

APPLICATIONS OF DERIVATIVE AND DIFFERENCE OPERATORS ON SOME SEQUENCES

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In this study, depending on the upper and the lower indices of the hyperharmonic number $h_n^{(r)}$, nonlinear recurrence relations are obtained. It is shown that generalized harmonic numbers and hyperharmonic numbers can be obtained from derivatives of the binomial coefficients. Taking into account of difference and derivative operators, several identities of the harmonic and hyperharmonic numbers are given. Negative-ordered hyperharmonic numbers are defined and their alternative representations are given.

1. INTRODUCTION

Harmonic numbers are longstanding subject of study and they are significant in various branches of analysis and number theory. The n -th harmonic number is the n -th partial sum of the harmonic series:

$$H_n := \sum_{k=1}^n \frac{1}{k}, \quad (n \in \mathbb{N} := \{1, 2, 3, \dots\}),$$

where the empty sum H_0 is conventionally understood to be zero. As a matter of fact, various generalizations of these numbers are also widely studied. Among many other generalizations we are interested in two important generalizations of these numbers, namely generalized harmonic numbers and hyperharmonic numbers.

For a positive integer n and an integer m the n -th generalized harmonic number of order m is defined by

$$H_n^{(m)} := \sum_{k=1}^n \frac{1}{k^m}.$$

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It is convenient to set $H_n^{(m)} = 0$ for $n \leq 0$. Hence for $m > 1$, $H_n^{(m)}$ is the n -th partial sum of the Riemann zeta function $\zeta(m)$.

Hyperharmonic numbers are another important generalization of harmonic numbers. For $r \in \mathbb{N}$, the n -th hyperharmonic number of order r is defined by [10, 18]

$$(1) \quad h_n^{(r)} = \sum_{k=1}^n h_k^{(r-1)}, \quad h_n^{(1)} := H_n.$$

Here we assume that $h_n^{(0)} := 1/n$, ($n \geq 1$) and $h_0^{(r)} := 0$, ($r \geq 0$). Hyperharmonic numbers are closely related to analytic number theory (see [2, 5, 11, 13, 19]), discrete mathematics and combinatorial analysis (see [3, 8, 10, 12]).

These numbers have an expression in terms of binomial coefficients and harmonic numbers [10, 18, 19]:

$$(2) \quad h_n^{(r)} = \binom{n+r-1}{r-1} (H_{n+r-1} - H_{r-1}).$$

Also we have the following representation [3, 12]:

$$h_n^{(r)} = \sum_{k=1}^n \binom{n+r-1-k}{r-1} \frac{1}{k}.$$

In Subsection 2.1, recurrence relations for both upper and lower indices of hyperharmonic numbers are given. Properties of the coefficients of these relations are examined.

For a function $f(x)$ that can be differentiated n -times define the derivative operator D_x by

$$D_x^n f(x) = \frac{d^n}{dx^n} f(x).$$

In Subsection 2.2 we focus on the following simple equation [16, 21]:

$$(3) \quad D_x \binom{x+n}{n} \Big|_{x=0} = H_n$$

which provides a connection between analysis and combinatorics. Paule and Schneider used (3) in [21] as a starting point, and they dealt with some important harmonic number identities. Also Chu and Donno [9] used the classical hypergeometric summation theorems to derive numerous identities involving harmonic numbers. In the paper [25], Sofo used the idea of the consecutive derivative operator of binomial coefficients to give integral representation for series containing binomial coefficients and harmonic numbers. Choi [8] showed how one can obtain identities about certain finite series involving binomial coefficients, harmonic numbers and generalized harmonic numbers by applying the usual differential operator to a known identity.

In [31], in terms of the telescoping method, Yan and Liu constructed a binomial identity first; then by applying the derivative operator to that identity, authors derived numerous interesting harmonic number identities.

In this subsection, we first obtain generalization of (3) for generalized harmonic numbers

$$D_x \binom{x + n^m}{n^m}_m \Big|_{x=0} = H_n^{(m)}.$$

For this generalization, the concept of “leaping binomial coefficients”

$$\binom{x + n^m}{n^m}_m = \frac{1}{(n!)^m} \prod_{i=1}^n (x + i^m)$$

are defined and the relationship between the leaping binomial coefficients and the classical binomial coefficients are obtained. After that, the generalization of (3) for hyperharmonic numbers

$$D_x \binom{x + n + r - 1}{n} \Big|_{x=0} = h_n^{(r)}$$

is also obtained. Using this generalization, dozens of new formulas containing harmonic, hyperharmonic and generalized harmonic numbers are given in the light of H. W. Gould’s book [16] about binomial coefficients.

The difference operator

$$(4) \quad \Delta f(x) = f(x+1) - f(x) \text{ and } \Delta^0 f(x) = f(x)$$

has a wide range of applications, in particular in discrete mathematics and differential equations theory (see [1, 6, 24]). This is the finite analog of the derivative operator [17]. In Subsections 2.3 and 2.4, the difference operator is primarily used to investigate the properties of harmonic and hyperharmonic numbers. Meanwhile, negative-ordered hyperharmonic numbers are presented and various representations are obtained.

In Subsection 2.5, Fibonacci numbers

$$F_n = F_{n-1} + F_{n-2} \text{ with } F_0 := 0, F_1 := 1$$

are examined with the help of the difference operator. We also encounter the Fibonacci numbers of negative indexed and give representation of these numbers.

In Subsection 2.6, results for hyperbolic functions are given to illustrate the prevalence of the difference operator in practice.

Finally, in the Appendix the reader can find two tables of summation formulas related to harmonic, hyperharmonic and generalized harmonic numbers. These formulas are applications of the results that we obtained in Subsection 2.2.

2. MAIN RESULTS

2.1 A symmetric identity and nonlinear first-order recurrences for hyperharmonic numbers

The following proposition shows a balance between the upper and the lower indices of hyperharmonic numbers.

