

APPROXIMATE SYMMETRIES AND SIMILARITY SOLUTIONS FOR WAVE EQUATIONS ON LIQUID FILMS

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We study the exact and approximate Lie symmetries for two equations which describe long waves with small amplitude on liquid films. Specifically, we study the 1+2 Benney-Luke and the 1+1 Benney-Lin equations, both from an exact and approximate perspective. To induce approximate symmetries, we show that terms involving derivatives higher than two are necessarily selected as the perturbation parameters. We construct conservation laws for both equations, and illustrate how the approximate point symmetries can be used to determine approximate similarity solutions.

1. INTRODUCTION

Long waves with small amplitude on liquid films can be described by approximations of the full water wave equations. Such long waves on liquid films were studied firstly by Benney [3] and later by Luke [4], Lin [19], Kawahara [16, 34] and others [1, 33, 25, 17, 31].

In this work we are interested in the algebraic properties of the 1+2 Benney-Luke equation

$$(1) \quad \Phi_{tt} - \Delta\Phi + \mu(a\Delta^2\Phi - b\Delta\Phi_{tt}) + \epsilon(\Phi_t\Delta\Phi + 2\nabla\Phi \nabla\Phi_t) = 0,$$

and of the 1+1 Benney-Lin equation

$$(2) \quad u_t + \beta_0 u_{xxxxx} + \beta_1 (u_{xxxx} + u_{xx}) + u_{xxx} + u_x + uu_x = 0.$$

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In both of these equations the parameters which introduce derivatives of order higher than two, are the approximate parameters. Indeed for equation (1) the approximate parameter is μ , while for (2) the approximate parameters are β_0 and β_1 .

We study the algebraic properties of these two equations by performing a complete symmetry analysis. More specifically, we determine the exact symmetries for these two equations and also the approximate point symmetries [2, 7, 27]. The theory of Lie symmetries is a powerful tool for the analysis and determination of analytical solutions for nonlinear differential equations, there is a plethora of applications in the literature, for instance see [35, 37, 30, 15, 24] and references therein.

Of interest, is the application of symmetries in gravity theories, where classifications of solutions are performed - refer to [11] and references therein. The study of conservation laws is also extensive, for example [8]. In [32], the conservation of Hamiltonians was analysed under a Galerkin Petrov time discretization scheme.

In approximate differential equations, usually the exact symmetries of the nonperturbative equation are lost. Though, the context of the application of symmetries is extended to the approximate symmetries [9, 36, 6, 18] in all cases. Approximate symmetries are used to define approximate invariants that are applied in the reconstruction of approximate solutions which are valid in a specific range of the space of independent variables. Some applications of approximate symmetries can be found in [12, 14, 29, 5, 10, 13] and references therein. The plan of the paper is as follows.

In Section 2 we briefly discuss the theory of exact Lie symmetries and of approximate symmetries. Sections 3 and 4 includes the main analysis of our work where we derive the symmetries (exact and approximate) for the two equations under consideration. The symmetries are applied in order to determine approximate similarity solutions and functions akin to integrating factors are used to construct conservation laws. Finally, we discuss our results and draw our conclusions in Section 5.

2. PRELIMINARIES

In this section we briefly discuss the main mathematical tools that we apply in this work. That is, we present the basic elements of the exact Lie point symmetries and of the approximate point symmetries.

2.1 Lie point symmetries

Let Φ be the map of a one-parameter point transformation such as

$$(3) \quad \Phi(\mathcal{G}(t, x)) = u(t, x),$$

with infinitesimal transformation (ε is the parameter of smallness)

$$(4) \quad t' = t + \varepsilon \xi^1(t, x, u),$$

$$(5) \quad x' = x + \varepsilon \xi^2(t, x, u),$$

$$(6) \quad u' = u + \varepsilon \eta(t, x, u),$$

and generator

$$(7) \quad X = \frac{\partial t'}{\partial \varepsilon} \partial_t + \frac{\partial x'}{\partial \varepsilon} \partial_x + \frac{\partial u'}{\partial \varepsilon} \partial_u.$$

Assume that $u(t, x)$ is a solution of the partial differential equation $\mathcal{G}(u^A, u_t^A, u_x^A \dots) = 0$. Then under the map Φ , the transformed function $u'(t', x')$ remains a solution of the differential equation $\mathcal{G}(u, u_t, u_x \dots) = 0$ when the following condition holds

$$(8) \quad \Phi(\mathcal{G}(u^A, u_t^A, u_x^A \dots)) = 0.$$

That is, the differential equation is invariant under the action of the map, Φ .

When the latter is true, we shall say that the infinitesimal transformation of the one-parameter point transformation, Φ , the vector field X , is a Lie symmetry for the differential equation $\mathcal{G}(u, u_t, u_x \dots) = 0$.

