i = -\lambda,\ \mu = 1,\ \gamma = 0$ *and* $y_j = 0,\ j = 1, ..., m$ *in* (14), we have

$$\begin{aligned}\sum_{n=0}^{\infty} {}_p\mathbb{M}_n^{(r)}(x,0;1,0,a,b;-\lambda)\frac{t^n}{n!} &= \frac{2^r}{(-\lambda b^t - a^t)^r} e^{xt}\ \ c^x \\ &= (-1)^{-r}\left(\frac{2}{\lambda b^t + a^t}\right)^r c^{xt} \\ &= (-1)^{-r}\sum_{n=0}^{\infty} \mathfrak{E}_n^{(r)}(x;a,b,c;\lambda)\frac{t^n}{n!}.\end{aligned}$$

Thus, equating the coefficients of t^n on both sides, we get
${}_p\mathbb{M}_n^{(r)}(x,0;o,a,b;-\lambda) = (-1)^{-r}\mathfrak{E}_n^{(r)}(x;a,b,c;\lambda)$.
(The generalized Apostol- Euler polynomials, see [**22**]).

10. Setting $\alpha_i = -\lambda,\ \mu = 1,\ \gamma = 1$ *and* $y_j = 0$ *in* (14), we have

$$\begin{aligned}\sum_{n=0}^{\infty} {}_p\mathbb{M}_n^{(r)}(x,0;1,a,b;-\lambda)\frac{t^n}{n!} &= \frac{(2t)^r}{(-\lambda b^t - a^t)^r} c^{xt} \\ &= \left(\frac{2t}{\lambda b^t + a^t}\right)^r c^{xt} \\ &= \sum_{n=0}^{\infty} \mathfrak{G}_n^{(r)}(x;a,b,c;\lambda)\frac{t^n}{n!}.\end{aligned}$$

Thus, equating the coefficients of t^n on both sides, we get
${}_p\mathbb{M}_n^{(r)}(x,0;1,a,b;-\lambda) = \mathfrak{G}_n^{(r)}(x;a,b,c;\lambda)$.
(The generalized Apostol- Genocchi polynomials, see [**22**]).

11. Setting $\alpha_i = -\lambda,\ b = e,\ a = 1,\ c = e,\ m = 1,\ \mu = 1,\ \gamma = 1$ *and*

$\phi(y_1) = e^{y_1 t^2}$ *in* (14), we have

$$\sum_{n=0}^{\infty} {}_p\mathbb{M}_n^{(r)}(x, y_1; 1, 1, e; -\lambda)\frac{t^n}{n!} = \frac{(2t)^r}{(-\lambda e^t - 1)^r} e^{xt}\ e^{y_1 t^2} = \left(\frac{2t}{\lambda e^t + 1}\right)^r e^{xt + y_1 t^2} = \sum_{n=0}^{\infty} {}_H G_n^{(r)}(x, y_1; \lambda)\frac{t^n}{n!}.$$

Thus, equating the coefficients of t^n on both sides, we get
${}_p\mathbb{M}_n^{(r)}(x, y_1; 1, 1, e; -\lambda) = {}_H G_n^{(r)}(x, y_1; \lambda)$.
(The generalized Apostol- Genocchi polynomials, see [**9**]).

12. Setting $\alpha_i = -\lambda,\ c = e,\ m = 2,\ \mu = 1,\ \gamma = 1$ *and* $\phi(y_1, y_2, t) = e^{y_1 t^2} C_0(y_2 t)$ *in* (14), we have

$$\sum_{n=0}^{\infty} {}_p\mathbb{M}_n^{(r)}(x, y_1, y_2; 1, a, b, e; -\lambda)\frac{t^n}{n!} = \frac{(2t)^r}{(-\lambda b^t - a^t)^r} e^{xt}\ e^{y_1 t^2} C_0(y_2 t) = \left(\frac{2t}{\lambda b^t + a^t}\right)^r e^{xt + y_1 t^2} C_0(y_2 t) = \sum_{n=0}^{\infty} {}_L G_n^{(r)}(x, y_1, y_2; a, b, e; \lambda)\frac{t^n}{n!}.$$

Thus, equating the coefficients of t^n on both sides, we get
${}_p\mathbb{M}_n^{(r)}(x, y_1, y_2; 1, a, b; -\lambda) = {}_L G_n^{(r)}(x, y_1, y_2; a, b, e; \lambda)$.
(A generalization of Apostol-type Laguerre-Genocchi polynomials, see [**9**]).