Proposition 1. *We have*

$$(5) \quad \sum_{s=1}^n h_s^{(r-1)} - \sum_{k=1}^r h_{n-1}^{(k)} = \frac{1}{n} = h_n^{(0)}.$$

Proof. From (1) we obtain

$$(6) \quad h_n^{(r-1)} = h_n^{(r)} - h_{n-1}^{(r)}.$$

Telescoping sum on r of (6) gives the statement. \square

Remark 2. *We can write the result (5) in the equivalent form*

$$h_n^{(r)} = \sum_{k=1}^r h_{n-1}^{(k)} + \frac{1}{n}$$

which is already obtained in [3]. In the light of this equation we get the following general result

$$h_n^{(r+s)} - h_n^{(s)} = \sum_{k=1}^r h_{n-1}^{(k+s)},$$

where $s \in \mathbb{N}$.

Since the hyperharmonic numbers of order r is an r -fold sum, some calculations involving these numbers can be difficult. For this reason, various studies have been performed in which some other representations of hyperharmonic numbers are given and recurrences are obtained. Now we give recurrence relations for the lower and the upper indices of $h_n^{(r)}$.

Proposition 3. *A recurrence with respect to the lower index n is*

$$(7) \quad h_n^{(r+1)} = \alpha h_{n-1}^{(r+1)} + \beta,$$

and a recurrence with respect to the upper index r is

$$(8) \quad (\alpha - 1) h_n^{(r+1)} = \alpha h_n^{(r)} - \beta,$$

where $\alpha = \alpha(n, r) = 1 + \frac{r}{n}$ and $\beta = \beta(n, r) = \frac{1}{n+r} \binom{n+r}{r}$.

Proof. Using the fact that $H_{n+r} = H_{n+r-1} + \frac{1}{n+r}$ and (2) it follows that

$$h_n^{(r+1)} = \frac{1}{n+r} \binom{n+r}{r} + \binom{n+r}{r} (H_{n+r-1} - H_r).$$

Considering the summation identity $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$ we get

$$h_n^{(r+1)} = \frac{1}{n+r} \binom{n+r}{r} + h_{n-1}^{(r+1)} + \binom{n+r-1}{r-1} (H_{n+r-1} - H_r).$$

Now, the following identity combines with (2) to give (7):

$$\binom{n+r-1}{r-1} = \frac{r}{n} \binom{n+r-1}{r}.$$

For the second recurrence we start with the equation

$$(9) \quad h_n^{(r+1)} = h_n^{(r)} + h_{n-1}^{(r+1)}.$$

On the other hand (7) gives

$$(10) \quad h_{n-1}^{(r+1)} = \frac{1}{\alpha} h_n^{(r+1)} - \frac{\beta}{\alpha}.$$

Considering (10) in (9) gives

$$h_n^{(r+1)} = h_n^{(r)} + \frac{1}{\alpha} h_n^{(r+1)} - \frac{\beta}{\alpha},$$

which can equally well be written

$$h_n^{(r+1)} = \frac{\alpha}{\alpha-1} h_n^{(r)} - \frac{\beta}{\alpha-1}.$$

□

Example 4. Let us show the usefulness of the Proposition 3. Fixing $n = 2$ in (7) gives

$$h_2^{(r+1)} = \left(1 + \frac{r}{2}\right) h_1^{(r+1)} + \frac{r+1}{2}.$$

Remembering that $h_1^{(r)} = 1$ for any order r , we get a general formula for $h_2^{(r)}$ as

$$h_2^{(r+1)} = r + 1 + \frac{1}{2}.$$

On the other hand if we fix $r = 1$ in (8), then we get a general formula for $h_n^{(2)}$ in terms of H_n as

$$h_n^{(2)} = (n+1) H_n - n.$$

Remark 5. *Let us observe*

$$n\alpha(n, r) = r\alpha(r, n)$$

and

$$\beta(n, r) = \frac{1}{n+r} \binom{n+r}{r} = \beta(r, n).$$

With the help of the equality

$$\beta(n, r) = \frac{1}{n+r} \binom{n+r}{r} = \frac{1}{r} \binom{n+r-1}{n},$$

we have the ordinary generating function of $\beta(k, r)$

$$\sum_{k=0}^{\infty} \beta(k, r) x^k = \frac{1}{r} \sum_{k=0}^{\infty} \binom{k+r-1}{k} x^k = \frac{1}{r(1-x)^r},$$

and also the ordinary generating function of $\alpha(k, r)$

$$\sum_{k=1}^{\infty} \alpha(k, r) x^k = \frac{x}{1-x} - r \ln(1-x).$$

The next proposition enables us to get a closed form evaluation for the finite sum of $\beta(k, r)$ in terms of α and β . Proof of it can be directly seen from the following basic properties of the binomial coefficients [17, p. 174]:

$$(11) \quad \binom{k+r}{r} = \frac{k+r}{r} \binom{k+r-1}{r-1}$$

and

$$(12) \quad \sum_{k=0}^n \binom{k+r-1}{k} = \binom{n+r}{n}.$$

Proposition 6. *Let $n, r \in \mathbb{N}$. Then*

$$\sum_{k=0}^n \beta(k, r) = \frac{\alpha(n, r) \beta(n, r)}{\alpha(n, r) - 1}$$

or equally

$$\sum_{k=0}^n \frac{1}{k+r} \binom{k+r}{r} = \frac{1}{r} \binom{n+r}{n}.$$

It is possible to generalize Proposition 6 by considering the concept of “falling factorial”. Recall that falling factorial is defined with the equation [17]:

$$x^n = x(x-1)(x-2)\dots(x-n+1).$$

Proposition 7. *Let $m, r \in \mathbb{N}$, then*

$$\sum_{k=0}^n \frac{1}{(k+r)^{\underline{m}}} \binom{k+r}{r} = \frac{1}{r^{\underline{m}}} \binom{n+r-m+1}{n}.$$

Proof. Because of (11) we have

$$\frac{1}{k+r} \binom{k+r}{r} = \frac{1}{r} \binom{k+r-1}{r-1}$$

and also

$$\frac{1}{(k+r)^{\underline{2}}} \binom{k+r}{r} = \frac{1}{r^{\underline{2}}} \binom{k+r-2}{r-2}.$$

Hence, in general we can write

$$\frac{1}{(k+r)^{\underline{m}}} \binom{k+r}{r} = \frac{1}{r^{\underline{m}}} \binom{k+r-m}{r-m}.$$

Summing both sides as

$$\sum_{k=0}^n \frac{1}{(k+r)^{\underline{m}}} \binom{k+r}{r} = \frac{1}{r^{\underline{m}}} \sum_{k=0}^n \binom{k+r-m}{r-m}$$

and employing (12) give the statement. \square

2.2 Hyperharmonic and generalized harmonic numbers via derivative operator

Here we generalize the identity (3) both for generalized harmonic numbers and hyperharmonic numbers.