Mathematically that is expressed as

$$(9) \quad X^{[n]}(\mathcal{G}) = 0,$$

or equivalently, if there exists a function ψ such that

$$(10) \quad X^{[n]}(\mathcal{G}) = \psi \mathcal{G}.$$

The vector field $X^{[n]}$ denotes the n^{th} -prolongation or extension of the symmetry vector in the space of variables $\{t, x, u, u_t, u_x, \dots\}$.

2.2 Conservation laws

The multipliers Λ - analogous to integrating factors, of \mathcal{G} are obtained from the relation

$$(11) \quad DT = \Lambda \mathcal{G},$$

where D refers to the total derivative [26]. The determining equations for the multipliers are then

$$(12) \quad E[\Lambda \mathcal{G}] = 0,$$

where E refers to the Euler operator. Once found, they give rise to a conserved vector T that satisfies the conservation relation

$$(13) \quad DT = 0,$$

along the solutions of \mathcal{G} . Eq. (13) is called a local conservation law. Conserved vectors may be derived systematically using (11) as the determining equation, however, conserved quantities may also be determined by a homotopy operator.

2.3 Approximate Symmetries

We will consider the approximation in the first-order of precision in the perturbation parameter ε [2]. An approximate equation

$$(14) \quad F(z, \varepsilon) \equiv F_0(z) + \varepsilon F_1(z) \approx 0, \quad z = (z^1, \dots, z^N),$$

is approximately invariant with respect to the one-parameter approximate transformation group

$$\bar{z}^i \approx h(z, \alpha, \varepsilon) \equiv h_0^i(z, \alpha) + \varepsilon h_1^i(z, \alpha), \quad i = 1, \dots, N,$$

(“ α, ε ” are two infinitesimal parameters) with the generator

$$(15) \quad X = X_0 + \varepsilon X_1 + O(\varepsilon^2),$$

if and only if,

$$(16) \quad \left(X_0 F_0(z) + \varepsilon (X_1 F_0(z) + X_0 F_1(z)) \right)_{Eq.(14)} = O(\varepsilon).$$

The determining equation (16) can be written as follows:

$$(17) \quad X_0 F_0(z) = \lambda(z) F_0(z),$$

$$(18) \quad X_1 F_0(z) + X_0 F_1(z) = \lambda(z) F_1(z).$$

The factor $\lambda(z)$ is determined by (17) and then substituted into (18), where the latter equation holds for $F_0(z) = 0$. Alternatively, one may evaluate (17) to obtain the exact symmetries, then find an auxiliary function H by virtue of (17), (18) and (14), that is

$$(19) \quad H = \frac{1}{\varepsilon} X_0 \left(F_0(z) + \varepsilon F_1(z) \right)_{F_0(z) + \varepsilon F_1(z) = 0}.$$

Thereafter X_1 is calculated by solving the determining equation for deformations

$$(20) \quad X_1 F_0(z)|_{F_0(z)=0} + H = 0.$$

Subsequently, approximate solutions may be calculated by choosing functionally independent invariants and thereafter proceeding with reductions of the equation.

3. SYMMETRY ANALYSIS OF THE BENNEY-LUKE EQUATION

In this Section we determine the exact Lie point symmetries and the approximate point symmetries for the Benney-Luke equation (1).

3.1 Exact Lie point symmetries

We apply the Lie theory and determine that the Benney-Luke equation is invariant under a five dimensional Lie algebra, consisting of the symmetry vector fields

$$(21) \quad X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = \partial_y, \quad X_4 = \partial_\Phi, \quad X_5 = -x\partial_y + y\partial_x.$$

In Table 1 we present the commutator of the Lie symmetry vectors. The admitted Lie algebra is the $2A_1 \oplus_s A_{2,1}$ in the Morozov-Mubarakzhanov classification scheme [20, 21, 22, 23]. In particular, the vector fields $\{X_2, X_3, X_5\}$ forms the E^2 Lie algebra. However, for specific values of the free parameters μ and ϵ there are some extra Lie point symmetries.

Table 1: Commutators of the admitted Lie point symmetries of equation (1)

$[\cdot, \cdot]$	\mathbf{X}_1	\mathbf{X}_2	\mathbf{X}_3	\mathbf{X}_4	\mathbf{X}_5
\mathbf{X}_1	0	0	0	0	0
\mathbf{X}_2	0	0	0	0	$-X_3$
\mathbf{X}_3	0	0	0	0	X_2
\mathbf{X}_4	0	0	0	0	0
\mathbf{X}_5		X_3	$-X_2$	0	0

Indeed, by neglecting all nonlinear terms, i.e. $\mu = \epsilon = 0$, equation (1) reduces to the linear 2+1 wave equation, that is, it is maximally symmetric and admits 15+1+infinity Lie point symmetries. Although when $\mu = 0$, and $\epsilon \neq 0$, aside from the Lie point symmetries (21) there exists the additional symmetry vector

$$(22) \quad X_6 = (\Phi\epsilon + 2t)\partial_\Phi - \epsilon t\partial_t.$$

The commutators of the admitted Lie point symmetries of equation (1) with $\mu = 0$ and $\epsilon \neq 0$ are presented in Table 2.