13. Setting $m = 1,\ \mu = 1 - k,\ \gamma = k$ *and* $\phi(y_1, t) = c^{y_1 t^2}$ *in* (14), we have

$$\sum_{n=0}^{\infty} {}_p\mathbb{M}_n^{(r)}(x, y_1; k, a, b, c; \bar{\alpha}_r)\frac{t^n}{n!} = \frac{t^{rk} 2^{r(1-k)}}{\prod_{i=0}^{r-1} (\alpha_i b^t - a^t)} c^{xt}\ c^{y_1 t^2} = (-1)^{-r} \frac{(-1)^r t^{rk}\ \ 2^{r(1-k)}}{\prod_{i=0}^{r-1} (\alpha_i b^t - a^t)} c^{xt + y_1 t^2} = (-1)^{-r} \sum_{n=0}^{\infty} {}_H M_n^{(r)}(x, y; a, b, c; \bar{\alpha}_r)\frac{t^n}{n!}.$$

Thus, equating the coefficients of t^n on both sides, we get
${}_p\mathbb{M}_n^{(r)}(x, y_1; k, a, b, c; \bar{\alpha}_r) = (-1)^{-r} {}_H M_n^{(r)}(x, y; a, b, c; \bar{\alpha}_r)$.
(A generalization of Hermite-Genocchi polynomials, see [**6**]).

14. Setting $c = b = e, y_j = 0,\ \mu = 1 - k,\ \gamma = k$ *and* $a = 1$ *in* (14), we have

$$\begin{aligned}\sum_{n=0}^{\infty} {}_p\mathbb{M}_n^{(r)}(x; k, 1, e, e; \bar{\alpha}_r)\frac{t^n}{n!} &= \frac{t^{rk}2^{r(1-k)}}{\prod_{i=0}^{r-1}(\alpha_i e^t - 1)}e^{xt} \\ &= \frac{(-1)^r t^{rk}\ \ 2^{r(1-=k)}}{\prod_{i=0}^{r-1}(1 - \alpha_i e^t)}e^{xt} \\ &= \sum_{n=0}^{\infty} \mathbb{M}_n^{(r)}(x; k, \bar{\alpha}_r)\frac{t^n}{n!}.\end{aligned}$$

Thus, equating the coefficients of t^n on both sides, we get
${}_p\mathbb{M}_n^{(r)}(x; k, 1, e, e; \bar{\alpha}_r) = \mathbb{M}_n^{(r)}(x; k, \bar{\alpha}_r)$.
(A unified family of generalized Apostol-Euler, Bernoulli and Genocchi polynomials, see [**4**]).

15. Setting $\alpha_i = -\lambda,\ c = b = e,\ a = 1$ *and* $\phi(y_1, y_2, ..., y_m, t) = e^{y_1 t}\ C_0(-y_2 t^2)$ in (14), we have

$$\begin{aligned}&\sum_{n=0}^{\infty} {}_p\mathbb{M}_n^{(r)}(x, y_1, y_2; \gamma, \mu, 1, e, e; -\lambda)\frac{t^n}{n!} \\ = &\ \frac{(t^\gamma 2^\mu)^r}{(-\lambda e^t - 1)^r}e^{xt+y^1t^2}C_0(-y_2t^2) \\ = &\ (-1)^{-r}\left(\frac{t^\gamma 2^\mu}{\lambda e^t + 1}\right)^r e^{xt+y_1t^2}C_0(-y_2t^2) \\ = &\ (-1)^{-r}\sum_{n=0}^{\infty} {}_sF_n^{(r)}(x, y_1, y_2; \lambda; \gamma, \mu)\frac{t^n}{n!}.\end{aligned}$$

Thus, equating the coefficients of t^n on both sides, we get
${}_p\mathbb{M}_n^{(r)}(x, y_1, y_2; \gamma, \mu, 1, e, e; -\lambda) = (-1)^{-r}{}_sF_n^{(r)}(x, y_1, y_2; \lambda; \gamma, \mu)$.
(The generalized Apostol type Legendre-based polynomials, see [**8**]).