The following definition plays a key role in generalizing (3) for generalized harmonic numbers.

Definition 8. *For any parameter x and positive integers m and n “leaping binomial coefficients” are defined by*

$$\begin{aligned} \binom{x+n^m}{n^m}_m &= \frac{(x+n^m)(x+(n-1)^m) \cdots (x+2^m)(x+1^m)}{(n!)^m} \\ &= \frac{1}{(n!)^m} \prod_{i=1}^n (x+i^m). \end{aligned}$$

Now we are ready to give a generalization of (3) for generalized harmonic numbers.

Proposition 9. *For any $m, n \in \mathbb{N}$ we have*

$$D_x \binom{x+n^m}{n^m}_m \Big|_{x=0} = H_n^{(m)}.$$

Proof. Let us observe the following equation

$$D_x \binom{x+n^m}{n^m}_m = \frac{(x+n^m)(x+(n-1)^m) \cdots (x+2^m)(x+1^m)}{(n!)^m} \times \left\{ \frac{1}{(x+n^m)} + \frac{1}{(x+(n-1)^m)} + \cdots + \frac{1}{(x+1^m)} \right\}.$$

Here evaluating both sides at $x = 0$ gives the statement. \square

Relation between the classical binomial coefficients and the leaping binomial coefficients is given by the following proposition.

Proposition 10. *For any positive integers m and $n \geq 2$ we have*

$$\binom{x+n^m}{n^m}_m = \frac{\frac{(n^m)!}{(n!)^m} \binom{x+n^m}{n^m}}{\prod_{i=2}^n \binom{x+i^m-1}{i^m-(i-1)^m-1} (i^m - (i-1)^m - 1)!}.$$

Proof. We start with the definition of the leaping binomial coefficients, and replace the terms

$$(x+i^m)(x+(i-1)^m)$$

by

$$\frac{(x+i^m)(x+i^m-1)(x+i^m-2) \cdots (x+(i-1)^m+1)(x+(i-1)^m)}{(x+i^m-1)(x+i^m-2) \cdots (x+(i-1)^m+1)}$$

for $i = 2, 3, \dots, n$. Reorganizing these terms we get

$$\binom{x+n^m}{n^m}_m = \frac{(n^m)!}{(n!)^m (n^m - (n-1)^m - 1)! \cdots (2^m - 1^m - 1)!} \times \frac{\binom{x+n^m}{n^m}}{\binom{x+n^m-1}{n^m-(n-1)^m-1} \binom{x+(n-1)^m-1}{(n-1)^m-(n-2)^m-1} \cdots \binom{x+2^m-1}{2^m-1^m-1}},$$

which completes the proof. \square

Some preparation is needed to generalize (3) for hyperharmonic numbers. With the help of the operator D_x and the classical gamma function

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt, \quad \operatorname{Re}(z) > 0,$$

the digamma function is defined by [1]:

$$\psi(z) = D_z \log(\Gamma(z)) = \frac{\Gamma'(z)}{\Gamma(z)}, \quad (z \in \mathbb{C} \setminus \{0, -1, -2, -3, \dots\}).$$

The rising factorial is defined by

$$z^{\bar{n}} = z(z+1)(z+2)\dots(z+n-1).$$

The rising factorial $z^{\bar{n}}$ is sometimes denoted by $(z)_n$ [1, 17]. It is closely related to the Euler's gamma function by the relation

$$z^{\bar{n}} = \frac{\Gamma(z+n)}{\Gamma(z)}.$$

Derivative of $z^{\bar{n}}$ turns out to be

$$(13) \quad D_z z^{\bar{n}} = z^{\bar{n}} (\psi(z+n) - \psi(z)).$$

Actually one can easily prove (13) by considering the equation

$$\psi(z+1) = \psi(z) + \frac{1}{z}.$$

Evaluating ψ at positive integers gives

$$(14) \quad \psi(n) = H_{n-1} - \gamma.$$

We recall that $\gamma = -\psi(1)$ is Euler-Mascheroni constant.

Considering (2) and (14) we also have generalization of (14) for hyperharmonic numbers as [14, 20]

$$(15) \quad h_n^{(r)} = \frac{r^{\bar{n}}}{n!} (\psi(n+r) - \psi(r)).$$

Now we are ready to give a generalization of (3) for hyperharmonic numbers.

Proposition 11. *Let $n \in \mathbb{N} \cup \{0\}$ and $r \in \mathbb{N}$. Then*

$$(16) \quad D_x \binom{x+n+r-1}{n} \Big|_{x=0} = h_n^{(r)}.$$

Proof. In the light of (13) we write

$$D_x \binom{x+n+r-1}{n} = D_x \frac{(x+r)^{\bar{n}}}{n!} = \frac{(x+r)^{\bar{n}}}{n!} (\psi(x+n+r) - \psi(x+r)).$$

Setting $x = 0$ in the above equation and considering (15) we get the desired result. \square

Remark 12. *Let us consider the binomial formula*

$$\sum_{n=0}^{\infty} \binom{x+n}{n} z^n = \frac{1}{(1-z)^{x+1}}$$

and in general

$$\sum_{n=0}^{\infty} \binom{x+n+r-1}{n} z^n = \frac{1}{(1-z)^{x+r}},$$

where $|z| < 1$ and for any $x \in \mathbb{C}$, $r \in \mathbb{N}$. Hence the generating functions of harmonic and hyperharmonic numbers are direct consequences of (16):

$$(17) \quad \sum_{n=0}^{\infty} H_n z^n = -\frac{\ln(1-z)}{1-z},$$

$$\sum_{n=0}^{\infty} h_n^{(r)} z^n = -\frac{\ln(1-z)}{(1-z)^r}.$$

Remark 13. Considering (16) with the following binomial equation [17, p. 174]

$$\sum_{k=0}^n \binom{x+k+r-2}{k} = \binom{x+n+r-1}{n}$$

gives (1).