We continue by using the above symmetry vectors in connection with evolutionary symmetries, in order to determine the conservation laws for the Benney-Luke equation.

Table 2: Commutators of the admitted Lie point symmetries of equation (1)

$[\cdot, \cdot]$	\mathbf{X}_1	\mathbf{X}_2	\mathbf{X}_3	\mathbf{X}_4	\mathbf{X}_5	\mathbf{X}_6
\mathbf{X}_1	0	0	0	0	0	$2X_5 - \epsilon X_1$
\mathbf{X}_2	0	0	0	0	$-X_3$	0
\mathbf{X}_3	0	0	0	0	X_2	0
\mathbf{X}_4	0	0	0	0	0	0
\mathbf{X}_5	0	X_3	$-X_2$	0	0	ϵX_5
\mathbf{X}_6	$\epsilon X_1 + 2X_5$	0	0	0	$-\epsilon X_5$	0

3.2 Conservation laws

Eq. (1) admits the multipliers

$$(1, \Phi_t, \Phi_x, \Phi_y, y\Phi_x - x\Phi_y),$$

that correspond to the following conservation laws.

- Conservation of mass

$$\begin{aligned} T^t &= -\frac{1}{2}\Phi\Phi_{xx}\epsilon - \frac{1}{2}\Phi\Phi_{yy}\epsilon - \Phi_x^2\epsilon - \Phi_y^2\epsilon + \Phi_{txx}b\mu + \Phi_{tyy}b\mu - \Phi_t, \\ T^x &= \frac{1}{2}\Phi\Phi_{tx}\epsilon - \frac{1}{2}\Phi_x\Phi_t\epsilon - \mu a\Phi_{xxx} - 2\mu a\Phi_{xyy} + \Phi_x, \\ T^y &= \frac{1}{2}\Phi\Phi_{ty}\epsilon - \frac{1}{2}\Phi_y\Phi_t\epsilon - \Phi_{yyy}\mu a + \Phi_y. \end{aligned}$$

- Conservation of energy

$$\begin{aligned} T^t &= -\frac{2}{3}\Phi\Phi_t\Phi_{xx}\epsilon - \frac{2}{3}\Phi\Phi_t\Phi_{yy}\epsilon - \frac{2}{3}\Phi_x^2\Phi_t\epsilon - \frac{2}{3}\Phi_y^2\Phi_t\epsilon - \frac{2}{3}\Phi\Phi_{tx}\Phi_x\epsilon \\ &\quad - \frac{2}{3}\Phi\Phi_{ty}\Phi_y\epsilon - \frac{1}{2}\Phi_t^2 - \frac{1}{2}\Phi_{xx}b\mu\Phi_{tt} - \frac{1}{2}\Phi_{yy}b\mu\Phi_{tt} + \frac{1}{2}\Phi_{txx}\Phi_t b\mu \\ &\quad + \frac{1}{2}\Phi_{tyy}\Phi_t b\mu + \frac{1}{2}\Phi\Phi_{ttxx}b\mu + \frac{1}{2}\Phi\Phi_{ttyy}b\mu - \frac{1}{2}\Phi\Phi_{xxxx}a\mu \\ &\quad - \Phi\Phi_{xxyy}a\mu - \frac{1}{2}\Phi\Phi_{yyyy}a\mu + \frac{1}{2}\Phi\Phi_{xx} + \frac{1}{2}\Phi\Phi_{yy}, \\ T^x &= \frac{2}{3}\Phi\Phi_t\Phi_{tx}\epsilon + \frac{2}{3}\Phi\Phi_{tt}\Phi_x\epsilon - \frac{1}{3}\Phi_x\Phi_t^2\epsilon - \frac{1}{2}\Phi\Phi_{tx} - \frac{1}{2}\Phi b\mu\Phi_{ttt} \\ &\quad + \frac{1}{2}\Phi\mu a\Phi_{txxx} + \frac{1}{2}\Phi_x\Phi_t + \frac{1}{2}\Phi_x b\mu\Phi_{ttt} - \frac{1}{2}\Phi_x\mu a\Phi_{txx} + \frac{1}{2}\Phi_{xx}\mu a\Phi_{tx} \\ &\quad - \frac{1}{2}\Phi_{xxx}\Phi_t\mu a - \Phi_{xyy}\Phi_t\mu a + \Phi_{yy}\mu a\Phi_{tx}, \\ T^y &= \frac{2}{3}\Phi\Phi_t\Phi_{ty}\epsilon + \frac{2}{3}\Phi\Phi_{tt}\Phi_y\epsilon - \frac{1}{3}\Phi_y\Phi_t^2\epsilon - \frac{1}{2}\Phi\Phi_{ty} - \frac{1}{2}\Phi b\mu\Phi_{ttt} \\ &\quad + \Phi\mu a\Phi_{txxy} + \frac{1}{2}\Phi\mu a\Phi_{tyyy} + \frac{1}{2}\Phi_y\Phi_t + \frac{1}{2}\Phi_y b\mu\Phi_{ttt} - \Phi_y\mu a\Phi_{txx} \\ &\quad + \frac{1}{2}\Phi_{yy}\mu a\Phi_{ty} - \frac{1}{2}\Phi_{yyy}\Phi_t\mu - \frac{1}{2}\Phi_y\mu a\Phi_{tyy}. \end{aligned}$$

