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CONFORMAL CURVATURE TENSORS IN A GENERALIZED RIEMANNIAN SPACE IN EISENHART SENSE

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Dedicated to Professor Svetislav Minčić on the occasion of his 90th birthday

In the present paper generalizations of conformal curvature tensor from Riemannian space are given for five independent curvature tensors in generalized Riemannian space (GR_N), i.e. when the basic tensor is non-symmetric.
In earlier works of S. Minčić and M. Zlatanović et al a special case has been investigated, that is the case when in the conformal transformation the torsion remains invariant (equitorsion transformation). In the present paper this condition is not supposed and for that reason the results are more general and new.

1. INTRODUCTION

The use of non-symmetric basic tensor and non-symmetric connection starts to be actual specially in relation with the works of A. Einstein [1] – [7] devoted to the Unified Field Theory (UFT). M. Prvanović [15] and S. Minčić [13] gave geometric interpretation of the torsion, curvature tensors and curvature pseudotensors of non-symmetric connection.

In the sense of the definition of L. P. Eisenhart [4], a generalized Riemannian space (GR_N) is a differentiable manifold endowed with non-symmetric basic tensor

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$(g_{ij} \neq g_{ji})$, where

$$(1) \quad \begin{aligned} g_{ij} &= g_{\underline{ij}} + g_{\underset{\vee}{ij}}, \quad det(g_{\underline{ij}}) = g \neq 0, \\ g_{\underline{ij}} &= \frac{1}{2}(g_{ij} + g_{ji}), \quad g_{\underset{\vee}{ij}} = \frac{1}{2}(g_{ij} - g_{ji}). \end{aligned}$$

For lowering of indices in GR_N one uses $g_{\underline{ij}}$, and for the rising ones $g^{\underline{ij}}$, where

$$(2) \quad g_{\underline{ij}} g^{\underline{ik}} = \delta^k_j, \quad det(g_{\underline{ij}}) \neq 0.$$

Christoffel's symbols in GR_N are given by:

$$(3) \quad \Gamma_{i.jk} = \frac{1}{2}(g_{ji,k} - g_{jk,i} + g_{ik,j}),$$

$$(4) \quad \Gamma^i_{jk} = g^{\underline{ip}} \Gamma_{p.jk} = \frac{1}{2} g^{\underline{ip}} (g_{jp,k} - g_{jk,p} + g_{pk,j}),$$

where , e.g., $g_{ji,k} = \frac{\partial g_{ji}}{\partial x^k}$.

In the works of S. Minčić [10], [11], [14], 12 curvature tensors are obtained in GR_N, and in S. Minčić [12] is proved that five among them are independent, because the others can be expressed by these 5 of 12 mentioned tensors and $R^i{}_{jmn}$, formed by $g_{\underline{ij}}$.

The cited independent curvature tensors in GR_N are according to [12]:

$$(5) \quad \underset{1}{R}{}^i{}_{jmn} = \Gamma^i_{jm,n} - \Gamma^i_{jn,m} + \Gamma^p_{jm}\Gamma^i_{pn} - \Gamma^p_{jn}\Gamma^i_{pm},$$

$$(6) \quad \underset{2}{R}{}^i{}_{jmn} = \Gamma^i_{mj,n} - \Gamma^i_{nj,m} + \Gamma^p_{mj}\Gamma^i_{np} - \Gamma^p_{nj}\Gamma^i_{mp},$$

$$(7) \quad \underset{3}{R}{}^i{}_{jmn} = \Gamma^i_{jm,n} - \Gamma^i_{nj,m} + \Gamma^p_{jm}\Gamma^i_{np} - \Gamma^p_{nj}\Gamma^i_{pm} + \Gamma^p_{nm}(\Gamma^i_{pj} - \Gamma^i_{jp}),$$

$$(8) \quad \underset{4}{R}{}^i{}_{jmn} = \Gamma^i_{jm,n} - \Gamma^i_{nj,m} + \Gamma^p_{jm}\Gamma^i_{np} - \Gamma^p_{nj}\Gamma^i_{pm} + \Gamma^p_{mn}(\Gamma^i_{pj} - \Gamma^i_{jp}),$$

$$(9) \quad \begin{aligned} \underset{5}{R}{}^i{}_{jmn} = \frac{1}{2}(&\Gamma^i_{jm,n} + \Gamma^i_{mj,n} - \Gamma^i_{jn,m} - \Gamma^i_{nj,m} \\ &+ \Gamma^p_{jm}\Gamma^i_{jn} + \Gamma^p_{mj}\Gamma^i_{np} - \Gamma^p_{jn}\Gamma^i_{mp} - \Gamma^p_{nj}\Gamma^i_{pm}). \end{aligned}$$

In [12] the following notations are introduced (omitting the indices on the left sides):

$$\mathcal{A} \equiv \mathcal{A}^i{}_{jmn} = \Gamma^i_{\underset{V}{jm};n}, \quad \mathcal{B} = \Gamma^p_{\underset{V}{jm}}\Gamma^i_{\underset{V}{pn}}, \quad \mathcal{C} = \Gamma^p_{\underset{V}{mn}}\Gamma^i_{\underset{V}{pj}} \tag{10}$$

$$\mathcal{A}' = \Gamma^i_{\underset{V}{jn};m}, \quad \mathcal{B}' = \Gamma^p_{\underset{V}{jn}}\Gamma^i_{\underset{V}{pm}}, \tag{11}$$

where with $;n$ the covariant derivative on x^n with respect to symmetric connection $\Gamma^i_{\underline{jk}}$ is denoted. In $\bar{\Gamma}^i_{\underset{V}{jm};n}$ the covariant derivative $;n$ is with respect of $\bar{\Gamma}^i_{\underline{jn}}$.