Remark 14. Let us recall the following binomial equation (see [17, p. 174]):

$$(18) \quad \sum_{j=0}^n \binom{x+j+r-1}{j} = \left(1 + \frac{n}{x+r}\right) \binom{x+n+r-1}{n}.$$

Applying D_x both sides of (18) and remembering that

$$\binom{n+r}{r} = \frac{n+r}{r} \binom{n+r-1}{r-1},$$

we get

$$(\alpha-1) h_{n-1}^{(r+1)} = h_n^{(r)} - \beta$$

which is an alternative proof of (8).

Remark 15. Before closing this subsection let us remind you that many equations can be obtained as the application of the results given in this section, you can find some of them in the Appendix.

2.3 Harmonic numbers via difference operator

The difference operator Δ (see (4)) is a linear operator, hence

$$\Delta [af(x) + bg(x)] = a\Delta f(x) + b\Delta g(x)$$

holds for any constants a and b . Applying Δ operator n -times to a suitable function f , we get [1, 6, 17, 22]:

$$(19) \quad \Delta^n f(x) = \sum_{i=0}^n (-1)^{n-i} \binom{n}{i} f(x+i).$$

Proposition 16. *We have*

$$\Delta^n (xf(x)) = x\Delta^n f(x) + n\Delta^{n-1} f(x+1).$$

Proof. By induction on n . □

Now, we use this result as a tool to investigate the properties of harmonic numbers.

Proposition 17. *For any $k \in \mathbb{N}$ we have*

$$(20) \quad \sum_{i=0}^k (-1)^{i+1} \binom{k}{i} H_{n+i} = \frac{(k-1)!}{(n+1)^{\bar{k}}}.$$

Proof. Setting $f(n) = H_n$ in (19) gives

$$\Delta^k H_n = \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} H_{n+i}.$$

On the other hand considering H_n with the difference operator gives

$$\Delta^k H_n = \frac{(-1)^{k+1} (k-1)!}{(n+1)^{\bar{k}}}.$$

Equality of these two equations completes the proof. □

As a result of Proposition 17 we have the following well-known equation [6, p. 34].

Corollary 18. *For $k \in \mathbb{N}$*

$$(21) \quad \sum_{i=1}^k (-1)^{i+1} \binom{k}{i} H_i = \frac{1}{k}.$$

Proof. Choosing $n = 0$ in Proposition 17 gives the result. □

Remark 19. *Let us consider the binomial transform [6, 26, 29]*

$$(22) \quad a_k = \sum_{i=0}^k \binom{k}{i} b_i \Leftrightarrow b_k = \sum_{i=0}^k (-1)^{k+i} \binom{k}{i} a_i.$$

In the light of (21) we can fix $a_k = H_k$ and

$$b_k = \begin{cases} \frac{(-1)^{k+1}}{k} & , k \in \mathbb{Z}^+ \\ 0 & , k = 0 \end{cases}.$$

Then from (22) we have the well-known formula (see [16, (1.45)] or [23, p. 53])

$$(23) \quad \sum_{i=1}^k (-1)^{i+1} \binom{k}{i} \frac{1}{i} = H_k$$

where $a_0 = H_0 = 0 = b_0$.

Now we give an identity for the binomial sum of harmonic numbers.

Proposition 20. *Let $k \geq 2$ be an integer. Then*

$$\sum_{i=0}^k (-1)^i \binom{k}{i} (n+i) H_{n+i} = \frac{(k-2)!}{(n+1)^{k-1}}.$$

Proof. Choosing $f(n) = H_n$ in Proposition 16 we get

$$\Delta^k (nH_n) = n\Delta^k H_n + k\Delta^{k-1} H_{n+1}.$$

Here by employing (20) we obtain RHS as

$$\Delta^k (nH_n) = n \frac{(-1)^{k+1} (k-1)!}{(n+1)^k} + k \frac{(-1)^k (k-2)!}{(n+2)^{k-1}}.$$

For the LHS we consider nH_n in (19) and proof follows. \square

Now we give a special case of Proposition 20 which is worth mentioning. This identity is known [28].

Corollary 21. *For an integer $k \geq 2$*

$$(24) \quad \sum_{i=1}^k (-1)^i \binom{k}{i} i H_i = \frac{1}{(k-1)}.$$

Remark 22. *Considering (24) with (22) we choose $a_k = kH_k$ and*

$$b_k = \begin{cases} \frac{(-1)^k}{(k-1)}, & k \geq 2 \\ 1, & k = 1 \\ 0, & k = 0 \end{cases}$$

to get

$$k(H_k - 1) = \sum_{i=2}^k \binom{k}{i} \frac{(-1)^i}{(i-1)}.$$

Corollary 23. *We have the following alternate sum of harmonic numbers and binomial coefficients*

$$(25) \quad \sum_{i=1}^{n-1} (-1)^{i+1} \binom{n+1}{i+1} H_i = \begin{cases} 2H_n, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

Proof. From Corollary 18 we have

$$\sum_{i=1}^k (-1)^{i+1} \binom{k}{i} H_i = \frac{1}{k}.$$

Here summing both sides from 1 to n we get

$$\sum_{k=1}^n \sum_{i=1}^k (-1)^{i+1} \binom{k}{i} H_i = \sum_{k=1}^n \frac{1}{k} = H_n.$$

By changing the order of sums this becomes

$$\sum_{i=1}^n (-1)^{i+1} H_i \sum_{k=i}^n \binom{k}{i} = H_n.$$

From equation [17]

$$(26) \quad \sum_{k=i}^n \binom{k}{i} = \binom{n+1}{i+1}$$

we find

$$\sum_{i=1}^n (-1)^{i+1} \binom{n+1}{i+1} H_i = H_n.$$

□

Remark 24. Equation (25) has been already discovered by Wang [30], but here we give an alternative proof of it. Besides, we also give one more proof of it in Appendix by using the equation (1.44) in Gould's book [16].