- Conservation of linear momentum in x

$$\begin{aligned}
T^t &= -\frac{1}{3}\Phi\Phi_x\Phi_{xx}\epsilon - \frac{1}{3}\Phi\Phi_x\Phi_{yy}\epsilon - \frac{2}{3}\Phi_x^3\epsilon - \frac{2}{3}\Phi_y^2\Phi_x\epsilon + \frac{1}{2}\Phi\Phi_{tx} \\
&\quad - \frac{1}{2}\Phi_x\Phi_t - \frac{1}{2}\Phi_{xx}b\mu\Phi_{tx} - 1/2\Phi_{yy}b\mu\Phi_{tx} + \frac{1}{2}\Phi_{txx}\Phi_xb\mu \\
&\quad + \frac{1}{2}\Phi_{tyy}\Phi_xb\mu, \\
T^x &= -\frac{1}{3}\Phi_x^2\Phi_t\epsilon + \frac{1}{2}\Phi_{xx}^2\mu a + \Phi_{yy}\mu a\Phi_{xx} - \Phi_{xyy}\Phi_x\mu a - \frac{1}{2}\Phi\Phi_{tt} \\
&\quad + \frac{1}{2}\Phi\Phi_{yy} + \frac{1}{2}\Phi_xb\mu\Phi_{tx} - \Phi_x\mu a\Phi_{xxx} - \frac{2}{3}\Phi\Phi_{ty}\Phi_y\epsilon + \frac{1}{2}\Phi\Phi_{ttyy}b\mu \\
&\quad - \Phi\Phi_{xyy}a\mu - \frac{1}{2}\Phi\Phi_{yyy}a\mu - \frac{1}{3}\Phi\Phi_t\Phi_{yy}\epsilon + \frac{1}{3}\Phi\Phi_{tx}\Phi_x\epsilon + \frac{1}{2}\Phi_x^2, \\
T^y &= \frac{1}{3}\Phi\Phi_x\Phi_{ty}\epsilon + \frac{2}{3}\Phi\Phi_{tx}\Phi_y\epsilon + \frac{1}{3}\Phi\Phi_t\Phi_{xy}\epsilon - \frac{1}{3}\Phi_y\Phi_x\Phi_t\epsilon - \frac{1}{2}\Phi\Phi_{xy} \\
&\quad - \frac{1}{2}\Phi b\mu\Phi_{txy} + \Phi\mu a\Phi_{xxy} + \frac{1}{2}\Phi\mu a\Phi_{yyy} + \frac{1}{2}\Phi_y\Phi_x + \frac{1}{2}\Phi_yb\mu\Phi_{tx} \\
&\quad - \Phi_y\mu a\Phi_{xxx} - \frac{1}{2}\Phi_y\mu a\Phi_{xyy} + \frac{1}{2}\Phi_{yy}\mu a\Phi_{xy} - \frac{1}{2}\Phi_{yyy}\Phi_x\mu a.
\end{aligned}$$