16. Setting $\alpha_i = \frac{\lambda}{u},\ c = b = e,\ a = 1,\ \mu = \gamma = 0$ *and* $\phi(y_1, y_2, ..., y_m, t) = \left(\frac{1-u}{u}\right)^r e^{y_1t^2+y_2t^3},\ u \neq 1$ *in* (14), we have

$$\begin{aligned}\sum_{n=0}^{\infty} {}_p\mathbb{M}_n^{(r)}(x, y_1, y_2; 0, 0, 1, e, e; \frac{\lambda}{u})\frac{t^n}{n!} &= \frac{e^{xt}}{\left(\frac{\lambda}{u}e^t - 1\right)^r}\left(\frac{1-u}{u}\right)^r e^{y_1t^2+y_2t^3} \\ &= \left(\frac{1-u}{\lambda e^t - u}\right)^r e^{xt+y_1t^2+y_2t^3} \\ &= \sum_{n=0}^{\infty} {}_H\xi_n^{(r)}(x, y_1, y_2; u, \lambda)\frac{t^n}{n!}.\end{aligned}$$

Thus, equating the coefficients of t^n on both sides, we get
${}_p\mathbb{M}_n^{(r)}(x, y_1, y_2; 0, 0, 1, e, e; \frac{\lambda}{u}) = {}_H\xi_n^{(r)}(x, y_1, y_2; u, \lambda)$.
(The three-variable Hermite-Apostol type Frobenius-Euler polynomials, see [**2**]).

17. Setting $\alpha_i = \lambda\ c = b = e,\ a = 1,\ \mu = k,\ \gamma = 1$ *and* $\phi(y_1, y_2, ..., y_m, t) = e^{yt^2} \prod_{j=1}^{m} (1 - y_j t)^{-\alpha_j}$ *in* (14), k even, we have

$$\begin{aligned} &\sum_{n=0}^{\infty} {}_p\mathbb{M}_n^{(r)}(x, y, y_1, ..., y_m; k, 1, e, e; \lambda)\frac{t^n}{n!} \\ = &\frac{(2^k t)^r}{(\lambda e^t - 1)^r}\, e^{xt + y_1 t^2} \prod_{j=1}^{m} (1 - y_j t)^{-\alpha_j} \\ = &\prod_{j=1}^{m} (1 - y_j t)^{-\alpha_j} \left(\frac{2^k t}{\lambda e^t + (-1)^{k+1}}\right)^r e^{xt + yt^2} \\ = &\sum_{n=0}^{\infty} T_{n,\lambda,k}^{(\alpha_1, \alpha_2, ..., \alpha_m, r)}(y_1, y_2, ..., y_m; x, y)t^n. \end{aligned}$$

Thus, equating the coefficients of t^n on both sides, we get
${}_p\mathbb{M}_n^{(r)}(x, y, y_1, ..., y_m; k, 1, e, e; \lambda) = \frac{1}{n!} T_{n,\lambda,k}^{(\alpha_1, \alpha_2, ..., \alpha_m, r)}(y_1, y_2, ..., y_m; x, y)$.
(The Legrange-based Apostol type Hermite polynomials, see [**12**]).

18. Setting $\alpha_i = \lambda\ c = b = e,\ a = 1,\ \mu = k,\ \gamma = 1$ *and* $\phi(y_1, y_2, ..., y_m, t) = e^{yt^2} \prod_{j=1}^{m} (1 - y_j t)^{-\alpha_j}$ *in* (14), k odd, we have

$$\begin{aligned} &\sum_{n=0}^{\infty} {}_p\mathbb{M}_n^{(r)}(x, y, y_1, ..., y_m; k, 1, e, e; \lambda)\frac{t^n}{n!} \\ = &\frac{(2^k t)^r}{(\lambda e^t - 1)^r}\, e^{xt + y_1 t^2} \prod_{j=1}^{m} (1 - y_j t)^{-\alpha_j} \\ = &\prod_{j=1}^{m} (1 - y_j t)^{-\alpha_j} \left(\frac{2^k t}{\lambda e^t + (-1)^{k+1}}\right)^r e^{xt + yt^2} \\ = &\sum_{n=0}^{\infty} T_{n,\lambda,k}^{(\alpha_1, \alpha_2, ..., \alpha_m, r)}(y_1, y_2, ..., y_m; x, y)t^n. \end{aligned}$$

Thus, equating the coefficients of t^n on both sides, we get
${}_p\mathbb{M}_n^{(r)}(x, y, y_1, ..., y_m; k, 1, e, e; \lambda) = \frac{1}{n!} T_{n,\lambda,k}^{(\alpha_1, \alpha_2, ..., \alpha_m, r)}(y_1, y_2, ..., y_m; x, y)$.
(The Legrange-based Apostol type Hermite polynomials, see [**12**]).