2.4 Hyperharmonic numbers via difference operator

By taking inspiration from (15), Mező [20] defined the hyperharmonic function:

$$(27) \quad h_z^{(w)} = \frac{z^{\bar{w}}}{z\Gamma(w)} (\psi(z+w) - \psi(w))$$

where $w, z+w \in \mathbb{C} \setminus (\mathbb{Z}^- \cup \{0\})$. Here \mathbb{Z}^- denotes the negative integers. In the light of this definition, Dil [14] gave formulas to evaluate some special values of $h_z^{(w)}$. However, all those evaluations are valid under the restriction of $w, z+w \in \mathbb{C} \setminus (\mathbb{Z}^- \cup \{0\})$, i.e. upper index can not be a negative integer. In this subsection firstly we are going to show a way to define "negative-ordered hyperharmonic numbers".

From (1) we have the recurrence

$$h_n^{(r)} = h_{n-1}^{(r)} + h_n^{(r-1)}.$$

To obtain lower-ordered hyperharmonic numbers in terms of higher-ordered ones, we can write it as

$$(28) \quad h_n^{(r-1)} = h_n^{(r)} - h_{n-1}^{(r)}.$$

Setting $r = 1$ gives

$$h_n^{(0)} = h_n^{(1)} - h_{n-1}^{(1)} = H_n - H_{n-1} = \frac{1}{n}.$$

For $r = 0$ in (28) gives

$$h_n^{(-1)} = h_n^{(0)} - h_{n-1}^{(0)} = \frac{1}{n} - \frac{1}{n-1} = \frac{-1}{n^2}.$$

Similarly, for $r = -1$ and $r = -2$ we have

$$h_n^{(-2)} = \frac{2}{n^3}$$

and

$$h_n^{(-3)} = \frac{-6}{n^4}$$

respectively. So by induction on r for $n > 1$ we have

$$h_n^{(-r)} = \frac{(-1)^r r!}{n^{r+1}}.$$

Hence we deduce the following definition.

Definition 25. For positive integers n and r , negative-ordered hyperharmonic number $h_n^{(-r)}$ is defined by

$$h_n^{(-r)} = \begin{cases} \frac{(-1)^r r!}{n^{r+1}}, & n > r \geq 1 \\ \sum_{k=0}^{n-1} \binom{r}{k} (-1)^k \frac{1}{n-k}, & r \geq n > 1 \\ 1, & n = 1 \end{cases}$$

Remark 26. There are two main reasons for giving the definition like this; the part $n > r \geq 1$ comes from the recursive definition of hyperharmonic numbers and the part $r \geq n > 1$ comes from the desire to define the generating function of the negative-ordered hyperharmonic numbers naturally. By considering the generating function of the hyperharmonic numbers (see (17)) we would like to define the generating function of the negative-ordered hyperharmonic numbers as follows:

$$\sum_{n=0}^{\infty} h_n^{(-r)} z^n = -(1-z)^r \ln(1-z).$$

One can think that for $n > r \geq 1$ we have two different representations of $h_n^{(-r)}$, but considering the equation (1.43) in [16]

$$\sum_{k=0}^r \binom{r}{k} (-1)^k \frac{1}{n-k} = \frac{(-1)^r}{(n-r) \binom{n}{r}}$$

we see that these two representations are equal.

Now we give some results which are obtained using the difference operator. We can consider $h_n^{(r)}$ either as a function of n or r , hence we get the following proposition which gives representations of hyperharmonic numbers with binomial coefficients.

Proposition 27. *Multiple difference of $h_n^{(r)}$ with respect to n (i.e. applying Δ operator k -times) gives*

$$(29) \quad h_{n+k}^{(r-k)} = \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} h_{n+i}^{(r)},$$

where $r \in \mathbb{Z}$ and $k, n \in \mathbb{N} \cup \{0\}$ and with respect to r gives

$$h_{n-k}^{(r+k)} = \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} h_n^{(r+i)},$$

where $r \in \mathbb{Z}$ and $k, n \in \mathbb{N}$ and $k \leq n$.

Proof. Proof can be seen from (19) and multiple difference of $h_n^{(r)}$ respect to the variables n and r . \square

Corollary 28. *For $k \in \mathbb{N}$*

$$h_{n-k}^{(k)} = \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} h_n^{(i)},$$

and

$$\sum_{i=0}^k (-1)^{k-i} \binom{k}{i} h_k^{(r+i)} = 0.$$

As an immediate result of (29), we give a new representation for the negative-ordered hyperharmonic numbers.

Corollary 29. *For $k, n \in \mathbb{N}$, the $(n+k)$ -th hyperharmonic number of order $-k$ is given by*

$$(30) \quad h_{n+k}^{(-k)} = \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} \frac{1}{n+i}.$$

Using the following result one can state the lower-ordered hyperharmonic numbers in terms of the higher-ordered hyperharmonic numbers.

Corollary 30. *We have*

$$h_k^{(r-k)} = \sum_{i=1}^k (-1)^{k-i} \binom{k}{i} h_i^{(r)}.$$

Corollary 31. *For non-negative integer k and n we have*

$$\frac{1}{k+1} = \sum_{i=0}^k \binom{k}{i} h_{i+1}^{(-i)},$$

and

$$(31) \quad H_n = \sum_{i=1}^n \binom{n}{i} h_i^{(1-i)}.$$

Proof. From (30) we have

$$h_{k+1}^{(-k)} = \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} \frac{1}{i+1}.$$

In the light of (22) we write

$$\frac{1}{k+1} = \sum_{i=0}^k \binom{k}{i} h_{i+1}^{(-i)}.$$

For the second equation, we write

$$\begin{aligned} \sum_{k=0}^{n-1} \frac{1}{k+1} &= \sum_{k=0}^{n-1} \sum_{i=0}^k \binom{k}{i} h_{i+1}^{(-i)} \\ &= \sum_{i=0}^{n-1} \sum_{k=i}^{n-1} \binom{k}{i} h_{i+1}^{(-i)}. \end{aligned}$$

With the help of (26) we get

$$\sum_{k=0}^{n-1} \frac{1}{k+1} = \sum_{i=0}^{n-1} \binom{n}{i+1} h_{i+1}^{(-i)}$$

which completes the proof. □

Remark 32. *By definition we have*

$$h_{i+1}^{(-i)} = \frac{(-1)^i}{i+1}.$$

Employing this in (31) gives an alternative proof of (23).