- Conservation of linear momentum in y

$$\begin{aligned}
T^t &= -\frac{1}{3}\Phi\Phi_{xx}\Phi_y\epsilon - \frac{1}{3}\Phi\Phi_y\Phi_{yy}\epsilon - \frac{2}{3}\Phi_x^2\Phi_y\epsilon - \frac{2}{3}\Phi_y^3\epsilon + \frac{1}{2}\Phi\Phi_{ty} \\
&\quad - \frac{1}{2}\Phi_y\Phi_t - \frac{1}{2}\Phi_{xx}b\mu\Phi_{ty} - \frac{1}{2}\Phi_{yy}b\mu\Phi_{ty} + \frac{1}{2}\Phi_{txx}\Phi_yb\mu \\
&\quad + \frac{1}{2}\Phi_{tyy}\Phi_yb\mu, \\
T^x &= \frac{1}{3}\Phi\Phi_{tx}\Phi_y\epsilon + \frac{2}{3}\Phi\Phi_x\Phi_{ty}\epsilon + \frac{1}{3}\Phi\Phi_t\Phi_{xy}\epsilon - \frac{1}{3}\Phi_y\Phi_x\Phi_t\epsilon - \frac{1}{2}\Phi\Phi_{xy} \\
&\quad - \frac{1}{2}\Phi b\mu\Phi_{txy} + \frac{1}{2}\Phi\mu a\Phi_{xxy} + 1/2\Phi_y\Phi_x + \frac{1}{2}\Phi_xb\mu\Phi_{ty} \\
&\quad - \frac{1}{2}\Phi_x\mu a\Phi_{xxy} + \frac{1}{2}\Phi_{xx}\mu a\Phi_{xy} + \Phi_{yy}\mu a\Phi_{xy} - \frac{1}{2}\Phi_y\mu a\Phi_{xxx} \\
&\quad - \Phi_y\mu a\Phi_{xyy}, \\
T^y &= -\frac{1}{3}\Phi_y^2\Phi_t\epsilon + \frac{1}{2}\Phi_{yy}^2\mu a + \frac{1}{2}\Phi_yb\mu\Phi_{ty} - \Phi_y\mu a\Phi_{xxy} - \Phi_y\Phi_{yyy}\mu a \\
&\quad - \frac{1}{2}\Phi\Phi_{tt} + \frac{1}{2}\Phi\Phi_{xx} + \frac{1}{3}\Phi\Phi_{ty}\Phi_y\epsilon + \frac{1}{2}\Phi\Phi_{txx}b\mu - \frac{1}{2}\Phi\Phi_{xxx}a\mu \\
&\quad - \frac{1}{3}\Phi\Phi_t\Phi_{xx}\epsilon - \frac{2}{3}\Phi\Phi_{tx}\Phi_x\epsilon + \frac{1}{2}\Phi_y^2.
\end{aligned}$$