Theorem 2. The multi-variable unified family of generalized Apostol-Euler, Bernoulli and Genocchi polynomials of order r ${}_p\mathbb{M}_n^{(r)}(x, y_1, y_2, ..., y_m, k, a, b; \bar{\alpha}_r)$ is given by the series:

$$(16)\quad {}_p\mathbb{M}_n^{(r)}(x,y_1,y_2,...,y_m,k,a,b;\bar{\alpha}_r)=\sum_{\ell=0}^{n}\binom{n}{\ell}{}_p\mathbb{M}_{n-\ell}^{(r)}(a,b,\bar{\alpha}_r)\ \ p_\ell(x,y_1,y_2,...,y_m).$$

Proof. Using Eq. (14), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}{}_p\mathbb{M}_n^{(r)}(x,y_1,y_2,...,y_m,k,a,b;\bar{\alpha}_r)\frac{t^n}{n!}\\
=\ &\frac{t^{r\gamma}2^{r\mu}}{\prod\limits_{i=0}^{r-1}(\alpha_i b^t-a^t)}\ \sum_{n=0}^{\infty}p_n(x,y_1,y_2,...,y_m)\frac{t^n}{n!}.\\
=\ &\sum_{\ell=0}^{\infty}{}_p\mathbb{M}_n^{(r)}(k,a,b;\bar{\alpha}_r)\frac{t^\ell}{\ell!}\\
&\sum_{n=0}^{\infty}p_n(x,y_1,y_2,...,y_m)\frac{t^n}{n!}.
\end{aligned}$$

.

By using Cauchy-product rule, we get

$$\sum_{n=0}^{\infty}{}_p\mathbb{M}_n^{(r)}(x,y;k,a,b;\bar{\alpha}_r)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}\frac{{}_p\mathbb{M}_{n-\ell}^{(r)}(k,a,b;\bar{\alpha}_r)}{(n-\ell)!\ell!}p_\ell(xl,y_1,y_2,...,y_m)t^n.$$

Equating the coefficients of the same powers of t , yields (16). □

Also, we obtain some basic properties for the polynomial ${}_p\mathbb{M}_n^{(r)}(x,y_1,y_2,...,y_m,k,a,b;\bar{\alpha}_r)$ by the following theorem:
Theorem 3. Let $a,b,c\in\mathbb{R}$ $(a\neq b)$ and $x\in R$. Then

$$\begin{aligned}
{}_p\mathbb{M}_n^{(r)}(x+z,y_1,y_2,...,y_m,k,a,b;\bar{\alpha}_r)\ &=\ \sum_{\ell=0}^{n}\binom{n}{\ell}x^{n-\ell}(\ln c)^{n-\ell}\\
(17)\qquad &\quad\ {}_p\mathbb{M}_\ell^{(r)}(z,y_1,y_2,...,y_m,k,a,b;\bar{\alpha}_r).
\end{aligned}$$

$$(18)\quad {}_p\mathbb{M}_n^{(r)}(x+r,y_1,y_2,...,y_m,k,a,b;\bar{\alpha}_r)={}_p\mathbb{M}_n^{(r)}(x,y_1,y_2,...,y_m,k,\frac{a}{c},\frac{b}{c};\bar{\alpha}_r).$$