By virtue of (10),(11) in [12] the following relations are proved

$$\underset{1}{R} = R + \mathcal{A} - \mathcal{A}' + \mathcal{B} - \mathcal{B}' \tag{12}$$

$$\underset{2}{R} = R - \mathcal{A} + \mathcal{A}' + \mathcal{B} - \mathcal{B}' \tag{13}$$

$$\underset{3}{R} = R + \mathcal{A} + \mathcal{A}' - \mathcal{B} + \mathcal{B}' - 2\mathcal{C} \tag{14}$$

$$\underset{4}{R} = R + \mathcal{A} + \mathcal{A}' - \mathcal{B} + \mathcal{B}' + 2\mathcal{C} \tag{15}$$

$$\underset{5}{R} = R + \mathcal{B} + \mathcal{B}'. \tag{16}$$

Conformally flat spaces satisfying certain conditions were considered by M. Prvanović, U. C. De, S. Bandhopadhyay at [16]. A. Velimirović at [19] has considered conformal equitorsion and concircular transformations.

Remark that at S. Minčić [12] $\underset{5}{R}$ is denoted with $\underset{2}{\widetilde{R}}$.

Analogously to the definition for conformal transformation in R_N, we give corresponding definition in GR_N.

Definition 1.1. *Conformal transformation in GR_N is transformation where basic tensor g_{ij} is changed by help of the next rule (K. Yano [20])*

$$\bar{g}_{ij}(x) = \rho^2(x) g_{ij}(x), \quad g_{ij} \neq g_{ji}, \tag{17}$$

where $\rho(x) = \rho(x^1, ..., x^N)$ is some differentiable function in GR_N.

Geometric objects are observed in common coordinate system, that we see, e.g., from (17). In that case by overline $^{-}$ the transformed object is denoted.

From $ds^2 = g_{ij}dx^i dx^j$ it follows that

$$\bar{ds}^2 = \rho^2 ds^2, \quad \bar{ds} = \rho ds. \tag{18}$$

Denoting

$$(ln\rho)_{,i} = \frac{\partial(ln\rho)}{\partial x^i} = \frac{1}{\rho}\rho_{,i} = \rho_i, \tag{19}$$

based on (3), we get

$$\bar{\Gamma}_{i.jk} = \rho^2(\Gamma_{i.jk} + g_{\underset{V}{ji}}\rho_k - g_{\underset{V}{jk}}\rho_i + g_{\underset{V}{ik}}\rho_j), \tag{20}$$

and $\bar{\Gamma}^i_{jk} = \bar{g}^{\underline{ip}}\bar{\Gamma}_{p.jk}$ is obtained by using the fact that it is

$$\bar{g}^{\underline{ij}} = \rho^{-2}g^{\underline{ij}} \quad . \tag{21}$$

2. CONFORMAL TENSOR OF THE 1^{ST} KIND IN GR_N

From the equations (omitted indices):

$$\underset{1}{R} = R + \mathcal{A} - \mathcal{A}' + \mathcal{B} - \mathcal{B}', \tag{22}$$

$$\underset{1}{\bar{R}} = \bar{R} + \bar{\mathcal{A}} - \bar{\mathcal{A}}' + \bar{\mathcal{B}} - \bar{\mathcal{B}}', \tag{23}$$

we obtain

$$\underset{1}{\bar{R}} - \bar{\mathcal{A}} + \bar{\mathcal{A}}' - \bar{\mathcal{B}} + \bar{\mathcal{B}}' = \underset{1}{R} - \mathcal{A} + \mathcal{A}' - \mathcal{B} + \mathcal{B}' + \bar{R} - R. \tag{24}$$

Because in conformal transformation in GR_N it is

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j\rho_k + \delta^i_k\rho_j - \rho^i g_{\underline{jk}} + g^{\underline{ip}}(g_{\underset{V}{jp}}\rho_k - g_{\underset{V}{jk}}\rho_p + g_{\underset{V}{pk}}\rho_j), \tag{25}$$

we have

$$\begin{aligned} &a) \quad \bar{\Gamma}^i_{\underset{V}{jk}} = \Gamma^i_{\underset{V}{jk}} + \xi^i_{jk}, \\ &b) \quad \xi^i_{jk} = g^{\underline{ip}}(g_{\underset{V}{jp}}\rho_k - g_{\underset{V}{jk}}\rho_p + g_{\underset{V}{pk}}\rho_j) = -\xi^i_{kj} \end{aligned} \tag{26}$$

(27) $$\bar{\Gamma}^i_{\underline{jk}} = \Gamma^i_{\underline{jk}} + \delta^i_j \rho_k + \delta^i_k \rho_j - \rho^i g_{\underline{jk}}.$$