Remark 33. *As a result we can state*

$$H_k = \sum_{i=1}^k (-1)^{k-i} \binom{k}{i} h_i^{(k+1)}.$$

Furthermore we obtain

$$\frac{1}{k} = \sum_{i=1}^k (-1)^{k-i} \binom{k}{i} h_i^{(k)},$$

and from this we have a double sum representation for harmonic numbers as

$$H_n = \sum_{1 \leq i \leq k \leq n} (-1)^{k-i} \binom{k}{i} h_i^{(k)}.$$

2.5 Fibonacci numbers via difference operator

In spite of the fact that Fibonacci numbers originally come from a counting problem, it is possible to define negative-indexed Fibonacci numbers as:

$$F_{-n} = (-1)^{n+1} F_n$$

where $n \in \mathbb{N}$. The recurrence relation

$$F_{-n} = F_{-n-1} + F_{-n-2}$$

also holds [4, 15].

Now we give binomial representations for the Fibonacci and the negative-indexed Fibonacci numbers.

Proposition 34. For non-negative integers k and n we have

$$F_{n-k} = \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} F_{n+i}.$$

Proof. (19) gives

$$\Delta^k F_n = \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} F_{n+i}.$$

On the other hand we calculate $\Delta^k F_n = F_{n-k}$, equality of these two results completes the proof. \square

By setting $n = 0$ in Proposition 34, we get the following result.

Corollary 35. For $k \in \mathbb{N}$ we have

$$(32) \quad F_{-k} = \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} F_i.$$

RHS of (32) is meaningful for $k > 0$ therefore we can consider (32) as a definition of the negative-indexed Fibonacci numbers. Calculations give the first few of them as $F_{-1} = 1$, $F_{-2} = -1$, $F_{-3} = 2, \dots$

Remark 36. *The negative-indexed Fibonacci numbers have been already discovered [4, 15]. Here we give another approach.*

By using (32), we give an alternative verification of the recurrence relation for the negative-indexed Fibonacci numbers.

Corollary 37. *For any $k \in \mathbb{N} \cup \{0\}$ we have*

$$F_{-k} = F_{-k-1} + F_{-k-2}.$$

Proof. Considering Corollary 35 we write

$$\begin{aligned} F_{-k-1} + F_{-k-2} &= F_{k+2} + \sum_{i=0}^{k+1} (-1)^{k-i} \left[\binom{k+2}{i} - \binom{k+1}{i} \right] F_i \\ &= F_{k+2} + \sum_{i=1}^{k+1} (-1)^{k-i} \binom{k+1}{i-1} F_i \\ &= \sum_{i=1}^{k+2} (-1)^{k-i} \binom{k+1}{i-1} F_i \end{aligned}$$

which can be written as

$$\begin{aligned} F_{-k-1} + F_{-k-2} &= \sum_{i=1}^{k+2} (-1)^{k-i} \left[\binom{k}{i-1} + \binom{k}{i-2} \right] F_i \\ &= \sum_{i=1}^{k+1} (-1)^{k-i} \binom{k}{i-1} F_i + \sum_{i=2}^{k+2} (-1)^{k-i} \binom{k}{i-2} F_i \\ &= \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} [F_{i+2} - F_{i+1}] \\ &= \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} F_i \\ &= F_{-k}. \end{aligned}$$

□

Remark 38. *Eventually for any $k \in \mathbb{Z}$ we have*

$$F_k = F_{k-1} + F_{k-2}.$$

Lastly we prove a correspondence between the negative- and the positive-indexed Fibonacci numbers.

Corollary 39. *The following identity holds:*

$$F_{-k} = (-1)^{k+1} F_k.$$

Proof. Considering (32) we obtain the result, by induction on k . \square

2.6 Hyperbolic functions via difference operator

Now we apply the difference operator to the hyperbolic functions.

Proposition 40. *Effects of Δ operator on $\sinh x$ and $\cosh x$ functions are given by*

$$\sum_{i=0}^k (-1)^{k-i} \binom{k}{i} \sinh(x+i) = \frac{1}{2e^x} \left(1 - \frac{1}{e}\right)^k \left(e^{2x+k} + (-1)^{k+1}\right)$$

and

$$\sum_{i=0}^k (-1)^{k-i} \binom{k}{i} \cosh(x+i) = \frac{1}{2e^x} \left(1 - \frac{1}{e}\right)^k \left(e^{2x+k} + (-1)^k\right).$$

Proof. Proof can be seen from (19) and multiple differences of $\sinh x$ and $\cosh x$ functions. \square

Remark 41. *As a consequence of Proposition 40 we have closed formulas for the alternate binomial sums of $\sinh x$ and $\cosh x$:*

$$\sum_{i=0}^k (-1)^{k-i} \binom{k}{i} \sinh(i) = \frac{(e-1)^k \left(e^k + (-1)^{k+1}\right)}{2e^k}$$

and

$$\sum_{i=0}^k (-1)^{k-i} \binom{k}{i} \cosh(i) = \frac{(e-1)^k \left(e^k + (-1)^k\right)}{2e^k}.$$

Now we have results on digamma function.

Proposition 42. *Considering the digamma function with the difference operator we get*

$$\frac{(k-1)!}{x^k} = \sum_{i=0}^k (-1)^i \binom{k}{i} \psi(x+i).$$

Proof. Proof can be seen from (19) and multiple differences of $\psi(x)$. \square

Remark 43. *Considering $k=1$ in Proposition 42 we give the following well-known result as a consequence:*

$$\psi(x+1) = \psi(x) + \frac{1}{x}.$$

3. APPENDIX

In Subsection 2.2, we have given the equation (16) as the generalization of the equation (3). In a similar fashion one can obtain the following equations:

$$(33) \quad D \binom{x+n}{n}^{-1} \Big|_{x=0} = -H_n,$$

$$(34) \quad D \binom{x+r-1+n}{n}^{-1} \Big|_{x=0} = - \binom{r-1+n}{n}^{-2} h_n^{(r)},$$

$$(35) \quad D \binom{x-1}{n} \Big|_{x=0} = (-1)^{n+1} H_n.$$

When these formulas are applied to appropriate sums containing the binomial coefficients, many equations containing harmonic numbers (and also generalizations of harmonic numbers) can be obtained. H.W. Gould gave a list of 500 binomial coefficient summations in [16]. Considering the equations (3), (16), (33), (34) and (35) together with that list we have several identities of harmonic, hyperharmonic and generalized harmonic numbers. Some of them are known but most of them are new. The following references also include formulas of these types: [6, 7, 8, 27, 28].