- Conservation of angular momentum

$$\begin{aligned}
T^t &= -\frac{1}{3}\Phi\Phi_x\Phi_{xx}\epsilon y - \frac{1}{3}\Phi\Phi_x\Phi_{yy}\epsilon y + \frac{1}{3}\Phi\Phi_{xx}\Phi_y\epsilon x + \frac{1}{3}\Phi\Phi_y\Phi_{yy}\epsilon x \\
&\quad - \frac{2}{3}\Phi_x^3\epsilon y + \frac{2}{3}\Phi_x^2\Phi_y\epsilon x - \frac{2}{3}\Phi_x\Phi_y^2\epsilon y + \frac{2}{3}\Phi_y^3\epsilon x + \frac{1}{2}\Phi\Phi_{tx}y \\
&\quad - \frac{1}{2}\Phi\Phi_{ty}x + \frac{1}{2}\Phi_t\Phi_yx - \frac{1}{2}\Phi_{tx}\Phi_{xx}b\mu y + \frac{1}{2}\Phi_{ty}\Phi_{xx}b\mu x \\
&\quad - \frac{1}{2}\Phi_{tx}\Phi_{yy}b\mu y + \frac{1}{2}\Phi_{ty}\Phi_{yy}b\mu x + \frac{1}{2}\Phi_{txx}\Phi_xb\mu y - \frac{1}{2}\Phi_{txx}\Phi_yb\mu x \\
&\quad + \frac{1}{2}\Phi_{tyy}\Phi_xb\mu y - \frac{1}{2}\Phi_{tyy}\Phi_yb\mu x - \frac{1}{2}\Phi_t\Phi_xy, \\
T^x &= \frac{1}{2}\Phi\Phi_{xy}x - \frac{1}{2}\Phi_x\Phi_yx - \frac{1}{2}\Phi\Phi_{tt}y + \frac{1}{2}\Phi\Phi_{yy}y + \frac{1}{2}\Phi\Phi_y + \frac{1}{2}\Phi_x^2y \\
&\quad - \frac{1}{3}\Phi\Phi_t\Phi_y\epsilon - \frac{1}{3}\Phi_t\Phi_x^2\epsilon y + \frac{1}{2}\Phi\Phi_{tty}b\mu - \frac{3}{2}\Phi\Phi_{xxy}a\mu + \Phi_x\Phi_{xy}a\mu \\
&\quad - \frac{1}{2}\Phi_{xx}\Phi_ya\mu - \Phi_y\Phi_{yy}a\mu - \frac{1}{3}\Phi\Phi_t\Phi_{xy}\epsilon x + \frac{1}{3}\Phi_t\Phi_x\Phi_y\epsilon x \\
&\quad - \frac{1}{3}\Phi\Phi_t\Phi_{yy}\epsilon y + \frac{1}{2}\Phi\Phi_{txy}b\mu x - \frac{1}{2}\Phi\Phi_{xxy}a\mu x + \frac{1}{2}\Phi_{txx}\Phi_xb\mu y \\
&\quad - \frac{1}{2}\Phi_{tty}\Phi_xb\mu x - \Phi_x\Phi_{xxx}a\mu y + \frac{1}{2}\Phi_x\Phi_{xxy}a\mu x - \frac{1}{2}\Phi_{xx}\Phi_{xy}a\mu x \\
&\quad + \Phi_{xx}\Phi_{yy}a\mu y - \Phi_{xy}\Phi_{yy}a\mu x + \frac{1}{2}\Phi_{xxx}\Phi_ya\mu x - \Phi_x\Phi_{xyy}a\mu y \\
&\quad + \Phi_{xyy}\Phi_ya\mu x + \frac{1}{2}\Phi\Phi_{tly}b\mu y - \Phi\Phi_{xly}a\mu y - \frac{1}{2}\Phi\Phi_{lyly}a\mu y \\
&\quad - \frac{1}{3}\Phi\Phi_{tx}\Phi_y\epsilon x - \frac{2}{3}\Phi\Phi_{ty}\Phi_x\epsilon x + \frac{1}{2}\Phi_{xx}^2a\mu y \\
&\quad - \frac{2}{3}\Phi\Phi_{ty}\Phi_y\epsilon y + \frac{1}{2}\Phi\Phi_{tx}\Phi_x\epsilon y, \\
T^y &= -\frac{1}{2}\Phi\Phi_{xy}y + \frac{1}{2}\Phi_x\Phi_yy + \frac{1}{2}\Phi\Phi_{tt}x - \frac{1}{2}\Phi\Phi_{xx}x - \frac{1}{2}\Phi\Phi_x - \frac{1}{2}\Phi_y^2x \\
&\quad - \Phi_{xxx}\Phi_ya\mu y + \Phi_{xxy}\Phi_ya\mu x - \frac{1}{2}\Phi_{xyy}\Phi_ya\mu y + \Phi_y\Phi_{yyy}a\mu x \\
&\quad + \frac{1}{2}\Phi_{xy}\Phi_{yy}a\mu y - \frac{1}{2}\Phi_x\Phi_{yyy}a\mu y - \frac{1}{2}\Phi\Phi_{txx}b\mu x + \frac{1}{2}\Phi\Phi_{xxx}a\mu x \\
&\quad + \Phi\Phi_{xxy}a\mu y + \frac{1}{2}\Phi\Phi_{xyyy}a\mu y + \frac{1}{2}\Phi_{txx}\Phi_yb\mu y - \frac{1}{2}\Phi_{tty}\Phi_yb\mu x \\
&\quad + \frac{1}{3}\Phi\Phi_t\Phi_x\epsilon + \frac{1}{3}\Phi_t\Phi_y^2\epsilon x - \frac{1}{2}\Phi\Phi_{txx}b\mu - \frac{1}{2}\Phi\Phi_{xyy}a\mu \\
&\quad + \Phi_{xy}\Phi_ya\mu - \frac{1}{2}\Phi_{yy}^2a\mu x + \frac{1}{2}\Phi_x\Phi_{yy}a\mu + \frac{1}{3}\Phi\Phi_{ty}\Phi_x\epsilon y \\
&\quad + \frac{2}{3}\Phi\Phi_{tx}\Phi_y\epsilon y + \frac{1}{3}\Phi\Phi_t\Phi_{xy}\epsilon y - \frac{1}{3}\Phi_t\Phi_x\Phi_y\epsilon y + \frac{1}{3}\Phi\Phi_t\Phi_{xx}\epsilon x \\
&\quad - \frac{1}{2}\Phi\Phi_{txy}b\mu y + \Phi\Phi_{xxx}a\mu - \frac{1}{3}\Phi\Phi_{ty}\Phi_y\epsilon x + \frac{2}{3}\Phi\Phi_{tx}\Phi_x\epsilon x.
\end{aligned}$$

3.3 Approximate Symmetries

If we treat $\mu \ll 1$ as a small parameter, then we obtain approximate symmetries

$$(23) \quad X = X^0 + \mu X^1 + O(\mu^2),$$

where

$$X^A = \xi_t^A(t, x, y, \Phi) \partial_t + \xi_x^A(t, x, y, \Phi) \partial_x + \xi_y^A(t, x, y, \Phi) \partial_y + \eta^A(t, x, y, \Phi) \partial_\Phi, \quad A = 0, 1$$

The X^0 are determined in the usual way but with $\mu = 0$. The X^1 are calculated by solving the determining equation for deformations

$$(24) \quad \begin{aligned} X^1(\Phi_{tt} - \Delta\Phi + \epsilon(\Phi_t \Delta\Phi + 2\nabla\Phi \cdot \nabla\Phi_t))|_{\Phi_{tt} - \Delta\Phi + \epsilon(\Phi_t \Delta\Phi + 2\nabla\Phi \cdot \nabla\Phi_t) = 0} \\ + H = 0, \end{aligned}$$

where

$$(25) \quad H = \frac{1}{\mu} X^0 \left(\Phi_{tt} - \Delta\Phi + \mu(a\Delta^2\Phi - b\Delta\Phi_{tt}) + \epsilon(\Phi_t \Delta\Phi + 2\nabla\Phi \cdot \nabla\Phi_t) \right) \text{ mod. Eq.(1).}$$