In the works of K. Yano [20], Part I, and N. Sinyukov [**17**] the conformal transformation of curvature tensor in Riemannian space R_N is considered and in Stanković et al [18], Zlatanović et al [21], A. Velimirović [19] "equitorsion" transformation is applied "because of non-possibility to find a generalization of conformal tensor in general case". In the following, we prove that it is possible. From (26) we have

$$\bar{T}^i_{jk} - T^i_{jk} = 2\xi^i_{jk}.$$

Definition 2.2. *Equitorsion deformation in GR_N is such one where it is*

(28) $$\bar{T}^i_{jk} = T^i_{jk} \Leftrightarrow \xi^i_{jk} = 0.$$

In the present work we do not use (28)*, that is we consider* **general conformal transformation in GR_N.**

In the Riemannian space R_N (Mikeš et al. [9], Eisenhart L. P. [8], Sinyukov, N.S. [17], Yano K. [20]) we have

(29) $$\bar{R}^i{}_{jmn} = R^i{}_{jmn} + \delta^i_m \rho_{jn} - \delta^i_n \rho_{jm} + \rho^i{}_m g_{\underline{jn}} - \rho^i{}_n g_{\underline{jm}},$$

where [20]

(30) $$\rho_{jm} = \rho_{j;m} - \rho_j \rho_m + \frac{1}{2}\rho_p \rho^p g_{\underline{jm}},$$

and by (;) the convariant derivative in relation with symmetric connection $\Gamma^i_{\underline{jk}}$ is denoted, while we have $\rho^p = g^{\underline{ip}} \rho_i$.

Using the value, given in (5), from (24),(29) we obtain

(31) $$\begin{aligned} &\underset{1}{\bar{R}}{}^i_{jmn} - \bar{\Gamma}^i_{\underset{V}{jm;n}} + \bar{\Gamma}^i_{\underset{V}{jn;m}} - \bar{\Gamma}^p_{\underset{V}{jm}} \bar{\Gamma}^i_{\underset{V}{pn}} + \bar{\Gamma}^p_{\underset{V}{jn}} \bar{\Gamma}^i_{\underset{V}{pm}} \\ &= \underset{1}{R}{}^i_{jmn} - \Gamma^i_{\underset{V}{jm;n}} + \Gamma^i_{\underset{V}{jn;m}} - \Gamma^p_{\underset{V}{jm}} \Gamma^i_{\underset{V}{pm}} + \Gamma^p_{\underset{V}{jn}} \Gamma^i_{\underset{V}{pm}} \\ &+ \delta^i_m \rho_{jn} - \delta^i_n \rho_{jm} + \rho^i{}_m g_{\underline{jn}} - \rho^i{}_n g_{\underline{jm}}. \end{aligned}$$

Contracting with $i = n$:

(32) $$\begin{aligned} &\underset{1}{\bar{R}}{}_{jm} - \bar{\Gamma}^i_{\underset{V}{jm;i}} + \bar{\Gamma}^i_{\underset{V}{ji;m}} - \bar{\Gamma}^p_{\underset{V}{jm}} \bar{\Gamma}^i_{\underset{V}{pi}} + \bar{\Gamma}^p_{\underset{V}{ji}} \bar{\Gamma}^i_{\underset{V}{pm}} \\ &= \underset{1}{R}{}_{jm} - \Gamma^i_{\underset{V}{jm;i}} + \Gamma^i_{\underset{V}{ji;m}} - \Gamma^p_{\underset{V}{jm}} \Gamma^i_{\underset{V}{pi}} + \Gamma^p_{\underset{V}{ji}} \Gamma^i_{\underset{V}{pm}} \\ &+ \rho_{jm} - N\rho_{jm} + \rho_{jm} - \rho^i{}_i g_{\underline{jm}}. \end{aligned}$$

Because in GR_N

$$\Gamma^i_{\underset{V}{ji}} = \bar{\Gamma}^i_{\underset{V}{ji}} = 0, \tag{33}$$

it follows that

$$\begin{aligned} &\underset{1}{\bar{R}}_{jm} - \bar{\Gamma}^i_{\underset{V}{jm;i}} + \bar{\Gamma}^p_{\underset{V}{ji}}\bar{\Gamma}^i_{\underset{V}{pm}} \\ &= \underset{1}{R}_{jm} - \Gamma^i_{\underset{V}{jm;i}} + \Gamma^p_{\underset{V}{ji}}\Gamma^i_{\underset{V}{pm}} + (2-N)\rho_{jm} - \rho^p{}_p g_{\underline{jm}}. \end{aligned} \tag{34}$$

Let us effect the composition of corresponding sides with

$$\rho^2 \bar{g}^{\underline{jm}} = g^{\underline{jm}} \tag{35}$$

and we obtain

$$\rho^2 \underset{1}{\bar{R}} - 0 + \rho^2 \bar{g}^{\underline{jm}} \bar{\Gamma}^p_{\underset{V}{ji}}\bar{\Gamma}^i_{\underset{V}{pm}} = \underset{1}{R} - 0 + g^{\underline{jm}}\Gamma^p_{\underset{V}{ji}}\Gamma^i_{\underset{V}{pm}} + (2-N)\rho^p{}_p - N\rho^p{}_p, \tag{36}$$

wherefrom, changing the repeated indices: $j \to s$, $i \to r$, $m \to t$.

$$\underset{1}{\rho}{}^p{}_p \equiv \rho^p{}_p = \frac{1}{2(N-1)}(\underset{1}{R} - \rho^2 \underset{1}{\bar{R}} + g^{\underline{st}}\Gamma^p_{\underset{V}{sr}}\Gamma^r_{\underset{V}{pt}} - \rho^2 \bar{g}^{\underline{st}}\bar{\Gamma}^p_{\underset{V}{sr}}\bar{\Gamma}^r_{\underset{V}{pt}}). \tag{37}$$