Remark 44. *In the following tables:*

1) *The numbers on the LHS are the numbers of formulas at the H.W. Gould's book [16]; whereas on the RHS are the equations we have obtained using these formulas.*

2) *There are some formulas containing hyperharmonic numbers with non-integer order. We recall that the meanings of these numbers are given by the hyperharmonic function defined by (27). For detailed calculation techniques of such numbers, see [14].*

(1.23)	$\sum_{k=0}^{\infty} \frac{H_k}{2^k} = 2 \ln 2$
(1.41)	$\sum_{k=1}^n (-1)^{k-1} \binom{n}{k} \frac{1}{k} = H_n$
(1.42)	$\sum_{k=1}^n (-1)^{k-1} \binom{n}{k} H_k = \frac{1}{n}$
(1.44)	$\sum_{k=1}^n (-1)^{k-1} \binom{n+1}{k+1} H_k = H_n$
(2.16)	$\sum_{k=1}^{\infty} \frac{H_k}{k!} = e \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!k}$
(3.2)	$\sum_{k=1}^n \binom{r-2+n-k}{r-2} H_k = h_n^{(r)}$

(3.36)	$2 \sum_{k=1}^{2n} (-1)^k H_k = H_n$
(3.95)	$\sum_{k=1}^n (-1)^k \binom{2n-2k}{n-k} \binom{2k}{k} \frac{1}{k} = 2^{2n} \left\{ h_n^{(\frac{1}{2})} - \binom{n-\frac{1}{2}}{n} H_n \right\}$
(3.100)	$\sum_{k=1}^n (-1)^k \binom{n+k}{2k} \binom{2k}{k} \frac{1}{k} = -2H_n$
(3.108)	$\sum_{k=0}^n \left\{ \binom{k+m+1}{m} H_k + h_m^{(k+2)} \right\} = \sum_{k=0}^m \left\{ \binom{k+n+1}{n} H_k + h_n^{(k+2)} \right\}$
(4.3)	$\sum_{k=1}^n (-1)^{k-1} \binom{n}{k} x^k H_k = n(1-x)^{n-1} + \sum_{k=1}^n \binom{n}{k} \frac{(x-k)^k (1-x+k)^{n-k}}{k}$
(6.19)	$\sum_{k=0}^n \binom{n}{k} \binom{r}{k} h_{n+r}^{(1-k)} = H_n + H_r$
(6.22)	$\sum_{k=1}^n (-1)^k \binom{2n}{k} \binom{2n-k}{n} \frac{2}{k} = \binom{2n}{n} \left\{ h_n^{(n+1)} - \binom{2n}{n} H_n \right\}$
(7.2)	$\sum_{k=1}^n \binom{n}{k} \binom{r}{k} H_k = \binom{r+n}{n} H_n - h_n^{(r+1)}$
(7.9)	$\sum_{k=1}^n (-1)^k \binom{n}{k} \binom{2k}{k}^{-1} \frac{2^{2k}}{(2k+1)} H_k = \binom{2n}{n}^{-1} \frac{2^{2n}}{(2n+1)} h_n^{(\frac{1}{2})}$
(7.13)	$\sum_{k=1}^n (-1)^{k-1} \binom{n}{k} \binom{2n-k}{n} H_k = \sum_{k=1}^n \binom{n}{k} \frac{2}{k}$
(7.15)	$\sum_{k=1}^n \binom{n}{k} \binom{r}{k} \left\{ (H_k)^2 + H_k^{(2)} \right\} = \binom{r+n}{n} \left\{ H_n^{(2)} - H_{n+r}^{(2)} + H_r^{(2)} \right\} + \left(\binom{r+n}{n} H_n - h_n^{(r+1)} \right) \left\{ H_n - H_{n+r} + H_r \right\}$
(12.9)	$\sum_{k=0}^n \binom{n}{k} \binom{x+k}{k}^{-1} \binom{x+k+1+n}{n}^{-1} \times \left\{ \frac{x+2k+1}{x+k+1} \left(H_k - \binom{x+k+1+n}{n}^{-1} h_n^{(x+k+2)} \right) - \frac{k}{(x+k+1)^2} \right\} = 0$
(12.9)	$\sum_{k=0}^n \binom{n}{k} \binom{y+k}{k} \binom{y+k+1+n}{n}^{-1} \times \left\{ \frac{y+2k+1}{y+k+1} \left(H_k + \binom{y+k+1+n}{n}^{-1} h_n^{(y+k+2)} \right) - \frac{k}{(y+k+1)^2} \right\} = H_n$
(Z.58)	$H_{2n} = 2^{2n} \binom{2n}{n}^{-1} \left\{ \binom{n-\frac{1}{2}}{n} H_n + h_n^{(\frac{1}{2})} \right\}$

The next table contains generalized version of the equations in the first table, by substitution $\binom{x+k}{k}$ with $\binom{x+r-1+k}{k}$ where $r > 1$.