Eq. (25) in this case is (B_{1-2} , constants)

$$(26) \quad \begin{aligned} H = 2B_2 \Phi_{xxxx} a\epsilon + 4B_2 \Phi_{xyyy} a\epsilon + 2B_2 \Phi_{yyyy} a\epsilon + 2B_1 \Phi_{xxtt} b \\ - 2B_1 \Phi_{xxxx} a - 4B_1 \Phi_{xyyy} a + 2B_1 \Phi_{yytt} b - 2B_1 \Phi_{yyyy} a. \end{aligned}$$

The solution of the system (24) provides (23), that is, the approximate symmetries

$$(27) \quad \begin{aligned} Y_1 = \mu\partial_t, \quad Y_2 = \mu\partial_x, \quad Y_3 = \mu\partial_y, \quad Y_4 = \mu\partial_\Phi, \quad Y_5 = \mu(-x\partial_y + y\partial_x), \\ Y_6 = \mu(\Phi\partial_\Phi + t\partial_t + x\partial_x + y\partial_y), \\ Y_7 = \mu\left((2t - \Phi\epsilon)\partial_\Phi + \epsilon t\partial_t\right), \end{aligned}$$

plus the exact symmetries (27). The commutators between the approximate symmetries are

$$\begin{aligned} [Y_1, Y_7] = \mu(\epsilon Y_1 + 2Y_4), \quad [Y_4, Y_7] = -\mu\epsilon Y_4, \quad [Y_3, Y_7] = [Y_2, Y_7] = [Y_5, Y_7] = 0, \\ [Y_6, Y_7] = [Y_5, Y_6] = 0, \quad [Y_1, Y_6] = \mu Y_1, \quad [Y_2, Y_6] = \mu Y_2, \quad [Y_3, Y_6] = \mu Y_3, \quad [Y_4, Y_6] = \mu Y_4. \end{aligned}$$

Meanwhile, if $\Phi = \Phi(t, x)$ we find through a similar calculation, the approximate symmetries

$$(28) \quad \begin{aligned} X_1, \quad X_2, \quad X_4, \quad Y_1, \quad Y_2, \\ Z_1 = \mu(\Phi\partial_\Phi + t\partial_t + x\partial_x), \\ Z_2 = \mu\left(2t\partial_\Phi + 2\epsilon t\partial_t + \epsilon x\partial_x\right). \end{aligned}$$

Now, approximate symmetries lead to approximate reductions and solutions.

A reduction by the symmetry combination $X_1 - X_4 + Y_1$ provides the approximate invariants

$$K(t, x, \Phi, \mu) = K^0(t, x, \Phi) + \mu K^1(t, x, \Phi) + O(\mu),$$

which leads to the system

$$(29) \quad K_t^0 - K_u^0 = 0,$$

$$(30) \quad K_t^1 - K_\Phi^1 = -K_t^0.$$

The solution of (29)-(30) gives two functionally independent invariants

$$(31) \quad \begin{aligned} K_1 &= K_1^0(t, x, \Phi) + \mu K_1^1(t, x, \Phi), \\ K_2 &= K_2^0(t, x, \Phi) + \mu K_2^1(t, x, \Phi). \end{aligned}$$

Eq. (29) has two functionally independent solutions

$$K_1^0 = x, \quad K_2^0 = \Phi + t.$$

A substitution of K_1^0 into (30) and taking $K_1^1 = 0$, we obtain one invariant

$$K_1 = x.$$

Next, a substitution of K_2^0 into (30) reveals that $K_2^1 = -t$. Hence the second invariant is

$$K_2 = \Phi + t - \mu t.$$

If we let $K_2 = A(K_1)$, we get the approximate invariant solution

$$(32) \quad \Phi(t, x) = A(x) + t(\mu - 1).$$

Substituting (32) into Eq. (1), yields

$$A(x) = c_1 + c_2 x + c_3 \sin\left(\frac{\sqrt{\epsilon\mu - \epsilon - 1} x}{\sqrt{a\mu}}\right) + c_4 \cos\left(\frac{\sqrt{\epsilon\mu - \epsilon - 1} x}{\sqrt{a\mu}}\right),$$

for c_{1-4} as constants.

In a similar procedure, but omitting the lengthy details, the linear combination $X_2 - X_4 + Y_2$ results in the approximate solution

$$\Phi(t, x) = c_1 t + c_2 - x + \mu x,$$

and the linear combination $X_2 - X_4 + Z_1$ leads to another approximate solution

$$\Phi(t, x) = \frac{c_1(t - tx\mu) + c_2 - x + \mu x^2}{1 - \mu x}.$$

Table 3: Commutators of the admitted Lie point symmetries of equation (2)

$[\cdot, \cdot]$	\mathbf{Y}_1	\mathbf{Y}_2	\mathbf{Y}_3
\mathbf{Y}_1	0	0	X_2
\mathbf{Y}_2	0	0	0
\mathbf{Y}_3	$-X_2$	0	0

4. SYMMETRY ANALYSIS OF THE BENNEY-LIN EQUATION

In the following section we determine the exact and approximate Lie point symmetries for the Benney-Lin equation (2).