This value we substitute in (34), from where

$$\begin{aligned} \underset{1}{\rho}_{jm} \equiv \rho_{jm} = \frac{1}{N-2}[&\underset{1}{R}_{jm} - \Gamma^r_{\underset{V}{jm;r}} + \Gamma^p_{\underset{V}{rj}}\Gamma^r_{\underset{V}{pm}} - \underset{1}{\bar{R}}_{jm} + \bar{\Gamma}^r_{\underset{V}{jm;r}} - \bar{\Gamma}^p_{\underset{V}{rj}}\bar{\Gamma}^r_{\underset{V}{pm}} \\ &+ \frac{1}{2(N-1)}(\rho^2 \underset{1}{\bar{R}} - \rho^2 \bar{g}^{\underline{st}}\bar{\Gamma}^p_{\underset{V}{rs}}\bar{\Gamma}^r_{\underset{V}{pt}} - \underset{1}{R} + g^{\underline{st}}\Gamma^p_{\underset{V}{rs}}\Gamma^r_{\underset{V}{pt}})g_{\underline{jm}}]. \end{aligned} \tag{38}$$

Introducing the notation $[m,n]$ for alternation on indices m,n, for example the equation (31) can be written in the form:

$$\begin{aligned} &\underset{1}{\bar{R}}{}^i{}_{jmn} - [\bar{\Gamma}^i_{\underset{V}{jm;n}} + \bar{\Gamma}^p_{\underset{V}{jm}}\bar{\Gamma}^i_{\underset{V}{pn}}]_{[m,n]} \\ &= \underset{1}{R}{}^i{}_{jmn} - [\Gamma^i_{\underset{V}{jm;n}} + \Gamma^p_{\underset{V}{jm}}\Gamma^i_{\underset{V}{pn}}]_{[m,n]} + [\delta^i_m \rho_{jn} + \rho^i{}_m g_{\underline{jn}}]_{[m,n]}. \end{aligned} \tag{39}$$

With respect of (38), we put into (39):

$$
\begin{aligned}
&\bar{\underset{1}{R}}{}^i_{jmn} - [\bar{\Gamma}^i_{\underset{V}{jm;n}} + \bar{\Gamma}^p_{\underset{V}{jm}}\bar{\Gamma}^i_{\underset{V}{pn}}]_{[m,n]} \\
&= \underset{1}{R}{}^i{}_{jmn} - [\Gamma^i_{\underset{V}{jm;n}} + \Gamma^p_{\underset{V}{jm}}\Gamma^i_{\underset{V}{pn}}]_{[m,n]} + \frac{1}{N-2}\{\delta^i_m[\underset{1}{R}_{jn} - \Gamma^r_{\underset{V}{jn;r}} - \Gamma^p_{\underset{V}{jr}}\Gamma^r_{\underset{V}{pn}} \\
&- \bar{\underset{1}{R}}_{jn} + \bar{\Gamma}^r_{\underset{V}{jn;r}} + \bar{\Gamma}^p_{\underset{V}{jr}}\bar{\Gamma}^r_{\underset{V}{pn}} + \\
&\frac{1}{2(N-1)}(\rho^2\bar{\underset{1}{R}} - \rho^2\bar{g}^{\underline{st}}\bar{\Gamma}^p_{\underset{V}{rs}}\bar{\Gamma}^r_{\underset{V}{pt}} - \underset{1}{R} + g^{\underline{st}}\Gamma^p_{\underset{V}{rs}}\Gamma^r_{\underset{V}{pt}})g_{\underline{jn}}]\}_{[m,n]} \\
&+ \frac{1}{N-2}\{g_{\underline{jn}}[\underset{1}{R}{}^i{}_m - g^{\underline{is}}\Gamma^r_{\underset{V}{sm;r}} - g^{\underline{is}}\Gamma^p_{\underset{V}{rs}}\Gamma^r_{\underset{V}{pm}} - \rho^2\bar{g}^{\underline{is}}(\bar{\underset{1}{R}}_{sm} - \bar{\Gamma}^r_{\underset{V}{sm;r}} - \bar{\Gamma}^p_{\underset{V}{rs}}\bar{\Gamma}^r_{\underset{V}{pm}}) \\
&+ \frac{1}{2(N-1)}(\rho^2\bar{\underset{1}{R}} - \rho^2\bar{g}^{\underline{st}}\bar{\Gamma}^p_{\underset{V}{rs}}\bar{\Gamma}^r_{\underset{V}{pt}} - \underset{1}{R} + g^{\underline{st}}\Gamma^p_{\underset{V}{rs}}\Gamma^r_{\underset{V}{pt}})\delta^i_m]\}_{[m,n]}.
\end{aligned} \tag{40}
$$

From (40) we conclude that the next theorem is valid:

Theorem 2.1. *The tensor*

$$
\begin{aligned}
\underset{1}{C}{}^i{}_{jmn} &= \underset{1}{R}{}^i{}_{jmn} - [\Gamma^i_{\underset{V}{jm;n}} + \Gamma^p_{\underset{V}{jm}}\Gamma^i_{\underset{V}{pn}}]_{[m,n]} \\
&+ \frac{1}{N-2}[\delta^i_m(\underset{1}{R}_{jn} - \Gamma^r_{\underset{V}{jn;r}} - \Gamma^p_{\underset{V}{jr}}\Gamma^r_{\underset{V}{pn}}) \\
&+ g_{\underline{jn}}(\underset{1}{R}{}^i{}_m - g^{\underline{is}}\Gamma^r_{\underset{V}{sm;r}} - g^{\underline{is}}\Gamma^p_{\underset{V}{rs}}\Gamma^r_{\underset{V}{pm}})]_{[m,n]} \\
&- \frac{1}{(N-2)(N-1)}(\underset{1}{R} - g^{\underline{st}}\Gamma^p_{\underset{V}{rs}}\Gamma^r_{\underset{V}{pt}})(\delta^i_m g_{\underline{jn}} - \delta^i_n g_{\underline{jm}})
\end{aligned} \tag{41}
$$

is an invariant of the conformal transformation in GR_N, that is

$$
\bar{\underset{1}{C}}{}^i{}_{jmn} = \underset{1}{C}{}^i{}_{jmn}. \tag{42}
$$

Definition 2.3. *The tensor $\underset{1}{C}{}^i{}_{jmn}$ we call conformal tensor of the 1^{st} kind in GR_N.*

3. CONFORMAL TENSOR OF THE 2^{ND} KIND IN GR_N

Analogously to the previous case, based on (13), now we have

$$
\bar{\underset{2}{R}} + \bar{\mathcal{A}} - \bar{\mathcal{A}}' - \bar{\mathcal{B}} + \bar{\mathcal{B}}' = \underset{2}{R} + \mathcal{A} - \mathcal{A}' - \mathcal{B} + \mathcal{B}' + \bar{R} - R. \tag{43}
$$

From (43), (29) we obtain

$$
\begin{aligned}
&\underset{2}{\bar{R}}{}^i{}_{jmn} + \bar{\Gamma}^i_{\underset{\vee}{jm};n} - \bar{\Gamma}^i_{\underset{\vee}{jn};m} - \bar{\Gamma}^p_{\underset{\vee}{jm}}\bar{\Gamma}^i_{\underset{\vee}{pn}} + \bar{\Gamma}^p_{\underset{\vee}{jn}}\bar{\Gamma}^i_{\underset{\vee}{pm}} \\
&= \underset{2}{R}{}^i{}_{jmn} + \Gamma^i_{\underset{\vee}{jm};n} - \Gamma^i_{\underset{\vee}{jn};m} - \Gamma^p_{\underset{\vee}{jm}}\Gamma^i_{\underset{\vee}{pn}} + \Gamma^p_{\underset{\vee}{jn}}\Gamma^i_{\underset{\vee}{pm}} \\
&+ \delta^i_m\rho_{jn} - \delta^i_n\rho_{jm} + \rho^i{}_m g_{\underline{jn}} - \rho^i{}_n g_{\underline{jm}}.
\end{aligned} \tag{44}
$$

The contraction with $i = n$ gives:

$$
\begin{aligned}
\underset{2}{\bar{R}}_{jm} + \bar{\Gamma}^i_{\underset{\vee}{jm};i} + \bar{\Gamma}^p_{\underset{\vee}{ji}}\bar{\Gamma}^i_{\underset{\vee}{pm}} &= \underset{2}{R}_{jm} + \Gamma^i_{jm;i} + \Gamma^p_{\underset{\vee}{ji}}\Gamma^i_{\underset{\vee}{pm}} \\
&+ (2-N)\rho_{jm} - \rho^p{}_p g_{\underline{jm}}.
\end{aligned} \tag{45}
$$

From here, componding with (35), it follows that

$$
\underset{2}{\rho}{}^p{}_p \equiv \rho^p{}_p = \frac{1}{2(N-1)}(\underset{2}{R} - \rho^2\underset{2}{\bar{R}} + g^{\underline{st}}\Gamma^p_{\underset{\vee}{sr}}\Gamma^r_{\underset{\vee}{pt}} - \rho^2\bar{g}^{\underline{st}}\bar{\Gamma}^p_{\underset{\vee}{sr}}\bar{\Gamma}^r_{\underset{\vee}{pt}}). \tag{46}
$$

This value we put into (45), and then one gets

$$
\begin{aligned}
&\underset{2}{\rho}_{jm} \equiv \rho_{jm} = \frac{1}{N-2}[\underset{2}{R}_{jm} + \Gamma^r_{\underset{\vee}{jm};r} + \Gamma^p_{\underset{\vee}{jr}}\Gamma^r_{\underset{\vee}{pm}} - \underset{2}{\bar{R}}_{jm} - \bar{\Gamma}^r_{\underset{\vee}{jm};r} - \bar{\Gamma}^p_{\underset{\vee}{jr}}\bar{\Gamma}^r_{\underset{\vee}{pm}} \\
&+ \frac{1}{2(N-1)}(\rho^2\underset{2}{\bar{R}} - \rho^2\bar{g}^{\underline{st}}\bar{\Gamma}^p_{\underset{\vee}{rs}}\bar{\Gamma}^r_{\underset{\vee}{pt}} - \underset{2}{R} + g^{\underline{st}}\Gamma^p_{\underset{\vee}{rs}}\Gamma^r_{\underset{\vee}{pt}})g_{\underline{jm}}].
\end{aligned} \tag{47}
$$

By further procedure, as in the previous case, one proves that the next theorem is valid.