Proof. For the first equation, from (14)

$$
\begin{aligned}
&\sum_{n=0}^{\infty} {}_p\mathbb{M}_n^{(r)}(x+z, y_1, y_2, ..., y_m, k, a, b; \bar{\alpha}_r)\frac{t^n}{n!}\\
= &\frac{t^{r\gamma} \ 2^{r\mu}}{\prod_{i=0}^{r-1}(\alpha_i b^t - a^t)} c^{(x+z)t} \ \phi(y_1, y_2, ..., y_m, t)\\
= &\frac{t^{r\gamma} \ 2^{r\mu}}{\prod_{i=0}^{r-1}(\alpha_i b^t - a^t)} c^{zt} \ \phi(y_1, y_2, ..., y_m, t) \ c^{xt}\\
= &\sum_{\ell=0}^{\infty} {}_p\mathbb{M}_\ell^{(r)}(z, y_1, y_2, ..., y_m, k, a, b; \bar{\alpha}_r)\\
&\sum_{n=0}^{\infty} \frac{(tx\ln c)^n}{n!},
\end{aligned}
$$

using Cauchy-product rule, we can easily obtain (17).
For the second equation (18), from (14)

$$
\begin{aligned}
&\sum_{n=0}^{\infty} {}_p\mathbb{M}_n^{(r)}(x+r, y_1, y_2, ..., y_m, k, a, b; \bar{\alpha}_r)\frac{t^n}{n!}\\
= &\frac{t^{r\gamma} \ 2^{r\mu}}{\prod_{i=0}^{r-1}(\alpha_i b^t - a^t)} c^{(x+r)t} \ \phi(y_1, y_2, ..., y_m, t)\\
= &\frac{t^{r\gamma} \ 2^{r\mu}}{\prod_{i=0}^{r-1}\left(\alpha_i (\frac{b}{c})^t - (\frac{a}{c})^t\right)} c^{xt} \ \phi(y_1, y_2, ..., y_m, t)\\
= &\sum_{n=0}^{\infty} {}_p\mathbb{M}_n^{(r)}(x, y_1, y_2, ..., y_m, k, \frac{a}{c}, \frac{b}{c}; \bar{\alpha}_r)\frac{t^n}{n!}.
\end{aligned}
$$

Equating the coefficient of $\frac{t^n}{n!}$ on both sides, yields (18). □

Corollary 1. If $z = 0$ in (17), we obtain the recurrence relation

$$
\begin{aligned}
(19) \qquad &{}_p\mathbb{M}_n^{(r)}(x, y_1, y_2, ..., y_m, k, a, b; \bar{\alpha}_r)\\
= &\sum_{\ell=0}^{n} \binom{n}{\ell} x^{n-\ell}(\ln c)^{n-\ell} {}_p\mathbb{M}_\ell^{(r)}(y_1, y_2, ..., y_m, k, a, b; \bar{\alpha}_r).\\
(20) \qquad = &\sum_{\ell=0}^{n} \binom{n}{n-\ell} x^{\ell}(\ln c)^{\ell} {}_p\mathbb{M}_{n-\ell}^{(r)}(y_1, y_2, ..., y_m, k, a, b; \bar{\alpha}_r).
\end{aligned}
$$

3. IMPLICIT SUMMATION FORMULAE FOR THE MULTI-VARIABLE UNIFIED FAMILY OF GENERALIZED APOSTOL-EULER, BERNOULLI AND GENOCCHI POLYNOMIALS

Theorem 4. Let $a,\ b>0$ and $a\neq b$. Then for $x,z\in\mathbf{R}$ and $n\geq 0$. The following implicit summation formula for ${}_p\mathbb{M}_n^{(r)}(x,y_1,y_2,...,y_m,k,a,b;\bar{\alpha}_r)$ holds true:

$$ {}_p\mathbb{M}_n^{(r)}(x+z,y_1,y_2,...,y_m,k,a,b;\bar{\alpha}_r) = $$

$$ \sum_{\ell=0}^{n} {}_p\mathbb{M}_\ell^{(r)}(x,y_1,y_2,...,y_m,k,a,b;\bar{\alpha}_r)\,(\ln c)^{n-\ell}\ z^{n-\ell} \tag{21} $$