4.1 Exact Lie point symmetries

The Benney-Lin equation (2) is invariant under the action of a three-dimensional point transformation with generators

$$Y_1 = \partial_t, \quad Y_2 = \partial_x \quad \text{and} \quad Y_3 = t\partial_x + \partial_u.$$

The commutators of these symmetry vectors are presented in Table 3, from where we can infer that the admitted Lie point symmetries form the $2A_1 \oplus_s A_1$ Lie algebra.

In the special case where $\beta_0 = \beta_1 = 0$, the Benney-Luke equation reduces to the KdV equation that is invariant by a four-dimensional Lie algebra [26]. At this point we mention that when $\beta_0 = 0$ and $\beta_1 \neq 0$ the Kawahara equation is recovered. However the admitted Lie algebra remain the same.

We omit the presentation of the similarity solutions for the Benney-Lin equation since recently in [28] the integrability of the Benney-Lin equation was studied by using Lie symmetries and singularity analysis.

4.2 Conservation laws

Here, we obtain only the conservation law multiplier 1, that leads to the conserved vector components

$$T^t = -u,$$

$$T^x = -\frac{1}{2}u^2 - u_x \beta_1 - \beta_1 u_{xxx} - u_{xxxx} \beta_0 - u - u_{xx}.$$

4.3 Approximate Symmetries

In the case that $\beta_0 = \beta_1 \ll 1$ is a small parameter, then we obtain approximate symmetries

$$(33) \quad X = X^0 + \beta_0 X^1 + O(\beta_0^2),$$

where this time

$$X^A = \xi_t^A(t, x, u) \partial_t + \xi_x^A(t, x, u) \partial_x + \eta^A(t, x, u) \partial_u, \quad A = 0, 1$$

The X^0 are determined when $\beta_0 = 0$. The X^1 are calculated by solving the determining equation for deformations

$$(34) \quad X^1 (u_t + u_{xxx} + u_x + uu_x) |_{u_t + u_{xxx} + u_x + uu_x = 0} + \bar{H} = 0,$$

where

$$(35) \quad \bar{H} = \frac{1}{\beta_0} X^0 \left(u_t + \beta_0 u_{xxxxx} + \beta_0 (u_{xxxx} + u_{xx}) + u_{xxx} + u_x + uu_x \right) \text{ mod. Eq.(2)}.$$

Eq. (35) in this case is (j_1 constant)

$$(36) \quad \bar{H} = j_1 (u_{xx} - u_{xxxx} - 2u_{xxxxx}).$$

The solution of the system (34) provides the approximate symmetries

$$(37) \quad \begin{aligned} P_1 = \partial_t, \quad P_2 = \partial_x, \quad P_3 = \partial_u + t\partial_x, \quad P_4 = \beta_0 P_1, \quad P_5 = \beta_0 P_2, \quad P_6 = \beta_0 P_3, \\ P_7 = \beta_0 \left((-2u - 2) \partial_u + 3t\partial_t + x\partial_x \right). \end{aligned}$$

The commutators between the approximate symmetries are

$$\begin{aligned} [P_1, P_7] = 3P_4, \quad [P_2, P_7] = P_3, \quad [P_3, P_7] = -2P_6, \quad [P_4, P_7] = 3\beta_0 P_4, \\ [P_5, P_7] = \beta_0 P_5, \quad [P_6, P_7] = -2\beta_0 P_6. \end{aligned}$$

An approximately invariant solution may be obtained from the symmetry combination $P_3 + P_5$. That is, we follow the same procedure as the previous section, to find the approximate solution

$$u(x, t) = \frac{x}{t} - \frac{\beta_0 x}{t^2} - 1 - \frac{j_2}{t} \exp(-\beta_0/t), \quad j_2 \text{ constant.}$$

5. CONCLUSIONS

In this work we studied two nonlinear equations which describe long-waves on liquid films by using the method of symmetry analysis. These two equations can be seen as (singular) perturbative equations and for this reason we focused on the approximate symmetries.

For the 2+1 Benney-Luke equation we determined that it admits five exact Lie point symmetries, while the exact conservation laws were derived. As far as the approximate symmetries for the Benney-Luke are concerned, we found seven approximate symmetries. These approximate results were applied to determine approximate similarity solutions.

The same procedure applied for the 1+1 Benney-Lin equation, where the exact symmetries were three and the approximate symmetries were again seven.

Again an approximate scaling similarity solution was derived by using these kind of symmetries.

We showed that approximate symmetries can play a significant role in the analysis of perturbative equations, while the approximate similarity solutions can be used to better understand the dynamics and the evolution of these systems. The physical interpretation of the results will be published in a future work.

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