Theorem 3.2. *The tensor*

$$
\begin{aligned}
&\underset{2}{C}{}^i{}_{jmn} = \underset{2}{R}{}^i{}_{jmn} - [\Gamma^i_{\underset{\vee}{jm};n} - \Gamma^p_{\underset{\vee}{jm}}\Gamma^i_{\underset{\vee}{pn}}]_{[m,n]} \\
&+ \frac{1}{N-2}[\delta^i_m(\underset{2}{R}_{jn} + \Gamma^r_{\underset{\vee}{jn};r} - \Gamma^p_{\underset{\vee}{jr}}\Gamma^r_{\underset{\vee}{pn}}) \\
&+ g_{\underline{jn}}(\underset{2}{R}{}^i{}_m + g^{\underline{is}}\Gamma^r_{\underset{\vee}{sm};r} - g^{\underline{is}}\Gamma^p_{\underset{\vee}{rs}}\Gamma^r_{\underset{\vee}{pm}})]_{[m,n]} \\
&- \frac{1}{(N-2)(N-1)}(\underset{2}{R} - g^{\underline{st}}\Gamma^p_{\underset{\vee}{rs}}\Gamma^r_{\underset{\vee}{pt}})(\delta^i_m g_{\underline{jn}} - \delta^i_n g_{\underline{jm}})
\end{aligned} \tag{48}
$$

is an invariant of the conformal transformation in GR_N. So we have

$$
\underset{2}{\bar{C}}{}^i{}_{jmn} = \underset{2}{C}{}^i{}_{jmn}. \tag{49}
$$

Definition 3.4. *The tensor $\underset{2}{C}{}^i{}_{jmn}$ we call a conformal tensor of the 2^{nd} kind in GR_N.*

4. CONFORMAL TENSOR OF THE 3^{RD} KIND IN GR_N

By virtue of (9), now we have

$$\underset{3}{\bar{R}} - \bar{\mathcal{A}} - \bar{\mathcal{A}}' + \bar{\mathcal{B}} - \bar{\mathcal{B}}' + 2\bar{\mathcal{C}} = \underset{3}{R} - \mathcal{A} - \mathcal{A}' + \mathcal{B} - \mathcal{B}' + 2\mathcal{C} + \bar{R} - R. \tag{50}$$

and, using (29) it follows that

$$\begin{aligned} &\underset{3}{\bar{R}}{}^i{}_{jmn} - \bar{\Gamma}^i_{\underset{V}{jm};n} - \bar{\Gamma}^i_{\underset{V}{jn};m} + \bar{\Gamma}^p_{\underset{V}{jm}}\bar{\Gamma}^i_{\underset{V}{pn}} - \bar{\Gamma}^p_{\underset{V}{jn}}\bar{\Gamma}^i_{\underset{V}{pm}} + 2\bar{\Gamma}^p_{\underset{V}{mn}}\bar{\Gamma}^i_{\underset{V}{pj}} \\ &= \underset{3}{R}{}^i{}_{jmn} - \Gamma^i_{\underset{V}{jm};n} - \Gamma^i_{\underset{V}{jn};m} + \Gamma^p_{\underset{V}{jm}}\Gamma^i_{\underset{V}{pn}} - \Gamma^p_{\underset{V}{jn}}\Gamma^i_{\underset{V}{pm}} + 2\Gamma^p_{\underset{V}{mn}}\Gamma^i_{\underset{V}{pj}} \\ &+ \delta^i_m\rho_{jn} - \delta^i_n\rho_{jm} + \rho^i{}_m g_{\underline{jn}} - \rho^i{}_n g_{\underline{jm}}, \end{aligned} \tag{51}$$

and contraction with $i = n$, we obtain:

$$\begin{aligned} \underset{3}{\bar{R}}_{jm} - \bar{\Gamma}^i_{\underset{V}{jm};i} + \bar{\Gamma}^p_{\underset{V}{mi}}\bar{\Gamma}^i_{\underset{V}{pj}} &= \underset{3}{R}_{jm} - \Gamma^i_{\underset{V}{jm};i} + \Gamma^p_{\underset{V}{mi}}\Gamma^i_{\underset{V}{pj}} \\ &+ (2-N)\rho_{jm} - \rho^p{}_p g_{\underline{jm}}. \end{aligned} \tag{52}$$

From here, componding by (35):

$$\underset{3}{\rho}{}^p{}_p \equiv \rho^p{}_p = \frac{1}{2(N-1)}(\underset{3}{R} - \rho^2\underset{3}{\bar{R}} + g^{\underline{st}}\Gamma^p_{\underset{V}{sr}}\Gamma^r_{\underset{V}{pt}} - \rho^2\bar{g}^{\underline{st}}\bar{\Gamma}^p_{\underset{V}{sr}}\bar{\Gamma}^r_{\underset{V}{pt}}). \tag{53}$$