Proof. Replacing x by $x+z$ in generating function (14) gives

$$ \begin{aligned} & \sum_{n=0}^{\infty} {}_p\mathbb{M}_n^{(r)}(x+z,y_1,y_2,...,y_m,k,a,b;\bar{\alpha}_r)\frac{t^n}{n!} \\ = & \frac{t^{r\gamma}2^{r\mu}}{\prod\limits_{i=0}^{r-1}(\alpha_i b^t-a^t)} c^{(x+z)t}\phi(y_1,y_2,...,y_m,t) \\ = & \frac{t^{r\gamma}2^{r\mu}}{\prod\limits_{i=0}^{r-1}(\alpha_i b^t-a^t)} c^{xt}\phi(y_1,y_2,...,y_m,t)c^{zt} \\ = & \sum_{\ell=0}^{\infty} {}_p\mathbb{M}_\ell^{(r)}(x,y_1,y_2,...,y_m,k,a,b;\bar{\alpha}_r)\frac{t^\ell}{\ell!} \\ & \sum_{n=0}^{\infty}\frac{(zt\ln c)^n}{n!} \\ = & \sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty} {}_p\mathbb{M}_\ell^{(r)}(x,y_1,y_2,...,y_m,k,a,b;\bar{\alpha}_r) \\ & (z\ln c)^n\frac{t^{n+\ell}}{n!\,\ell!}, \end{aligned} $$

which by replacing n by $n-m$ in the right hand side and then equating the coefficients of the same powers of t in both sides of the last equation yields (21). □

Theorem 5. Let $a,\ b>0$ and $a\neq b$. Then for $x,z\in\mathbf{R}$ and $n\geq 0$. The following implicit summation formula for ${}_p\mathbb{M}_n^{(r)}(x,y_1,y_2,...,y_m,k,a,b;\bar{\alpha}_r)$ holds true:

$$ \begin{aligned} {}_p\mathbb{M}_{n+\ell}^{(r)}(z,y_1,y_2,...,y_m,k,a,b;\bar{\alpha}_r) = & \sum_{p,q=0}^{n,\ell}\binom{n}{p}\binom{\ell}{q}(\ln c)^{p+q}(z-x)^{p+q} \\ & {}_p\mathbb{M}_{n+\ell-p-q}^{(r)}(x,y_1,y_2,...,y_m,k,a,b;\bar{\alpha}_r). \end{aligned} \tag{22} $$

Proof. Replacing t by $t+u$ in the generating function (14) and using the following rule [**19**]

$$\sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} f(m+n)\frac{x^n}{n!}\frac{y^m}{m!}, \tag{23}$$

in the left hand side becomes

$$\sum_{n,\ell=0}^{\infty} {}_p\mathbb{M}_{n+\ell}^{(r)}(x,y_1,y_2,...,y_m,k,a,b;\bar{\alpha}_r)\frac{t^n}{n!}\frac{u^\ell}{\ell!} =$$

$$\frac{(t+u)^{r\gamma}2^{r\mu}}{\prod_{i=0}^{r-1}(\alpha_i b^{t+u}-a^{t+u})}\, c^{x(t+u)}\phi(y_1,y_2,...,y_m,t+u).$$

Rewriting Eq. (24) as

$$c^{-x(t+u)}\sum_{n,\ell=0}^{\infty} {}_p\mathbb{M}_{n+\ell}^{(r)}(x,y_1,y_2,...,y_m,k,a,b;\bar{\alpha}_r)\frac{t^n}{n!}\frac{u^\ell}{\ell!} =$$

$$\frac{(t+u)^{r\gamma}2^{r\mu}}{\prod_{i=0}^{r-1}(\alpha_i b^{t+u}-a^{t+u})}\,\phi(y_1,y_2,...,y_m,t+u).$$

Replacing x by z in the above equation and equating the resulting equation to the above equation, we get

$$c^{(z-x)(t+u)}\sum_{n,\ell=0}^{\infty} {}_p\mathbb{M}_{n+\ell}^{(r)}(x,y_1,y_2,...,y_m,k,a,b;\bar{\alpha}_r)\frac{t^n}{n!}\frac{u^\ell}{\ell!} =$$

$$\sum_{n,\ell=0}^{\infty} {}_p\mathbb{M}_{n+\ell}^{(r)}(z,y_1,y_2,...,y_m,k,a,b;\bar{\alpha}_r)\frac{t^n}{n!}\frac{u^\ell}{\ell!}. \tag{24}$$