(1.23)	$\sum_{k=0}^{\infty} \frac{h_k^{(r)}}{2^k} = 2^r \ln 2$
(1.41)	$\sum_{k=1}^n (-1)^{k-1} \binom{n}{k} \frac{k}{(k+r-1)^2} = \binom{r-1+n}{n}^{-2} h_n^{(r)}$

(1.42)	$\sum_{k=1}^n (-1)^{k-1} \binom{n}{k} \binom{k+r-1}{k}^{-2} h_k^{(r)} = \frac{1}{n+r-1}$
(1.44)	$\sum_{k=1}^n (-1)^{k-1} \binom{n+1}{k+1} \binom{r-1+k}{k}^{-2} h_k^{(r)} = \sum_{k=1}^n \frac{k}{(k+r-1)^2}$
(2.16)	$\sum_{k=1}^{\infty} \binom{r-1+k}{k}^{-2} \frac{h_k^{(r)}}{k!} = e \sum_{k=0}^{\infty} \frac{(-1)^k}{(r+k)^2 k!}$
(3.2)	$\sum_{k=1}^n \binom{m+n-k}{m} h_k^{(r)} = h_n^{(m+r+1)}$
(3.36)	$2 \sum_{k=1}^{2n} (-1)^k \binom{r-1+2n-k}{r-1} h_k^{(r)} = h_n^{(r)}$
(3.95)	$\sum_{k=1}^n (-1)^k \binom{2n-2k}{n-k} \binom{2k}{k} \frac{k}{(r-1+k)^2}$ $= 2^{2n} \binom{r-1+n}{n}^{-1} \left\{ h_n^{(r-\frac{1}{2})} - \binom{r-1+n}{n}^{-1} \binom{r-\frac{3}{2}+n}{n} h_n^{(r)} \right\}$
(3.100)	$\sum_{k=1}^n (-1)^k \binom{n+k}{2k} \binom{2k}{k} \frac{k}{(r-1+k)^2}$ $= (-1)^n \binom{r-1+n}{n}^{-1} \left\{ h_n^{(r-n-1)} - \binom{r-2}{n} \binom{r-1+n}{n}^{-1} h_n^{(r)} \right\}$
(3.108)	$\sum_{k=0}^n \left\{ \binom{k+r+m}{m} h_k^{(r)} + \binom{k+r-1}{k} h_m^{(k+r+1)} \right\}$ $= \sum_{k=0}^m \left\{ \binom{k+r+n}{n} h_k^{(r)} + \binom{k+r-1}{k} h_n^{(k+r+1)} \right\}$
(4.3)	$\sum_{k=1}^n (-1)^k \binom{n}{k} (x+r-1)^{k-1} \binom{r-1+k}{k}^{-1} \left\{ k - (x+r-1) \binom{r-1+k}{k}^{-1} h_k^{(r)} \right\}$ $= \sum_{k=1}^n \binom{n}{k} (x-k)^k (1-x+k)^{n-k} \frac{k}{(r-1+k)^2}$
(6.19)	$\sum_{k=0}^n \binom{n}{k} \binom{r}{k} h_{n+r}^{(j-k)} = \binom{n+j-1}{n} h_r^{(j)} + \binom{r+j-1}{r} h_n^{(j)}$
(6.22)	$\sum_{k=1}^n (-1)^k \binom{2n}{k} \binom{2n-k}{n}^2 \frac{k}{(r-1+k)^2}$ $= \binom{2n}{n} \binom{r-1+n}{n}^{-1} \left\{ h_n^{(n+r)} - \binom{2n+r-1}{n} \binom{r-1+n}{n}^{-1} h_n^{(r)} \right\}$
(7.2)	$\sum_{k=1}^n \binom{n}{k} \binom{m}{k} \binom{r-1+k}{k}^{-2} h_k^{(r)}$ $= \binom{r-1+n}{n}^{-1} \left\{ \binom{r-1+m+n}{n} \binom{r-1+n}{n}^{-1} h_n^{(r)} - h_n^{(m+r)} \right\}$
(7.9)	$\sum_{k=1}^n (-1)^k \binom{n}{k} \binom{2k}{k}^{-1} \frac{2^{2k}}{(2k+1)} h_k^{(r)} = \binom{2n}{n}^{-1} \frac{2^{2n}}{(2n+1)} h_n^{(r-\frac{1}{2})}$
(7.13)	$\sum_{k=1}^n (-1)^{k-1} \binom{n}{k} \binom{2n-k}{n} \binom{m-1+k}{k}^{-2} h_k^{(m)} = \sum_{k=1}^n \binom{n}{k}^2 \frac{k}{(k+m-1)^2}$
(7.29)	$\frac{\partial}{\partial j} S_{j+r-1}^n \Big _{j=0} = \frac{\partial}{\partial j} F \left(-2n, \frac{1}{2}, n+j+r; 4 \right) \Big _{j=0}$ $= \sum_{k=0}^{2n} (-1)^{k+1} \binom{2n}{k} \binom{2k}{k} \binom{n+r-1+k}{k}^{-2} h_k^{(n+r)}$
(7.30)	$\frac{\partial}{\partial j} R_{j+r-1}^n \Big _{j=0} = \frac{\partial}{\partial j} F \left(-2n-1, \frac{1}{2}, n+j+r; 4 \right) \Big _{j=0}$

	$= \sum_{k=0}^{2n+1} (-1)^{k+1} \binom{2n+1}{k} \binom{2k}{k} \binom{n+r-1+k}{k}^{-2} h_k^{(n+r)}$
(12.9)	$\sum_{k=0}^n \binom{n}{k} \binom{x+k}{k}^{-1} \binom{x+r+k+n}{n}^{-1} \binom{r-1+k}{k} \left\{ \binom{r-1+k}{k}^{-1} \frac{x+r+2k}{x+r+k} h_k^{(r)} - \binom{x+r+k+n}{n}^{-1} \frac{x+r+2k}{x+r+k} h_n^{(x+k+r+1)} - \frac{k}{(x+r+k)^2} \right\} = 0$
(12.9)	$\sum_{k=0}^n \binom{n}{k} \binom{y+k}{k} \binom{r+y+k+n}{n}^{-1} \binom{r-1+k}{k}^{-1} \left\{ \binom{r-1+k}{k}^{-1} \frac{r+y+2k}{r+y+k} h_k^{(r)} + \binom{r+y+k+n}{n}^{-1} \frac{r+y+2k}{r+y+k} h_n^{(y+k+r+1)} + \frac{k}{(r+y+k)^2} \right\} = h_n^{(r)} \binom{r-1+n}{n}^{-2}$
(Z.58)	$h_{2n}^{(2r-1)} = 2^{2n} \binom{2n}{n}^{-1} \left\{ \binom{n+r-\frac{3}{2}}{n} h_n^{(r)} + \binom{n+r-1}{n} h_n^{(r-\frac{1}{2})} \right\}$

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