This value we put into (52), from where it is obtained

$$\begin{aligned} \underset{3}{\rho}_{jm} \equiv \rho_{jm} = &\frac{1}{N-2}[\underset{3}{R}_{jm} - \Gamma^r_{\underset{V}{jm};r} + \Gamma^p_{\underset{V}{jr}}\Gamma^r_{\underset{V}{pm}} - \underset{3}{\bar{R}}_{jm} + \bar{\Gamma}^r_{\underset{V}{jm};r} - \bar{\Gamma}^p_{\underset{V}{jr}}\bar{\Gamma}^r_{\underset{V}{pm}} \\ &+ \frac{1}{2(N-1)}(\rho^2\underset{3}{\bar{R}} - \rho^2\bar{g}^{\underline{st}}\bar{\Gamma}^p_{\underset{V}{rs}}\bar{\Gamma}^r_{\underset{V}{pt}} - \underset{3}{R} + g^{\underline{st}}\Gamma^p_{\underset{V}{rs}}\Gamma^r_{\underset{V}{pt}})g_{\underline{jm}}]. \end{aligned} \tag{54}$$

Further, the equation (51) we write in the form ((m,n) means a symmetrization, and $[m,n]$ an alternation on m, n, without division):

$$\begin{aligned} &\underset{3}{R}{}^i{}_{jmn} - \bar{\Gamma}^i_{\underset{V}{jm};n(m,n)} + \bar{\Gamma}^p_{\underset{V}{jm}}\bar{\Gamma}^i_{\underset{V}{pn}[m,n]} + 2\bar{\Gamma}^p_{\underset{V}{mn}}\bar{\Gamma}^i_{\underset{V}{pj}} \\ &= \underset{3}{R}{}^i{}_{jmn} - \Gamma^i_{\underset{V}{jm};n(m,n)} + \Gamma^p_{\underset{V}{jm}}\Gamma^i_{\underset{V}{pn}[m,n]} + 2\Gamma^p_{\underset{V}{mn}}\Gamma^i_{\underset{V}{pj}} \\ &+ (\delta^i_m\underset{3}{\rho}_{jn} + \underset{3}{\rho}{}^i{}_m g_{\underline{jn}}) \end{aligned} \tag{55}$$

and then exchange in this equation by virtue of (54).

In this manner, we have proved:

Theorem 4.3. *The tensor*

$$
\begin{aligned}
\underset{3}{C}^i{}_{jmn} &= \underset{3}{R}^i{}_{jmn} - \underset{\vee}{\Gamma}^i_{jm;n(m,n)} + \underset{\vee}{\Gamma}^p_{jm}\underset{\vee}{\Gamma}^i_{pn[m,n]} + 2\underset{\vee}{\Gamma}^p_{mn}\underset{\vee}{\Gamma}^i_{pj}\\
&+ \frac{1}{N-2}[\delta^i_m(\underset{3}{R}_{jn} - \underset{\vee}{\Gamma}^r_{jn;r} + \underset{\vee}{\Gamma}^p_{jr}\underset{\vee}{\Gamma}^r_{pn})\\
&+ g_{\underline{jn}}(\underset{3}{R}^i{}_m - g^{\underline{is}}\underset{\vee}{\Gamma}^r_{sm;r} + g^{\underline{is}}\underset{\vee}{\Gamma}^p_{sr}\underset{\vee}{\Gamma}^r_{pm})]_{[m,n]}\\
&- \frac{1}{(N-2)(N-1)}(\underset{3}{R} - g^{\underline{st}}\underset{\vee}{\Gamma}^p_{rs}\underset{\vee}{\Gamma}^r_{pt})(\delta^i_m g_{\underline{jn}} - \delta^i_n g_{\underline{jm}})
\end{aligned}
\tag{56}
$$

is an invariant of the conformal transformation in GR_N, i.e. we have

$$
\underset{3}{\bar{C}}^i{}_{jmn} = \underset{3}{C}^i{}_{jmn}. \tag{57}
$$

Definition 4.5. *The tensor $\underset{3}{C}^i{}_{jmn}$ we call a conformal tensor of the 3^{rd} kind in GR_N.*

5. CONFORMAL TENSOR OF THE 4^{TH} KIND IN GR_N

Using (10), by procedure as like in the previous case, the next theorem is proved.

Theorem 5.4. *The tensor*

$$
\begin{aligned}
\underset{4}{C}^i{}_{jmn} &= \underset{4}{R}^i{}_{jmn} - \underset{\vee}{\Gamma}^i_{jm;n(m,n)} + \underset{\vee}{\Gamma}^p_{jm}\underset{\vee}{\Gamma}^i_{pn[m,n]} - 2\underset{\vee}{\Gamma}^p_{mn}\underset{\vee}{\Gamma}^i_{pj}\\
&+ \frac{1}{N-2}[\delta^i_m(\underset{4}{R}_{jn} - \underset{\vee}{\Gamma}^r_{jn;r} + \underset{\vee}{\Gamma}^p_{jr}\underset{\vee}{\Gamma}^r_{pn})\\
&+ g_{\underline{jn}}(\underset{4}{R}^i{}_m - g^{\underline{is}}\underset{\vee}{\Gamma}^r_{sm;r} + g^{\underline{is}}\underset{\vee}{\Gamma}^p_{sr}\underset{\vee}{\Gamma}^r_{pm})]_{[m,n]}\\
&- \frac{1}{(N-2)(N-1)}(\underset{4}{R} - g^{\underline{st}}\underset{\vee}{\Gamma}^p_{rs}\underset{\vee}{\Gamma}^r_{pt})(\delta^i_m g_{\underline{jn}} - \delta^i_n g_{\underline{jm}})
\end{aligned}
\tag{58}
$$

is an invariant of the conformal transformation in GR_N, i.e. we have

$$
\underset{4}{\bar{C}}^i{}_{jmn} = \underset{4}{C}^i{}_{jmn}. \tag{59}
$$

Definition 5.6. *The tensor $\underset{4}{C}^i{}_{jmn}$ we call a* **conformal tensor of the 4^{th} kind in** *GR_N.*