On expanding exponential function in (24) gives

$$\sum_{N=0}^{\infty}\frac{[(z-x)(t+u)\ln c]^N}{N!}\sum_{n,\ell=0}^{\infty} {}_p\mathbb{M}_{n+\ell}^{(r)}(x,y_1,y_2,...,y_m,\gamma,a,b;\bar{\alpha}_r)\frac{t^n}{n!}\frac{u^\ell}{\ell!} =$$

$$\sum_{n,\ell=0}^{\infty} {}_p\mathbb{M}_{n+\ell}^{(r)}(z,y_1,y_2,...,y_m,k,a,b;\bar{\alpha}_r)\frac{t^n}{n!}\frac{u^\ell}{\ell!}, \tag{25}$$

using Eq. (23) in the left hand side of Eq. (25), we get

$$\sum_{p,q=0}^{\infty}\frac{(\ln c)^{p+q}(z-x)^{p+q}\, t^p\, u^q}{p!\, q!}\sum_{n,\ell=0}^{\infty} {}_p\mathbb{M}_{n+\ell}^{(r)}(x,y_1,y_2,...,y_m,k,a,b;\bar{\alpha}_r)\frac{t^n}{n!}\frac{u^\ell}{\ell!} =$$

$$\sum_{n,\ell=0}^{\infty} {}_p\mathbb{M}_{n+\ell}^{(r)}(z,y_1,y_2,...,y_m,k,a,b;\bar{\alpha}_r)\frac{t^n}{n!}\frac{u^\ell}{\ell!}. \tag{26}$$

Now replacing n by $n-p$, ℓ by $\ell-q$ and using Cauchy-product rule in the left hand side of (26), we get

$$\sum_{n,\ell=0}^{\infty}\sum_{p,q=0}^{n,\ell}\frac{(\ln c)^{p+q}(z-x)^{p+q}}{p!\ q!}{}_p\mathbb{M}_{n+\ell-p-q}^{(r)}(x,y_1,y_2,...,y_m,k,a,b;\bar{\alpha}_r)\frac{t^n}{(n-p)!}\frac{u^\ell}{(\ell-q)!}=$$

$$\sum_{n,\ell=0}^{\infty} {}_p\mathbb{M}_{n+\ell}^{(r)}(z,y_1,y_2,...,y_m,k,a,b;\bar{\alpha}_r)\frac{t^n}{n!}\frac{u^\ell}{\ell!}. \tag{27}$$

Finally, equating the coefficients of the similar powers of t and u in the above equation, yields (22). □

Theorem 6. The following identity holds true, when $\alpha_i,\ i=0,1,...,r-1$

$$\begin{aligned} {}_p\mathbb{M}_n^{(r)}(r-x,y_1,y_2,...,y_m,k,a,b;\bar{\alpha}_r) &= \frac{(-1)^{r\gamma+n}}{\prod_{i=0}^{r-1}\alpha_i}\sum_{\ell=0}^{n}\binom{n}{\ell}\left(r\ln\left(\frac{ab}{c}\right)\right)^{n-\ell} \\ &\quad {}_p\mathbb{M}_\ell^{(r)}(x,y_1,y_2,...,y_m;k,a,b;\frac{1}{\bar{\alpha}_r}). \end{aligned} \tag{28}$$

Proof. From (14)

$$\begin{aligned} & {}_p\mathbb{M}_n^{(r)}(r-x,y_1,y_2,...,y_m,k,a,b;\bar{\alpha}_r) \\ = & \frac{t^{r\gamma}\ 2^{r\mu}}{\prod_{i=0}^{r-1}(\alpha_i b^t-a^t)}\ c^{(r-x)t}\ \phi(y_1,y_2,...,y_m,t) \\ = & \frac{(-1)^{r\gamma}\ (-t)^{r\gamma}\ 2^{r\mu}}{\prod_{i=0}^{r-1}\alpha_i\ \prod_{i=0}^{r-1}\left(\frac{b^{-t}}{\alpha_i}-a^{-t}\right)}\ c^{-xt}\ c^{rt}\phi(y_1,y_2,...,y_m,t)\ \ c^{xt} \\ = & \frac{(-1)^{r\gamma}}{\prod_{i=0}^{r-1}\alpha_i}\left(\frac{ba}{c}\right)^{-rt}\sum_{\ell=0}^{\infty}{}_p\mathbb{M}_\ell^{(r)}(x,y_1,y_2,...,y_m,k,a,b;\frac{1}{\bar{\alpha}_r})\frac{-t^\ell}{\ell!} \\ = & \frac{(-1)^{r\gamma}}{\prod_{i=0}^{r-1}\alpha_i}\sum_{j=0}^{\infty}\frac{(r\ln(\frac{ab}{c}))^j}{j!}(-t)^j \\ & \sum_{\ell=0}^{\infty}{}_p\mathbb{M}_\ell^{(r)}(x,y_1,y_2,...,y_m,k,a,b;\frac{1}{\bar{\alpha}_r})\frac{(-t)^\ell}{\ell!}. \end{aligned}$$

Hence, we can easily obtain (28) □

Theorem 7. The following implicit summation formula for ${}_p\mathbb{M}_n^{(r)}(x, y_1, y_2, ..., y_m, k, a, b; \bar{\alpha}_r)$ holds true:

$$
{}_p\mathbb{M}_n^{(r)}(x+1, y_1, y_2, ..., y_m, k, a, b; \bar{\alpha}_r) =
$$

$$
\sum_{\ell=0}^{n} \binom{n}{\ell} (\ln c)^{\ell} \; {}_p\mathbb{M}_{n-\ell}^{(r)}(x, y_1, y_2, ..., y_m, k, a, b; \bar{\alpha}_r) \tag{29}
$$

Proof. From Eq. (14), we have

$$
\begin{aligned}
& \sum_{n=0}^{\infty} {}_p\mathbb{M}_n^{(r)}(x+1, y_1, y_2, ..., y_m, k, a, b; \bar{\alpha}_r) \frac{t^n}{n!} \\
= & \frac{t^{r\gamma} 2^{r\mu}}{\prod_{i=0}^{r-1} (\alpha_i b^t - a^t)} \; c^{(x+1)t} \phi(y_1, y_2, ..., y_m, t), \\
= & \frac{t^{r\gamma} 2^{r\mu}}{\prod_{i=0}^{r-1} (\alpha_i b^t - a^t)} \; c^{xt} \; \phi(y_1, y_2, ..., y_m, t) \; c^t, \\
= & \sum_{n=0}^{\infty} {}_p\mathbb{M}_n^{(r)}(x, y_1, y_2, ..., y_m, k, a, b; \bar{\alpha}_r) \frac{t^n}{n!} \\
& \sum_{\ell=0}^{\infty} \frac{(t \ln c)^{\ell}}{\ell!}.
\end{aligned}
$$

By using Cauchy-product rule, then

$$
\sum_{n=0}^{\infty} {}_p\mathbb{M}_n^{(r)}(x+1, y_1, y_2, ..., y_m, k, a, b; \bar{\alpha}_r) \frac{t^n}{n!} =
$$

$$
\sum_{n=0}^{\infty} \sum_{\ell=0}^{n} \binom{n}{\ell} (\ln c)^{\ell} \; {}_p\mathbb{M}_{n-\ell}^{(r)}(x, y_1, y_2, ..., y_m, k, a, b; \bar{\alpha}_r) \frac{t^n}{n!}.
$$

Equating the coefficients of t^n on both sides, yields (29). □

Theorem 8. For $n, \; j \in \mathbb{N}_0$, we have the relationship

$$
{}_p\mathbb{M}_n^{(r)}(x, y_1, y_2, ..., y_m, k, a, b; \bar{\alpha}_r) =
$$

$$
\sum_{j=0}^{n} (x)_{\underline{j}} \sum_{\ell=j}^{n} \binom{n}{n-\ell} (\ln c)^{\ell} \; S(\ell, j) \, {}_p\mathbb{M}_{n-\ell}^{(r)}(y_1, y_2, ..., y_m, k, a, b; \bar{\alpha}_r). \tag{30}
$$

between the multi-Variable Unified Family of Generalized Apostol-Euler, Bernoulli and Genocchi polynomials and Stirling number of second kind.

Proof. Using (20) and from definition of Stirling number of second kind, see [**3, 20**], we have

$$_p\mathbb{M}_n^{(r)}(x, y_1, y_2, ..., y_m, k, a, b; \bar{\alpha}_r) =$$

$$\sum_{\ell=0}^{n} \binom{n}{n-\ell} (\ln c)^{\ell} \sum_{j=0}^{l} (x)_{\underline{\ell}} S(\ell, j\ )_p\mathbb{M}_{n-\ell}^{(r)}(y_1, y_2, ..., y_m, k, a, b; \bar{\alpha}_r).$$

By using Cauchy-product rule, yields (30). □

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