6. CONFORMAL TENSOR OF THE 5^{TH} KIND IN GR_N

By the procedure from previous cases, on the base of (11),(5),(6), we obtain

$$\begin{aligned}(60)\quad &\underset{5}{\bar R}{}^i{}_{jmn} - \bar\Gamma^p_{\underset{V}{jm}}\bar\Gamma^p_{\underset{V}{pn}} - \bar\Gamma^p_{\underset{V}{jn}}\Gamma^i_{\underset{V}{pm}}\\ &= \underset{5}{R}{}^i{}_{jmn} - \Gamma^p_{\underset{V}{jm}}\Gamma^i_{\underset{V}{pn}} - -\Gamma^p_{jn}\Gamma^i_{pm} \quad +\delta^i_m\rho_{jn} - \delta^i_n\rho_{jm} + \rho^i{}_m g_{\underline{jn}} - \rho^i{}_n g_{\underline{jm}}.\end{aligned}$$

Contracting by $i = n$, we have:

$$(61)\qquad \underset{5}{\bar R}_{jm} - \bar\Gamma^p_{\underset{V}{ji}}\bar\Gamma^i_{\underset{V}{pm}} = \underset{5}{R}_{jm} - \Gamma^p_{\underset{V}{ji}} + \Gamma^i_{\underset{V}{pm}} + (2-N)\rho_{jm} - \rho^p{}_p g_{\underline{jm}}.$$

Componding (61) with (35), it follows that

$$(62)\qquad \underset{5}{\rho}{}^p{}_p \equiv \rho^p{}_p = \frac{1}{2(N-1)}(\underset{5}{R} - \rho^2\underset{5}{\bar R} - g^{\underline{st}}\Gamma^p_{\underset{V}{sr}}\Gamma^r_{\underset{V}{pt}} + \rho^2\bar g^{\underline{st}}\bar\Gamma^p_{\underset{V}{sr}}\bar\Gamma^r_{\underset{V}{pt}}).$$

Putting this value into (62), it is obtained

$$\begin{aligned}(63)\quad &\underset{5}{\rho}_{jm} \equiv \rho_{jm} = \frac{1}{N-2}[\underset{5}{R}_{jm} - \Gamma^p_{\underset{V}{jr}}\Gamma^r_{\underset{V}{pm}} - \underset{5}{\bar R}_{jm} + \bar\Gamma^p_{\underset{V}{jr}}\bar\Gamma^r_{\underset{V}{pm}}\\ &+ \frac{1}{2(N-1)}(\rho^2\underset{5}{\bar R} - \rho^2\bar g^{\underline{st}}\bar\Gamma^p_{\underset{V}{sr}}\bar\Gamma^r_{\underset{V}{pt}} - \underset{5}{R} + g^{\underline{st}}\Gamma^p_{\underset{V}{sr}}\Gamma^r_{\underset{V}{pt}})g_{\underline{jm}}]\end{aligned}$$

Analogously to previous cases, we prove that the next theorem is valid.

Theorem 6.5. *The tensor*

$$\begin{aligned}(64)\quad &\underset{5}{C}{}^i{}_{jmn} = \underset{5}{R}{}^i{}_{jmn} - \Gamma^p_{\underset{V}{jm}}\Gamma^i_{\underset{V}{pn}} - \Gamma^p_{\underset{V}{jn}}\Gamma^i_{\underset{V}{pm}}\\ &+ \frac{1}{N-2}[\delta^i_m(\underset{5}{R}_{jn} - \Gamma^p_{\underset{V}{jr}} + \Gamma^r_{\underset{V}{pn}}) + g_{\underline{jn}}(\underset{5}{R}{}^i{}_m - g^{\underline{is}}\Gamma^p_{\underset{V}{rs}}\Gamma^r_{\underset{V}{pm}})]_{[m,n]}\\ &- \frac{1}{(N-2)(N-1)}(\underset{5}{R} - g^{\underline{st}}\Gamma^p_{\underset{V}{rs}}\Gamma^r_{\underset{V}{pt}})(\delta^i_m g_{\underline{jn}} - \delta^i_n g_{\underline{jm}})\end{aligned}$$

is an invariant of the conformal transformation in GR_N, that is

$$(65)\qquad \underset{5}{\bar C}{}^i{}_{jmn} = \underset{5}{C}{}^i{}_{jmn}.$$

Definition 6.7. *The tensor $\underset{5}{C}{}^i{}_{jmn}$ is called conformal tensor of the 5^{th} kind in GR_N.*

Conclusion. All conformal tensors $\underset{\theta}{C}{}^i{}_{jmn}, (\theta = 1, ..., 5)$ in GR_N, that are obtained in the present work, in the Riemannian space R_N(i.e when $g_{ij} = g_{ji}$) reduce to known conformal tensor $C^i{}_{jmn}$. In GR_N are obtained the tensors $\underset{\theta}{C}{}^i{}_{jmn}$ from other authors, Stanković et al [18], Zlatanović et al [21], in the case of equitorsion transformation, i.e. if $\bar{T}^i_{jk} = T^i_{jk}$. That means that in the present work the general case is investigated.

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